Gravitational radiation from distant encounters and from head-on collisions of black holes: The zero-frequency limit

Larry Smarr*†

Princeton University Observatory, Princeton, New Jersey 08540

(Received 12 April 1976; revised manuscript received 7 January 1977)

The zero-frequency limit (ZFL) of the energy spectrum ($dE/d\omega d\Omega$) for the gravitational radiation emitted during the scattering or collision of two particles is investigated. If the asymptotic trajectories have constant velocities, at least one of which is nonzero, then the ZFL of the spectrum is flat and can be easily calculated. These calculations are made for the cases of distant encounters or head-on collisions of two compact objects, and comparisons to previous methods are made. It is found that the ZFL not only gives the exact low-frequency results, but that it provides an estimate of the total energy radiated, its polarization, and its angular distribution. Applied to the high-velocity collision ($V_\infty \sim 1$) of two equal-mass black holes it predicts an isotropic angular distribution of gravitational radiation with an efficiency of order unity

I. INTRODUCTION

When two bodies gravitationally scatter or collide, the accelerations involved cause gravitational radiation to be emitted. If the incoming and outgoing trajectories asymptotically have constant velocities, at least one of which is nonzero, then a technique exists[1-6,29] to calculate exactly the zero-frequency limit (ZFL) of $dE/d\omega d\Omega$, the energy radiated per unit frequency per steradian. One finds that the energy spectrum is flat as $\omega \to 0$, and therefore the ZFL also gives an estimate of the total energy radiated as well as of its angular distribution. This estimate is obtained by multiplying the ZFL by a cutoff frequency ω_c, whose value depends on some physical cutoff in the particular problem. Further, since this is the very-long-wavelength limit ($\omega^{-1} \gg$ size of the interaction region), the details of the internal structure of the objects as well as the details of their scattering are irrelevant. Therefore this limit should reliably describe even the scattering or collision of two black holes subject only to the above-stated restrictions on their asymptotic trajectories. I use the ZFL technique in Sec. II to calculate the zero-frequency gravitational bremsstrahlung produced by distant encounters or head-on collisions of two objects. In Sec. III, the exact ZFL answers and the approximate values for the total energy radiated are compared with previous calculations[7-12] of these two problems using other techniques.

II. THE ZERO-FREQUENCY LIMIT

The ZFL technique was originally derived from quantum arguments,[13] but it is equivalent to a purely classical calculation[5] (see Appendix). I shall use the notation of Feynman[1] since his method seems the easiest to apply. Let P^N represent the 4-momentum of the Nth particle ($P^N \cdot P^N = -m_N^2$), q^α the propagation 4-vector of the outgoing gravitational radiation, and $\epsilon_{\rm I}^{\mu\nu}$ the polarization tensor of type I (+ or ×).[14] Then the Feynman-Weinberg-DeWitt (FWD) amplitude for emission of such gravitational radiation from a classical scattering problem is

$$a_{\rm I} = \omega \sum_N \eta_N (P_\mu^N \epsilon_{\rm I}^{\mu\nu} P_\nu^N)/(P_\alpha^N q^\alpha) \qquad (2.1)$$

where $\eta_N = +1$ (-1) for incoming (outgoing) particles. The frequency ω of the radiation emitted in the direction $\vec{n}$, where $\vec{n} \cdot \vec{n} = 1$, is given by $q_\alpha = \omega(1, \vec{n})$. The analogous amplitude for the emission of electromagnetic radiation is

$$a_{\rm I}^{\rm em} = \omega \sum_N \eta_N (P_\mu^N \epsilon_{\rm I}^\mu)/(P_\alpha^N q^\alpha)\,, \qquad (2.2)$$

where $\epsilon_{\rm I}^\mu$ is a polarization vector of the appropriate type.

The energy radiated by gravitational radiation per unit frequency per steradian at zero frequency is given by ($c = 1$)

$$\left(\frac{dE_0}{d\omega d\Omega}\right)_{\rm I} = \frac{G}{2\pi^2} |a_{\rm I}|^2\,. \qquad (2.3)$$

This is the key formula I will use in the calculations below. One sees that the energy spectrum is flat as $\omega \to 0$, which suggests there exists some cutoff frequency ω_c above which $dE/d\omega$ rapidly drops to zero. Using an ω_c picked from physical considerations for the particular problem, one can then use (2.3) to get an *estimate* of the total energy radiated in polarization I into solid angle Ω:

$$\frac{\Delta E_{\rm I}}{d\Omega} \simeq \frac{G\omega_c}{2\pi^2} |a_{\rm I}|^2\,. \qquad (2.4)$$

The total energy is then obtained by integrating over angle and summing over polarization:

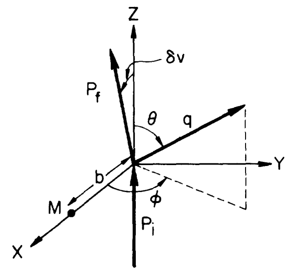

FIG. 1. The coordinate system used for the gravitational scattering of a small mass m by a large mass M with impact parameter b. The initial momenta P_i of m lies along the z axis. The final momenta P_f has been deflected through an angle δv into the XZ plane. During the process, gravitational radiation is emitted in the (θ, ϕ) direction with momentum $\vec{q} = \omega \vec{n}$.

$$\Delta E \simeq \frac{G\omega_c}{2\pi^2} \sum_{I} \int |a_I|^2 d\Omega \ . \tag{2.5}$$

Again, the values given by (2.4) and (2.5) should be considered estimates correct to order unity, while the ZFL, Eq. (2.3), should give the exact answer at zero frequency. From here on I set $c = G = 1$.

Turn now from the frequency dependence to the angular dependence. Using (2.1) with $P = \gamma m(1, \vec{v})$ and $\vec{v} \cdot \vec{n} = \cos\theta$, one finds $a \sim \sin^2\theta/(1 - v\cos\theta)$. As FWD point out, this means the amplitude for gravitational-radiation emission, in the limit $v \to 1$, does not show the sharp forward peaking $(\theta \simeq 0)$ that electromagnetism does. This is because the amplitude for emission of electromagnetic radiation (2.2) goes as $\sim \sin\theta/(1 - v\cos\theta)$. The presence of s momenta in the numerator for emission of a spin-s massless particle (scalar, photon, graviton) means the angular dependence is very different for different spin in the ZFL. The question arises as to whether these general comments on the nature of the coupling between matter and radiation fields in the ZFL allow one to conclude that gravitational radiation can never be beamed at zero frequency. To determine the answer I have carried out several model calculations using the ZFL.

