Inverse scattering problem in nonrelativistic S-matrix theory

James T. Cushing
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556
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On the basis of unitarity and the Mandelstam representation, the inverse scattering problem is solved in a nonrelativistic S-matrix framework. Specifically, it is shown that the s-wave scattering amplitude, $A_0(s)$, will uniquely determine the full scattering amplitude, $A(s,t)$, provided no more than one subtraction in t is required and provided only the s wave has bound states. Generalizations of this result are also discussed.

I. INTRODUCTION

A standard form of the inverse scattering problem in nonrelativistic potential scattering theory is that of determining a local central potential, $V(r)$, which will reproduce a given partial-wave amplitude, $A_l(s)$. Here l is a fixed value of the angular momentum and s is a continuous energy variable. The Gel'fand-Levitan construction procedure states, basically, that if $A_l(s)$ has no bound states in it, then $V(r)$ is uniquely determined, whereas if n bound states are present, then an n-parameter family of equivalent local potentials is determined.[1,2] The dynamical equation used in this procedure is the nonrelativistic Schrödinger equation.

The purpose of this paper is to obtain a result similar to the Gel'fand-Levitan result, but this time within the framework of S-matrix theory and the Mandelstam representation, thus avoiding use of the Schrödinger equation. Such an approach may be generalizable to a relativistic, crossing-symmetric case since both S-matrix theory and the Mandelstam representation admit of straightforward relativistic forms. On the other hand, we do not have a Schrödinger-type equation to carry out the Gel'fand-Levitan construction relativistically. As a further motivation for the present work, we point out that a few years ago it was noticed that if, for the relativistic, crossing-symmetric $\pi\pi$ system, the partial-wave amplitudes, $A_l(s)$, were so correlated that one partial-wave amplitude at all energies would uniquely determine all other partial-wave amplitudes, then isotopic spin would be required as a symmetry for this system.[3]

In Sec. II we summarize some relevant facts about the Mandelstam representation and unitarity, and in Sec. III we present an argument for the uniqueness of the Mandelstam weight functions given the s-wave scattering amplitude, $A_0(s)$. Section IV contains a few comments about the foregoing demonstration and about difficulties in the full crossing-symmetric problem.

II. MANDELSTAM REPRESENTATION AND UNITARITY

The connection between S-matrix theory and the Schrödinger equation was investigated some years ago by Blankenbecler *et al.*[4] and more recently by Frederiksen *et al.*[5] For our discussion of the inversion problem in nonrelativistic S-matrix theory we restrict ourselves to the single-channel representation[4,5]

$$A(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt'\,\sigma(t')}{(t'-t)}+\frac{1}{\pi^2}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}\frac{dt'\,\rho(s',t')}{(s'-s)(t'-t)}\,, \tag{1}$$

where $s=4(k^2+m^2)$, $t=-2k^2(1-\cos\theta)$. Such a representation is the single-channel analog of the crossing-symmetric expression valid for the $\pi\pi$ system and is expected in nonrelativistic Schrödinger theory for a local potential, $V(r)$, which is a superposition of Yukawa potentials.[4] It is important to realize that the locality assumption is in fact contained in the region of support for the double spectral function, $\rho(s,t)$. That is, for a nonlocal separable potential in Schrödinger theory $A(s,t)$ has another cut in s for negative values of s.[6]

Since we are interested in the eventual generalization of this discussion to the relativistic case, we shall use relativistic kinematics throughout. Our normalizations are such that the differential cross section in the center-of-momentum frame is

$$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2 s}\,|A(s,\cos\theta)|^2\,, \tag{2}$$

and the optical theorem is

$$\sigma_{\text{tot}}(s)=\frac{1}{[s(s-4m^2)]^{1/2}}\,\text{Im}A(s,\theta=0)\,. \tag{3}$$

The elastic unitarity condition becomes

$$\mathrm{Im}A(k,(\vec{k}_f-\vec{k}_i)^2)=\frac{1}{64\pi^2}\left(\frac{s-4m^2}{s}\right)^{1/2}\int d\Omega_{\hat{k}'}A^*(k,(\vec{k}_f-\vec{k}')^2)\times A(k,(\vec{k}'-\vec{k}_i)^2). \quad (4)$$