First, consider a point particle of mass m and velocity v which is gravitationally scattered by a much heavier particle of mass M. If there is a large impact parameter b, then the angle of scattering[15] is

$$\Delta\theta = \left(\frac{r_{\rm Sch}}{b}\right)\left(\frac{1+v^2}{v^2}\right) , \tag{2.6}$$

where $r_{\rm Sch} = 2M$ is the Schwarzschild radius of the large mass M. If m is initially moving up the z axis while M is fixed on the x axis, then the initial and final momenta of m are

$$P^i = \gamma m(1, 0, 0, v), \qquad P^f = \gamma m(1, \delta, 0, v) \tag{2.7}$$

where $\delta \equiv v\Delta\theta$. Let the radiation be emitted along $\vec{n}$ in the (θ, ϕ) direction[16] (see Fig. 1). The two polarization tensors are constructed from the unit vectors $(\vec{e}_\theta, \vec{e}_\phi)$ by

$$\epsilon_+ = \frac{1}{\sqrt{2}} (\vec{e}_\theta \vec{e}_\theta - \vec{e}_\phi \vec{e}_\phi) , \qquad \epsilon_\times = \frac{1}{\sqrt{2}} (\vec{e}_\theta \vec{e}_\phi + \vec{e}_\phi \vec{e}_\theta) . \tag{2.8}$$

Transforming $(\vec{e}_\theta, \vec{e}_\phi)$ into $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and using (2.1), the two amplitudes are (the emission from mass M can be neglected in lowest order)

$$a_+ = \frac{\gamma m\, \delta\{v \sin\theta \cos\phi[2\cos\theta - v(1+\cos^2\theta)] - \delta(1 - v\cos\theta)(\cos^2\theta\cos^2\phi - \sin^2\phi)\}}{\sqrt{2}\,(1 - v\cos\theta)(1 - v\cos\theta - \delta\sin\theta\cos\phi)} , \tag{2.9}$$

$$a_\times = \frac{2\gamma m\delta \sin\phi(v\sin\theta - \delta\cos\theta\cos\phi)}{\sqrt{2}\,(1 - v\cos\theta - \delta\sin\theta\cos\phi)} . \tag{2.10}$$

To proceed I will keep only the terms first order in δ. This effectively means considering only small-angle, large-impact-parameter scattering. Precisely, I require for a given b and δ that

$$(r_{\rm Sch}/b)^{1/2} < v < 1 - \tfrac{1}{2}\delta^2 , \qquad \theta > \delta v^{-1} . \tag{2.11}$$

The energy E_0, radiated per unit frequency interval per steradian at zero frequency in each polarization is now obtained using (2.3), (2.9), and (2.10):

$$\left(\frac{dE_0}{d\omega d\Omega}\right)_+ = \left(\frac{\gamma^2 m^2 v^2 \delta^2}{4\pi^2}\right) \times \frac{\sin^2\theta[2\cos\theta - v(1+\cos^2\theta)]^2}{(1 - v\cos\theta)^4} \cos^2\phi,$$

$$\left(\frac{dE_0}{d\omega d\Omega}\right)_\times = \left(\frac{\gamma^2 m^2 v^2 \delta^2}{\pi^2}\right) \frac{\sin^2\theta}{(1 - v\cos\theta)^2} \sin^2\phi . \tag{2.12}$$

Integrating over angles yields

$$\left(\frac{dE_0}{d\omega}\right)_+ = \frac{\gamma^2 m^2 v^2 \delta^2}{4\pi}\left(\frac{1}{v^5}\right) \times \left[8v - \tfrac{16}{3}v^3 - 4(1-v^2)\ln\left(\frac{1+v}{1-v}\right)\right],$$

$$\left(\frac{dE_0}{d\omega}\right)_\times = \frac{2\gamma^2 m^2 v^2 \delta^2}{\pi}\left(\frac{1}{v^3}\right)\left[-2v + \ln\left(\frac{1+v}{1-v}\right)\right]. \quad (2.13)$$

These ZFL spectra will be compared with other published bremsstrahlung calculations in Sec. III.

The other model calculation is that of the head-on collision of two particles to form one particle at rest. In this case, no restriction will be placed on the masses of the two particles. If mass m_1 has velocity v_1 and mass m_2 has velocity v_2, then the initial momenta of m_1 and m_2 are

$$P_1^i = \gamma_1 m_1(1, 0, 0, v_1),$$
$$P_2^i = \gamma_2 m_2(1, 0, 0, -v_2). \quad (2.14)$$

Here there is no contribution from the final particle at rest. Using conservation of momentum ($\gamma_1 m_1 v_1 = \gamma_2 m_2 v_2$) with (2.1) and (2.8), I find the amplitudes to be

$$a_+ = \frac{1}{\sqrt{2}}(\gamma_1 m_1 v_1)\sin^2\theta\left[\frac{v_1+v_2}{(1-v_1\cos\theta)(1+v_2\cos\theta)}\right],$$
$$a_\times = 0. \quad (2.15)$$

The vanishing of $a_\times$ is a general property of a nonrotating axisymmetric system, as is the independence of a_+ from the azimuthal angle ϕ. The result for $dE_0/d\omega d\Omega = (dE_0/d\omega d\Omega)_+$ is

$$\left(\frac{dE_0}{d\omega d\Omega}\right) = \frac{\gamma_1^2 m_1^2 v_1^2 \sin^4\theta}{4\pi^2} \times \left[\frac{v_1+v_2}{(1-v_1\cos\theta)(1+v_2\cos\theta)}\right]^2. \quad (2.16)$$

For comparison with previous calculations, I consider the two limiting cases: First, if $m_1 \equiv m \ll m_2 \equiv M$, $v_1 \equiv v \gg v_2$, $m\gamma \ll M$, then

$$\frac{dE_0}{d\omega d\Omega} = \frac{\gamma^2 m^2 v^4}{4\pi^2}\frac{\sin^4\theta}{(1-v\cos\theta)^2}, \quad (2.17)$$

$$\frac{dE_0}{d\omega} = \frac{\gamma^2 m^2}{2\pi v}\left[8v - \tfrac{16}{3}v^3 - 4(1-v^2)\ln\left(\frac{1+v}{1-v}\right)\right], \quad (2.18)$$

and second, if $m_1 = m_2 = m$, $v_1 = v_2 = v$, then

$$\frac{dE_0}{d\omega d\Omega} = \frac{\gamma^2 m^2 v^4}{\pi^2}\frac{\sin^4\theta}{(1-v^2\cos^2\theta)^2}, \quad (2.19)$$

$$\frac{dE_0}{d\omega} = \frac{2\gamma^2 m^2}{\pi}\left\{2 + (1-v^2)\left[1 - \frac{1}{2v}(3+v^2)\ln\left(\frac{1+v}{1-v}\right)\right]\right\}. \quad (2.20)$$

III. COMPARISON WITH OTHER TECHNIQUES

A number of calculations exist in the literature of the gravitational radiation produced by distant encounters or head-on collisions of particles.[7-12] Because these calculations involve various regimes of velocity or mass ratio, a fairly broad comparison can be made with the results of the preceding section. The purpose of the comparison is threefold. First, it will demonstrate the validity of the ZFL, if the ZFL gives answers which agree to order unity with more complicated but better established methods of calculation. Second, the ZFL results contain more details about the radiation at low frequencies (for instance polarization) than any previously published work. Third, the ZFL provides a framework in which a number of different calculational techniques can be compared and contrasted.

The bremsstrahlung from scattering will be treated first. Consider the low-velocity limit of (2.13). Adding polarizations I find the ZFL:

$$\frac{dE_0}{d\omega} = \frac{32 m^2 M^2}{5\pi b^2}, \quad v \to 0, \quad \omega \to 0. \quad (3.1)$$

This agrees exactly with the ZFL of Ruffini and Wheeler's[8] calculation of low-velocity scattering using the Landau-Lifshitz quadrupole-moment technique. They do not calculate the energy carried by each polarization, but I find $(dE_0/d\omega)_\times = 5(dE_0/d\omega)_+$ as $v \to 0$. They were able to obtain the spectrum $dE/d\omega$ and the total energy radiated ΔE. Dividing ΔE by (3.1) they obtain an effective cutoff frequency ω_c such that

$$\Delta E = \left(\frac{dE_0}{d\omega}\right)\omega_c, \quad \omega_c \simeq 3.8 v b^{-1}. \quad (3.2)$$

Their figure 30 of $dE/d\omega$ shows the characteristic type of spectrum one expects for processes in which the ZFL is useful: nonzero $dE_0/d\omega$ at zero frequency and a cutoff which is a factor of order unity (here 3.8) times a characteristic frequency (vb^{-1}) for the problem. The angular distribution of the ZFL equation (2.12) may be compared with Peters's[7,16] Eq. (3.16) which gives $dE/d\Omega$ obtained by taking the $v \ll 1$ limit of his perturbation calculation. Since his $dE/d\Omega$ contains all frequencies and both polarizations, the ZFL $dE_0/d\omega d\Omega$, Eq. (2.12), must be multiplied by ω_c, (3.2) above, and summed over the two polarizations. Figure 2 then plots both $dE/d\Omega$ from Peters's work and from this paper for $v = 0.1$. The agreement between them in the YZ plane ($\phi = \pi/2$) is very good (this is the dominant $\times$ polarization), while the XZ plane ($\phi = 0$), shows the higher frequencies Peters includes contribute an angular dependence which tends to smooth out the basic 4-petal quadrupole

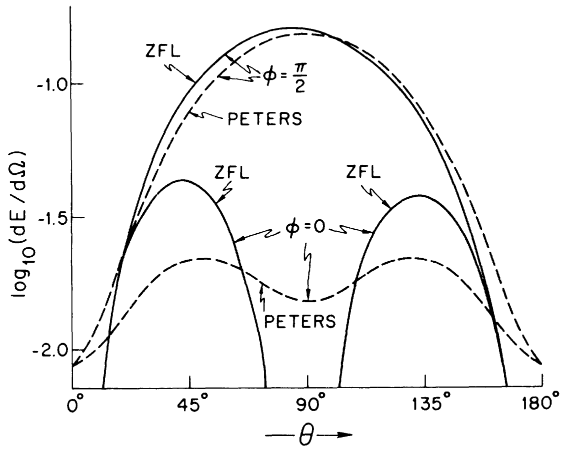

FIG. 2. The angular distribution of gravitational radiation emitted during the low-velocity ($v = 0.1$) scattering of two particles. $\text{Log}_{10}(dE/d\Omega)$ is plotted against θ for $\phi = 0$ (the XZ plane) and $\phi = \pi/2$ (the YZ plane). Both the results of the ZFL (this paper) and the fully relativistic perturbation technique (Peters) are graphed. The units of $dE/d\Omega$ are $M^2m^2b^{-3}$ ($G = c = 1$).

shape. Nonetheless, it is clear that the angular distribution which arises solely from the ZFL dominantes the higher-frequency contributions in this low-velocity limit.

The agreement continues in the ultrarelativistic limit $v \rightarrow 1$. Again to compare with Peters, I need to choose a cutoff frequency ω_c. From Peters's Eq. (4.4) one finds that the radiation time during closest approach goes as γ^{-1}, suggesting an $\omega_c = K\gamma b^{-1}$, where K is a constant of order unity. Multiplying (2.12) by ω_c (with $K \sim 1$) and summing polarizations, I obtain the ZFL estimate of $dE/d\Omega$ which is plotted in Fig. 3 ($\phi = 0$) and Fig. 4 ($\phi = \pi/2$) for $v = 0.1$, 0.75, and 0.99. [Note the $v = 0.1$ plot disagrees with Fig. 2 by the factor 3.8 in Eq. (3.2).] The sharp forward peaking that Peters finds is quite evident. One can directly compare Peters's Fig. 3 with Fig. 3 of this paper. It seems that the angular smoothing induced by the inclusion of the higher frequencies, noted for the low-v limit, carries over to the high-v limit in the XZ plane, with the ZFL still producing quite distinct backlobes. However, the envelope of the ZFL angular distribution is quite close to the shape Peters finds. In addition, I find the peak value in the XZ plane of $(dE/d\Omega$ goes as $0.7K\gamma^5$ while Peters's Fig. 4 indicates that he finds $(dE/d\Omega)_{\text{peak}} \sim 2.5\gamma^5$ in the units $M^2m^2b^{-3}$. While Peters does not graph $dE/d\Omega$ for the YZ plane, he does comment that it is peaked as well. In Fig. 4, one can clearly see the transformation of the YZ-plane quadrupole pattern to a beamed pattern as v goes from 0 to 1. As in the low-v limit the $\times$ polarization dominates the + polariza-