Here, as elsewhere, we use interchangeably the notations $A(s,t,u)$, $A(s,t)$, $A(s,\cos\theta)$, $A(k,\cos\theta)$, and $A(k,(\vec{k}_f-\vec{k}_i)^2)$ as suits our purpose. If we define partial-wave amplitudes as

$$A_l(s)=\tfrac{1}{2}\int_{-1}^{1}dz\,A(s,z)P_l(z)\ , \quad (5)$$

so that

$$A(s,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(\cos\theta)\ , \quad (6)$$

then the unitarity condition becomes

$$\mathrm{Im}A_l(s)=\frac{1}{16\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}|A_l(s)|^2 \quad (7)$$

with

$$A_l(s)=16\pi\left(\frac{s}{s-4m^2}\right)^{1/2}e^{i\delta_l}\sin\delta_l(s)\ . \quad (8)$$

If we use Eq. (1) to carry out the partial-wave projections, we obtain

$$A_l(s)=\frac{2}{\pi(s-4m^2)}\int_{4m^2}^{\infty}dt'\,\sigma(t')\,Q_l\left(1+\frac{t'}{2k^2}\right)+\frac{2}{\pi^2(s-4m^2)}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\frac{\rho(s',t')}{(s'-s)}Q_l\left(1+\frac{t'}{2k^2}\right), \quad (9)$$

and for the Born amplitude we define

$$A_l^B(s)=\frac{1}{2\pi k^2}\int_{4m^2}^{\infty}dt'\,\sigma(t')\,Q_l\left(1+\frac{t'}{2k^2}\right). \quad (10)$$

In our case these partial-wave amplitudes satisfy dispersion relations of the form

$$A_l(s)=A_l^B(s)+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'\,\mathrm{Im}A_l(s')}{(s'-s)}+\frac{1}{\pi}\int_{-\infty}^{-12m^2}\frac{ds'\,[\mathrm{Im}A_l(s')-\mathrm{Im}A_l^B(s')]}{(s'-s)} \quad (11)$$

The Born amplitude, $A_l^B(s)$, has *only* a left-hand cut and it runs from $-\infty$ to 0.

When the Mandelstam representation, Eq. (1), is combined with elastic unitarity, Eq. (4), one obtains the well-known result[4]

$$\begin{aligned}\rho(s,t)=\frac{1}{16\sqrt{s}}\Bigg(&\int_{4m^2}^{\infty}\frac{dt_1}{\pi}\int_{4m^2}^{\infty}\frac{dt_2}{\pi}\frac{\sigma(t_1)\sigma(t_2)H(t-\bar{t})}{K^{1/2}(s,t;t_1,t_2)}\\&+2\mathrm{P}\int_{4m^2}^{\infty}\frac{ds_1}{\pi}\int_{4m^2}^{\infty}\frac{dt_1}{\pi}\int_{4m^2}^{\infty}\frac{dt_2}{\pi}\frac{\rho(s_1,t_1)\sigma(t_2)H(t-\bar{t})}{(s_1-s)K^{1/2}(s,t;t_1,t_2)}\\&+\int_{4m^2}^{\infty}\frac{ds_1}{\pi}\int_{4m^2}^{\infty}\frac{ds_2}{\pi}\int_{4m^2}^{\infty}\frac{dt_1}{\pi}\int_{4m^2}^{\infty}\frac{dt_2}{\pi}\frac{\rho(s_1,t_1)\rho(s_2,t_2)H(t-\bar{t})}{(s_1-s+i\epsilon)(s_2-s-i\epsilon)K^{1/2}(s,t;t_1,t_2)}\Bigg)\ ,\end{aligned} \quad (12)$$

where

$$\bar{t}(s;t_1,t_2)=t_1+t_2+\frac{2t_1t_2}{s-s_0}+2(t_1t_2)^{1/2}\left[1+\frac{t_1+t_2}{s-s_0}+\frac{t_1t_2}{(s-s_0)^2}\right]^{1/2}, \quad (13)$$

$$K(s,t;t_1,t_2)=t_1^{\,2}+t_2^{\,2}+t^2-2(tt_1+tt_2+t_1t_2)-\frac{tt_1t_2}{k^2}\ . \quad (14)$$

Here $\bar{t}$ is that value of t (for fixed s and given t_1 and t_2) such that $K(s,t;t_1,t_2)=0$. It is important to realize that, owing to the step function $H(t-\bar{t})$ in the integrals of Eq. (12), the t_1-t_2 double integrals run only over that region in which $K(s,t;t_1,t_2)>0$. As we shall discuss explicitly below, this means that for any finite values of s and t these double integrals extend over only a *finite* region in the t_1-t_2 plane. It is precisely this property in Eq. (12) which allows us to solve our problem.