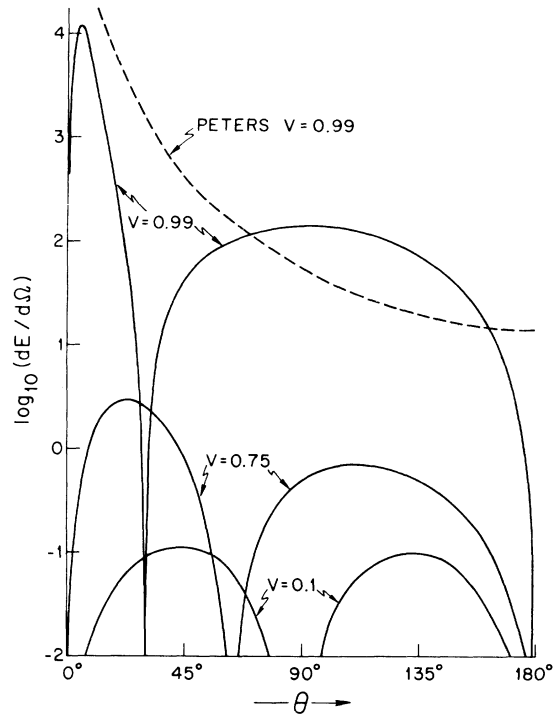

FIG. 3. The change in $\log_{10}(dE/d\Omega)$ versus θ in the XZ plane for $v = 0.1$, 0.75, 0.99. The result of Peters (Ref. 7) is shown for comparison for $v = 0.99$. The units for $dE/d\Omega$ are as in Fig. 2.

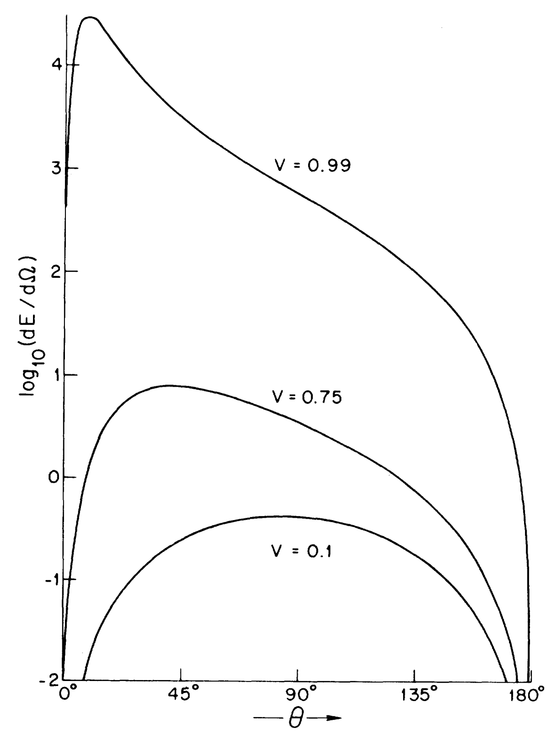

FIG. 4. As for Fig. 3, except in the YZ plane.

tion so that the most intense radiation occurs in the YZ plane.

Finally, I estimate the total energy radiated, ΔE, by integrating $\omega_c(dE_0/d\Omega d\omega)$ over angles. Using (2.13), I find

$$\Delta E_+ = \frac{8Km^2M^2\gamma^3}{3\pi b^3} ,\qquad \Delta E_\times = \frac{8Km^2M^2\gamma^3}{\pi b^3}\ln(4\gamma^2) . \tag{3.3}$$

Here again the $\times$ polarization is dominant with $\Delta E_\times/\Delta E_+ = 3\ln(4\gamma^2)$. Equation (3.3), obtained using the ZFL, casts new light on the apparent disagreement between the results of Peters,[7] who used perturbation theory, and the results of Matzner and Nutku,[9] who used the method of virtual quanta. Peters finds that $\Delta E \sim \gamma^3$, while Matzner and Nutku find $\Delta E \sim \gamma^3 \ln(4\gamma^2)$. If ω_c is independent of angle, then the ZFL analysis shows that Matzner and Nutku are correct in their discovery of the logarithm term.[17] However, my results for the numerical coefficient of ΔE are in much closer agreement with Peters than with Matzner and Nutku:

$$\left.\begin{array}{l}\Delta E_{\rm Peters} \sim 25\\ \Delta E_{\rm MN} \sim 256\ln(4\gamma^2)\\ \Delta E_{\rm ZFL} \sim 10K\ln(4\gamma^2)\end{array}\right\} \times \frac{m^2M^2\gamma^3}{b^3} . \tag{3.4}$$

The ZFL seems to be a complementary calculation to Matzner and Nutku's since their virtual-quanta technique breaks down at low frequencies.[9] The above comparison demonstrates that the ZFL can predict sharp forward peaking as $v \to 1$.

That this is not always necessary is made clear by studying the model calculation for the head-on collision of two particles. I investigated such a problem earlier[18] because it serves as a model for the high-speed collision of two black holes. The details of low-velocity collisions can be calculated on digital computers by solving the axisymmetric Einstein equations for the spacetime of the collision.[19, 20] This complicated procedure is necessary when $v_\infty \le 0$, because then the acceleration of the holes toward each other, as well as the details of the horizons coalescing, will determine the angular pattern and total energy content of the radiation. However, in the case $v_\infty \simeq 1$, it may well be that the ZFL technique can be applied.

One limit of a head-on collision is that for which one mass m is very much smaller than the other mass M. The relativistic spacetime perturbation version of this collision has been calculated[11] for the case where $v_\infty > 0$. This may be compared with the ZFL result (2.17). As $v \to 0$, $dE/d\omega d\Omega$ shows the $\sin^4\theta$ quadrupole dependence characteristic of low-velocity infall. Furthermore, $dE_0/d\omega \sim v_\infty^4$, as pointed out in Ref. 11. Thus, $dE_0/d\omega \to 0$ for parabolic infall ($v_\infty \to 0$), even though $\Delta E \ne 0$. This demonstrates that if both the initial and final velocities of the particles are zero, the ZFL method fails and the gravitational radiation emitted is dominated by the details of the collision.

On the other hand, as $v_\infty \to 1$ the low frequencies dominate the spectrum (see $dE/d\omega$ in Figs. 3 and 4 of Ref. 11). In contrast to the distant-encounter scattering, one can see from the $v \to 1$ limit of (2.17), plotted in Fig. 5, that no sharp forward peaking occurs in the angular distribution of the head-on collision. As for the ZFL limit of $dE/d\omega$ itself, Eq. (2.18), one can compare the $dE_0/d\omega$ calculated in Ref. 11 for the lowest multipole moments. Summing the values for $l = 2, 3, 4,$ one finds the fully relativistic calculation[11] yields $dE_0/d\omega \lesssim 5.9m^2$ for $\gamma = 4$ and $dE_0/d\omega \lesssim 13.5\, m^2$ for $\gamma = 6$. My Eq. (2.18) gives $dE_0/d\omega = 4.9m^2$ and $12.9m^2$, respectively. This indicates very close agreement as $v \to 1$.

Turning finally to the total energy radiated as $v \to 1$, one finds from Ref. 11 that $\omega_c \sim (2M)^{-1}$ independent of v_∞. Presumably this is because the characteristic collision time is determined by the large black hole whose radius is $2M$. Using this ω_c to multiply (2.18), I find the limit as $v \to 1$

$$\Delta E \sim 0.2\left(\frac{\gamma M}{M}\right)\gamma mc^2 , \tag{3.5}$$

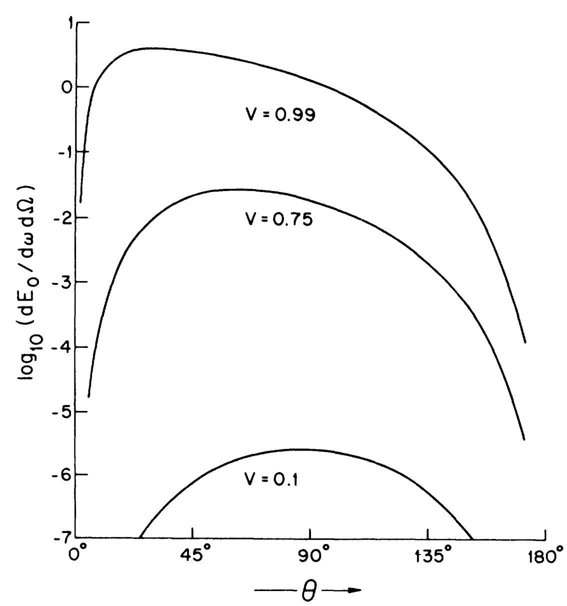

FIG. 5. $\text{Log}_{10}(dE_0/d\omega d\Omega)$ versus θ for a head-on collision of two particles with $m_1 \ll m_2$ and $v = 0.1,\ 0.75,\ 0.99$. The units for $dE_0/d\omega d\Omega$ are m_1^2.

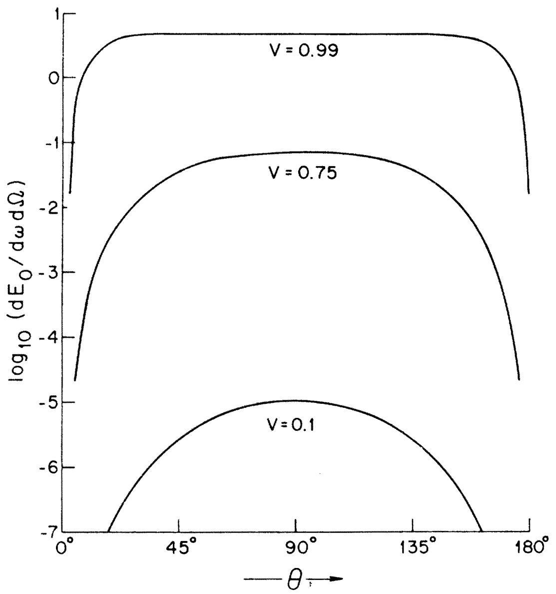

FIG. 6. Same as Fig. 5, except $m_1 = m_2$.

which means the radiation efficiency $\Delta E/\gamma M \ll 1$ since $\gamma m \ll M$ by assumption. The γ^2 dependence predicted analytically by the ZFL is seen to agree well with the numerical results graphed in Fig. 2 of Ref. 11.

Since the ZFL seems to give good agreement in the $m_1 \ll m_2$ limit, I now consider the $m_1 = m_2$ case. The low-v limit is essentially as above. In the high-velocity limit of Eq. (2.19), one finds that not only is there no sharp forward peaking, but that instead the angular distribution tends to an isotropic pattern[18] (see Fig. 6). This is due to the sum and interference of two patterns like Fig. 5, since now both particles contribute equally. The $v \to 1$ limit of Eq. (2.20) yields the ZFL of the spectrum

$$\frac{dE_Q}{d\omega} \to \frac{4\gamma^2 m^2}{\pi}, \tag{3.6}$$

a factor of 3 greater than for the above case with $m_1 \ll m$.

Until recently, there has been no calculation available with which to compare this ZFL result. Adler and Zeks[12] have calculated the time-reversed version of two equal masses flying apart from a central object initially at rest. The technique they use, the purely classical approach as given by Weinberg,[5] is completely equivalent to the ZFL. They thus agree exactly with Eq. (3.6). To estimate the total energy radiated one has to estimate ω_c, the cutoff frequency. Dimensionally ω_c goes as m^{-1}. If energy conservation is not to be grossly violated, the efficiency $\Delta E/2m\gamma$ must be less than unity. This, together with the fact that the characteristic scale is $m\gamma$, leads to $\omega_c \sim (Km\gamma)^{-1}$. This gives an efficiency estimate of

$$\frac{\Delta E}{2m\gamma} = \left(\frac{dE_Q}{d\omega}\right)\left(\frac{\omega_c}{2m\gamma}\right) \simeq \frac{2}{\pi K} = \frac{0.6}{K}. \tag{3.7}$$

These calculations lead to the following speculative picture of the high-speed collision of two equal-mass black holes. As they approach each other at roughly constant velocity ($v_\infty \sim 1$), they see each other's gravitational field Lorentz-contracted[21] from spheres into plane waves with axial symmetry in the plane. As the plane waves collide, they tend to focus themselves somewhat, although not nearly as severely as homogeneous plane waves,[22] and become outgoing gravitational waves with roughly isotropic distribution. Since most of the energy is in the initial kinetic energy of the holes, this gets converted, with on the order of unity efficiency, into escaping radiation leaving the small rest-mass energy behind in the final spherical hole.