More specifically, since $\sigma(t)$ is nonvanishing only for $t\geq 4m^2$, we see from the $H(t-\bar{t})$ in Eq. (12) that all the integrals vanish unless $t>\bar{t}(s;4m^2,4m^2)$; that is, below the boundary curve

$$t=\frac{16m^2s}{s-4m^2}\equiv b_1(s)\ . \quad (15)$$

Therefore, $\rho(s,t)$ for $s>4m^2$ vanishes below this curve and, hence, necessarily for $t<16m^2$, since the asymptote of $b_1(s)$ is $16m^2$ as $s\to\infty$. Only the first integral contributes in Eq. (12) until

$$t > \bar{t}(s; 16m^2, 4m^2) \equiv b_2(s), \qquad (16)$$

which boundary curve we denote by $t = b_2(s)$. The asymptote of this curve is $t = 36m^2$ as $s \to \infty$. For $b_1(s) \leq t < b_2(s)$, only the first integral contributes and gives $\rho(s,t)$ *exactly* in this region of the s-t plane. The third integral contributes when

$$t > \bar{t}(s; 16m^2, 16m^2) \equiv b_3(s). \qquad (17)$$

The asymptote of the curve is $t = 64m^2$ as $s \to \infty$. For $b_2(s) \leq t < b_3(s)$, only the first two integrals contribute. Moreover, the values of $\rho(s_1, t_1)$ required to compute the second integral have already been determined, at least up to $t_1 = 36m^2$.

The general boundary curve for $K(s, t; t_1, t_2) = 0$ is given by

$$(s - 4m^2)[t - t_1 - t_2 - 2(t_1t_2)^{1/2}][t - t_1 - t_2 + 2(t_1t_2)^{1/2}] = 4t_1t_2t. \qquad (18)$$

Similarly, the upper limits on all the t_1-t_2 integrals in Eq. (12) are determined (for given values of s and t) by $K(s_1t; t_1, t_2) = 0$ as

$$t_2 = \left[(t_1 - t)^2 + 4tt_1\left(1 + \frac{(t + t_1)}{(s - s_0)} + \frac{tt_1}{(s - s_0)^2}\right)\right]^{1/2} \pm \left[4tt_1\left(1 + \frac{(t + t_1)}{(s - s_0)} + \frac{tt_1}{(s - s_0)^2}\right)\right]^{1/2} \qquad (19)$$

Both roots are positive, except when $t_1 = t$, in which case $t_2 = 0$.

As long as s and t remain in the region $b_1(s) \leq t \leq b_2(s)$, only the first integral in Eq. (12) contributes and $\sigma(t)$ need only be known in the region $4m^2 \leq t \leq 16m^2$. This integral extends only over a finite region in the t_1-t_2 plane.

Once the second integral begins to contribute, only $\rho(s, t)$ in the region $b_1(s) \leq t \leq b_2(s)$ is required until the t_1-t_2 boundary curve [cf. Eq. (19)] moves out to the point

$$t_2 = 4m^2, \quad t_1 = 36m^2$$

corresponding to the curve [cf. Eq. (18)]

$$(s - 4m^2)(t - 64m^2)(t - 16m^2) = 576m^4t. \qquad (20)$$

This must be compared with the boundary curve for the third integral in Eq. (12), $b_3(s)$ [cf. Eq. (18)]

$$(s - 4m^2)(t - 64m^2) = 1024m^4. \qquad (21)$$

If these curves were to intersect, that would require

$$t - 16m^2 = \tfrac{9}{16}t$$

or

$$t = (\tfrac{16}{7})16m^2 < 4 \times 16m^2,$$

which is impossible since both curves lie *above* $t = 64m^2$. Furthermore, the curve of Eq. (20) lies below the curve of Eq. (21) since the asymptotic values of these as $s \to 4m^2$ are, respectively,

$$t \to \frac{576m^4}{s - 4m^2},$$

$$t \to \frac{1024m^4}{s - 4m^2}.$$

We see that once the second integral in Eq. (12) requires $\rho(s, t)$ in the region $t > b_2(s)$, $\rho(s, t)$ has already been determined everywhere in the region $t \leq 64m^2$. Furthermore, the third integral has not yet begun to contribute.

The general case is easily seen now. To begin consider just the first two terms in Eq. (12).

(i) $n = 2$: $4m^2 \leq t_1 \leq 16m^2 \to \rho$ for $t \leq 36m^2$,

(ii) $n = 3$: $t_1 \leq 36m^2 \to \rho$ for $t \leq 64m^2$,

(iii) $n = 4$: $t_1 \leq 64m^2 \to \rho$ for $t \leq 100m^2$.