Penrose[23] has shown, by finding trapped surfaces in the initial data for a speed-of-light collision, that if "cosmic censorship"[24] holds (no naked singularities form) then the efficiency must be strictly less than 50%. The ZFL estimate, Eq. (3.7), shows there is a chance that this hypothesis is violated (if $K < 1.3$), but since the ZFL cannot determine ω_c exactly it cannot decide the issue. Fortunately, since the above discussion was submitted, D'Eath,[25] using an elegant and ingenious perturbation method, has been able to calculate the detailed structure of high-speed collisions.[28] He finds that cosmic censorship is not violated since $K \sim 2.6$, i.e., the efficiency is $\sim 25\%$. Furthermore, his calculations confirm the ZFL prediction [Eq. (3.6) and Fig. 6] that the angular distribution becomes isotropic as $v_\infty \to 1$. This result is important because it demonstrates that the ZFL method can accurately discriminate between situations where beaming occurs (bremsstrahlung) and where it does not (collisions). Also, the above case shows the ZFL can predict certain aspects of the radiation before complicated calculations are made.

Note that the hyperbolic collision ($v_\infty \simeq 1$) is qualitatively different from a parabolic one ($v_\infty = 0$). In the former situation, the gravitational radiation is produced by the field far from the black holes, and the efficiency is determined by the asymptotic trajectories. The black holes play little role in either the generation or propagation of the main part of the radiation. Just the opposite seems to be the case for a parabolic or bound collision. Here the ZFL fails precisely because the details of the interaction region determine the radiation.

Thus, it seems that detailed study of the hyperbolic collision will not shed light on the parabolic case and vice versa. In particular, the upper limit on the efficiency for hyperbolic collisions (50%) obtained[23] by use of the Hawking black-hole-area-increase theorem,[26] is within a factor 2 of the real efficiency, whereas for the parabolic case the actual efficiency seems to be orders of magnitude lower[20] than the Hawking[27] upper limit of 29%.

IV. SUMMARY

In summary, I have shown that the very simple analytic procedure of calculating the exact $dE_0/d\omega d\Omega$ at zero frequency reproduces the main qualitative, and in most cases quantitative, features of fully relativistic numerical calculations for which this procedure is appropriate. Because one is concerned only with very long wavelengths, the details of the scattering process are irrelevant, and only the asymptotic trajectories need be considered. There is a great enhancement at low frequencies, and the spectrum $dE/d\omega$ is flat (independent of ω) as $\omega\to 0$. This means that an estimate, good to order unity, of the *total* energy radiated, as well as its angular distribution, can be obtained by estimating the cutoff frequency ω_c and multiplying it by the ZFL of $dE/d\omega d\Omega$.

The procedure is useful whenever the asymptotic trajectories have constant velocity, at least one of which is nonzero. Thus, it works for both high-velocity ($v\to 1$) and low-velocity ($v\to 0$) distant encounters, where the scattering is through a small angle. It also works for head-on collisions when the velocity at infinity $v_\infty\to 1$. Even for low v, it recovers most of the qualitative features. It is found that beaming may (distant encounters) or may not (head-on collisions) occur, even though the amplitude for the emission of gravitational radiation from a single particle trajectory is *not* beamed for high v. From a quantum viewpoint, this is because one sums the amplitudes (2.1) for emission from all lines and then squares the result to obtain (2.3), thus introducing interference terms which cause the beaming. From a classical viewpoint, the difference can be understood by considering Lorentz-frame transformations. In the distant-encounter case, one can go to a frame in which the velocity is small (the moving-particle frame) and the angular distribution is quadrupole. When one transfers to the fixed frame in the laboratory, a strong headlight effect occurs. In the head-on collision, there is *no* frame in which the acceleration is small, so there is no large velocity shift between frames and consequently no beaming.

An intriguing aspect of the ZFL method is that it can be applied even when the interaction occurs in a strong-gravitational-field region, e.g., a head-on collision of two black holes. It will be interesting to see if there are other examples of strong-field high-velocity problems which can be treated using the ZFL.

ACKNOWLEDGMENTS

I thank B. S. DeWitt for suggesting the problem and R. Bontz, R. H. Price, and W. H. Press for very valuable criticism. I am grateful to P. L. Chrzanowski, P. D'Eath, R. A. Matzner, C. H. Taubes, and K. S. Thorne for stimulating discussions, and to N. M. J. Woodhouse for helpful comments on the manuscript. Financial support during revision of this paper was provided by the Harvard Society of Fellows.

APPENDIX

Because the main references[1-6] for the fundamental ZFL equation (2.3) are quantum mechanical, the following classical derivation was worked out in collaboration with Robert Bontz and Richard Price. Let us consider the metric perturbation $h_{\mu\nu}$ induced at the field point x by a particle with 4-velocity u_ν at a point x' on its trajectory. Since we are interested in the long-wavelength limit, we assume that the field point is far from the particle and that spacetime is nearly flat there. In the particle's rest frame the perturbation is that due to the gravitational field of a static particle of mass M. From Ref. 14 (MTW), p. 441, this means that

$$\bar{h}_{00}=\frac{4M}{r}\ ,\quad \bar{h}_{0j}=\bar{h}_{jk}=0\ , \tag{A1}$$

where $\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h$. However, we need the perturbation in the rest frame of the observer. By a Lorentz transformation one finds

$$\bar{h}_{\mu\nu}=\frac{4M u_\mu u_\nu}{-k\cdot u}\ . \tag{A2}$$

The denominator occurs because in the particle's rest frame

$$-k\cdot u=k^0=|x^0-x'^0|\equiv r_{\rm ret}\ , \tag{A3}$$

where $k=r_{\rm ret}(1,\hat{n})$ is the null vector joining the point x and x'.

The energy flux through a 2-sphere of radius $r_{\rm ret}$ is given by [MTW equation (35.23)]

$$\frac{dE}{d\Omega dt}=\frac{{r_{\rm ret}}^2}{32\pi}\ \langle \dot{h}^{TT}_{jk}\dot{h}^{TT}_{jk}\rangle_{\rm av}\ \ , \tag{A4}$$

where the TT gauge implies only $h_{\theta\theta}$, $h_{\phi\phi}$, and $h_{\theta\phi}$ are nonzero. Using the polarization tensors defined by Eq. (2.8), this can be rewritten as

$$\frac{dE}{d\Omega dt}=\frac{r_{\rm ret}^2}{32\pi}\langle(\dot{h}_{jk}\epsilon_+^{jk})^2+(\dot{h}_{jk}\epsilon_\times^{jk})^2\rangle_{\rm av}\ . \tag{A5}$$

Define the amplitudes $\dot{\mathcal{B}}_{\rm I}$ and their Fourier transforms $\dot{B}_{\rm I}(\omega)$ by

$$\dot{\mathcal{B}}_{\rm I}\equiv\dot{h}_{jk}\epsilon_{\rm I}^{jk}\ ,\qquad \dot{B}_{\rm I}\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\ \dot{\mathcal{B}}_{\rm I}\,e^{i\omega t}, \tag{A6}$$

noting that $\dot{B}_{\rm I}(-\omega)=\dot{B}_{\rm I}^*(\omega)$ since $\dot{\mathcal{B}}_{\rm I}$ is real. Parseval's theorem then allows us to rewrite Eq. (A5) as

$$\frac{dE}{d\Omega}=\frac{r_{\rm ret}^2}{32\pi}\sum_{\rm I}\int_{-\infty}^{\infty}dt|\dot{\mathcal{B}}_{\rm I}|^2=\frac{r_{\rm ret}^2}{16\pi}\sum_{\rm I}\int_0^{\infty}|\dot{B}_{\rm I}(\omega)|^2d\omega\ . \tag{A7}$$

The ZFL then yields

$$\frac{dE_0}{d\Omega d\omega}=\frac{r_{\rm ret}^2}{16\pi}\sum_{\rm I}|\dot{B}_{\rm I}(0)|^2\ . \tag{A8}$$

The final step is evaluating $\dot{B}_{\rm I}(0)$:

$$\begin{aligned}\dot{B}_{\rm I}(0)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\dot{\mathcal{B}}_{\rm I}\,dt\\&=\frac{1}{\sqrt{2\pi}}[\mathcal{B}_{\rm I}(t=+\infty)-\mathcal{B}_{\rm I}(t=-\infty)]\\&=\frac{1}{\sqrt{2\pi}}[(h_{jk}\epsilon_{\rm I}^{jk})_{t=+\infty}-(h_{jk}\epsilon_{\rm I}^{jk})_{t=-\infty}]\ .\end{aligned} \tag{A9}$$

Here it is explicitly seen that the ZFL depends *only* on the asymptotic ($t\to\pm\infty$) trajectories (through h_{jk}). Plugging in h_{jk} from Eq. (A2) gives

$$\frac{dE_0}{d\omega d\Omega}=\frac{r_{\rm ret}^2}{16\pi}\sum_{\rm I}\left(\frac{1}{\sqrt{2\pi}}\sum_N\eta_N\frac{4P_\mu^NP_\nu^N}{k\cdot P^N}\epsilon_{\rm I}^{\mu\nu}\right) \tag{A10}$$

where the η_N and P_μ^N are as defined in Sec. II. Since $k=r_{\rm ret}(1,\vec{n})=r_{\rm ret}\omega^{-1}q$, where $q=\omega(1,\vec{n})$, Eq. (2.3) follows immediately.

Weinberg's classical derivation[5] leads him to the following formula (his equation 10.4.22):

$$\frac{dE}{d\Omega d\omega}=\frac{G\omega^2}{2\pi^2}\sum_{N,M}\frac{(P_N\cdot P_M)^2-\frac{1}{2}m_N^2m_N^2}{(P_N\cdot q)(P_M\cdot q)}\ , \tag{A11}$$

where the η_N, η_M are dropped for convenience. Here the two polarization states have been summed over, but it is not apparent that his result is equivalent to my Eq. (2.3) summed over I. However, as shown below, if one assumes momentum conservation

$$\sum_N P_N=0\ , \tag{A12}$$

the two formulas are identical.

First, split $\underline{P}_N$ into orthogonal pieces:

$$\begin{aligned}\underline{P}_N&=P_N^0\underline{u}+\vec{P}_N\\&=P_N^0\underline{u}+\vec{P}_N^q+\vec{P}_N^\perp\ .\end{aligned} \tag{A13}$$

My notation is that an arrow over the letter denotes a three-vector, while an underline denotes a four-vector. The time axis lies in the direction of $\underline{u}$ and the spatial part of $\vec{P}_N$ is split into a piece in the direction of $\underline{q}$: $\vec{P}_N^q\equiv(\vec{n}\cdot\vec{P}_N)\vec{n}$ and a piece transverse to $\underline{q}$: $\vec{P}_N^\perp\equiv\vec{P}_N-\vec{P}_N^q$. Now since $\underline{P}_N\cdot\underline{q}=\omega(P_N^0+|\vec{P}_N^0|)$, Eq. (A13) can be rewritten as

$$\underline{P}_N=(\omega^{-1}\underline{P}_N\cdot\underline{q}-|\vec{P}_N^q|)\underline{u}+\vec{P}_N^q+\vec{P}_N^\perp\ . \tag{A14}$$

Weinberg requires us to consider $(\underline{P}_N\cdot\underline{P}_M)^2$ or

$$(\underline{P}_N\cdot\underline{P}_M)^2=[-(\omega^{-1}\underline{P}_N\cdot\underline{q}-|\vec{P}_N^q|)(\omega^{-1}\underline{P}_M\cdot\underline{q}-|\vec{P}_M^q|)+\vec{P}_N^q\cdot\vec{P}_M^q+\vec{P}_N^\perp\cdot\vec{P}_M^\perp]^2\ . \tag{A15}$$

Expanded out, this is a sum of terms each containing four momenta. Every time a term involving $\underline{P}_N\cdot\underline{q}$ appears in the numerator of (A11) it cancels an identical term in the denominator, leaving one term involving a piece of P_N and two P_M. The sum N then vanishes by momentum conservation [Eq. (A12)]. For instance, a typical term is

$$\sum_{N,M}\frac{(\omega^{-1}\underline{P}_N\cdot\underline{q})|\vec{P}_M^q|\vec{P}_N^q\cdot\vec{P}_M^q}{(\underline{P}_N\cdot\underline{q})(\underline{P}_M\cdot\underline{q})}=\omega^{-1}\sum_N\vec{P}_N^q\cdot\sum_M\frac{|\vec{P}_M^q|\vec{P}_M^q}{\underline{P}_M\cdot\underline{q}}=0\ . \tag{A16}$$

Thus, we can drop all terms in Eq. (A15) involving $\underline{P}_N\cdot\underline{q}$. Further, the terms $\vec{P}_N^q\cdot\vec{P}_M^q=|\vec{P}_N^q|\,|\vec{P}_M^q|$ cancel the terms $-|\vec{P}_M^q||\vec{P}_M^q|$, leaving only

$$(\underline{P}_N\cdot\underline{P}_M)^2=(\vec{P}_N^\perp\cdot\vec{P}_M^\perp)^2 \tag{A17}$$

if momentum conservation (A12) holds. Similarly one can show that

$$(m_Nm_M)^2=|\vec{P}_N^\perp|^2\,|\vec{P}_M^\perp|^2\ . \tag{A18}$$

To make the final step, sum over polarization states in Eq. (2.3). By introducing the basis set in Eq. (2.8) one finds

$$\begin{aligned}P^M\cdot\epsilon_+\cdot P^N&=\frac{1}{\sqrt{2}}[(P_N^\theta)^2-(P_N^\phi)^2]\ ,\\P^N\cdot\epsilon_\times\cdot P^N&=\sqrt{2}\ P_N^\theta P_N^\phi\ ,\end{aligned} \tag{A19}$$

and a short calculation demonstrates that

$$\sum_{\rm I}\left(\sum_N P^N\cdot\epsilon_{\rm I}\cdot P^N\right)\left(\sum_M P^M\cdot\epsilon_{\rm I}\cdot P^M\right)=\sum_{N,M}[(\vec{P}_N^\perp\cdot\vec{P}_M^\perp)^2-\tfrac{1}{2}(|\vec{P}_N^\perp|\,|P_M^\perp|)^2]\ , \tag{A20}$$

where $\vec{P}_N^\perp=P_M^\theta\vec{e}_\theta+P_M^\phi\vec{e}_\phi$. This completes the proof of equivalence. In practice, it seems much easier to use the ZFL equation (2.3) than Weinberg's Eq. (A11) and one obtains polarization information automatically.

*Work supported in part by the National Science Foundation under Grant No. MPS74-18970.

† Present address: Center for Astrophysics, Harvard University, Cambridge, Mass. 02138.

[1]R. P. Feynman, letter to V. F. Weisskopf, 1961 (unpublished).

[2]S. Weinberg, Phys. Lett. 9, 357 (1964).

[3]S. Weinberg, Phys. Rev. 135, B1049 (1964).

[4]S. Weinberg, Phys. Rev. 140, B516 (1964).

[5]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[6]B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).

[7]P. C. Peters, Phys. Rev. D 1, 1559 (1970).

[8]R. Ruffini and J. A. Wheeler, in *The Significance of Space Reseach for Fundamental Physics*, edited by A. F. Moore and V. Hardy (ESRO, Paris, 1971).

[9]R. A. Matzner and Y. Nutku, Proc. R. Soc. London A336, 285 (1974)

[10]S. J. Kovács and K. S. Thorne, Caltech Report No. OAP-482

[11]R. Ruffini, Phys. Rev. D 7, 2972 (1973).

[12]R. J. Adler and B. Zeks, Phys. Rev. D 12, 3007 (1975).

[13]If one considers the general quantum amplitude for emission of a graviton line from a Feynman diagram representing the scattering process (Refs. 1–4, 6), then it is found that soft-graviton emission ($\omega \to 0$) from external particle lines is infrared divergent and completely swamps emission from internal lines in the ZFL. Thus, keeping only leading terms in ω^{-1} leads to the amplitude (2.1). There seems to be a numerical discrepancy of a factor of 4 between (2.3) and Eq. (13) of Ref. 1.

[14]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 952.

[15]A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton Univ. Press, Princeton, 1975), p. 415.

[16]The angles I use are different from those used in Ref. 7.

[17]Dr. P. C. Peters has made the following points: (1) My use of an angle-independent cutoff frequency $\omega_c \simeq K\gamma b^{-1}$ (third paragraph of Sec. III) may not be in agreement with his more detailed (unpublished) calculations. He suggests $\omega_c \simeq \gamma^{-1}(1 - v\cos\theta)^{-1}$ as an alternative. (2) If $\omega_c = \omega_c(\theta)$, then the angular integration over $d\Omega$ will give $\Delta E \sim \gamma^3$ with no $\ln(4\gamma^2)$ as I find with my ω_c. (3) In addition to the gravitational radiation due to the kinematics of any particle collision, which can be calculated using the ZFL, there may also be radiation produced by stresses keeping the particles in orbit [see P. C. Peters, Phys. Rev. D 5, 2477 (1972); R. H. Price and V. Sandberg, *ibid.* 8, 1640 (1973)].

[18]L. Smarr, unpublished work, 1973.

[19]L. Smarr, A. Čadež, B. S. DeWitt, and K. Eppley, Phys. Rev. D 14, 2443 (1976).

[20]L. Smarr, paper given at the 8th Texas Symposium on Relativistic Astrophysics, 1976 (unpublished); Ann. Acad. Sci. (N.Y.) (to be published).

[21]F. A. E. Pirani, Proc. R. Soc. London A252, 96 (1959).

[22]K. A. Khan and R. Penrose, Nature (London) 229, 185 (1971); P. Szekeres, J. Math. Phys. 13, 286 (1972).

[23]R. Penrose, private communication.

[24]R. Penrose, Ann. Acad. Sci. (N.Y.) 224, 125 (1973).

[25]P. D. D'Eath, report, 1976 (unpublished).

[26]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, England, 1973).

[27]S. W. Hawking, Phys. Rev. Lett. 26, 1344 (1971).

[28]See also G. E. Curtis, Ph.D. thesis, Oxford, 1975 (unpublished).

[29]I thank Kip Thorne for pointing out that a discussion of estimating gravitational radiation from impulsive events, using a classical ZFL approach, appears on pp. 112–114 of J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962).