The boundary curves of the successive regions here pass through the points [cf. Eq. (19)] $t_2 = 4m^2 \equiv s_0$, $t_1 = n^2s_0$, $n \geq 2$. The corresponding boundary curves in the s-t plane are [cf. Eq. (18)]

$$(s - s_0)[t - (n + 1)^2s_0][t - (n - 1)^2s_0] = 4n^2s_0{}^2t \qquad (22)$$

with asymptotes

$$t \underset{s \to s_0}{\simeq} \frac{4n^2s_0{}^2}{s - s_0}, \quad t \underset{s \to \infty}{\longrightarrow} (n + 1)^2s_0. \qquad (23)$$

The boundary curves do not cross each other [cf. Eq. (13)].

The curves for the third integral in Eq. (12) come into play successively on the curves passing through

$$t_1 = t_2 = N^2s_0, \quad N \geq 2$$

corresponding to

$$(s - s_0)[t - (2N)^2s_0] = 4N^4s_0{}^2 \qquad (24)$$

with asymptotes

$$t \underset{s \to s_0}{\simeq} \frac{4N^4s_0{}^2}{s - s_0}, \quad t \underset{s \to \infty}{\longrightarrow} (2N)^4s_0 \qquad (25)$$

The boundary curves of Eqs. (22) and (23) approach the same asymptotes as $s \to \infty$ when

$$2N = n + 1$$

or when N given by

$$N=\frac{n+1}{2}\,,\quad n=2,3,\ldots$$

is an integer. This occurs for $n=3,5,7,\ldots$. If these curves with common asymptotes were to cross it would require that

$$(s-s_0)[t-(n+1)^2]\,s_0=\frac{4n^2s_0^{\,2}t}{t-(n-1)^2s_0}$$
$$=\frac{(2N)^4}{4}\,s_0^{\,2}$$
$$=\frac{(n+1)^4}{4}\,s_0^{\,2}$$

or that

$$t=\frac{(n+1)^4(n-1)^2s_0}{(n+1)^4-16n^2}$$
$$=(n+1)^2\,\frac{(n^2-1)s_0}{(n+1)^4-16n^2}<(n+1)^2s_0\,.$$

This value of t is not accessible since $t\geq(n+1)^2s_0$. Finally, the asymptote (25) is *above* that of (23). That is, the boundaries for new contributions to the third integral in Eq. (12) always remain above those for the corresponding curves for the second integral.

Hence, the iteration regions never overlap and at each iteration of Eq. (12) only previously determined values of $\rho(s,t)$ are required.

III. CONSTRUCTION PROCEDURE

We are now in a position to solve our inversion problem. We take as given $A_0(s)$, the $l=0$ partial-wave amplitude known for all values of s. That is, we ask whether or not knowledge of $A_0(s)$ will uniquely determine $\sigma(t)$ and $\rho(s,t)$. Since $A_0(s)$ is assumed to be an analytic function of s (with the usual left-hand and right-hand cuts), values of $A_0(s)$ [or of $\delta_0(s)$; cf. Eq. (8)] for any continuous range of values of s in principle determine $A_0(s)$ everywhere in the s plane.

With the aid of[7]

$$Q_l(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_l(y)dy}{x-y}$$

for the Legendre function of the second kind we obtain

$$Q_l(x+i\epsilon)-Q_l(x-i\epsilon)=-i\pi P_l(x)H(1-x)H(1+x)\,.$$

Consequently, from Eq. (9) we have

$$\mathrm{Im}[(s-4m^2)A_l(s)]=\int_{4m^2}^{4m^2-s}dt'\sigma(t')P_l\left(1+\frac{2t'}{s-4m^2}\right)+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{4m^2-s}\frac{dt'\rho(s',t')}{s'-s}\;P_l\left(1+\frac{2t'}{s-4m^2}\right)$$
$$+\frac{2}{\pi}H(s-4m^2)\int_{4m^2}^{\infty}dt'\rho(s,t')Q_l\left(1+\frac{2t'}{s-4m^2}\right)\,. \tag{26}$$

If we write $t=4m^2-s$ and define (for $l=0$)

$$g(t)\equiv\mathrm{Im}[tA_0(4m^2-t)]\,,$$

then we have

$$g(t)=\int_{4m^2}^{t}dt'\sigma(t')+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{t}\frac{dt'\rho(s',t')}{s'-4m^2+t}+\frac{2}{\pi}H(-t)\int_{4m^2}^{\infty}dt'\rho(4m^2-t,t')Q_0\left(1-\frac{2t'}{t}\right).$$

For $t\geq 4m^2$ we can write

$$\frac{dg(t)}{dt}=\sigma(t)+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-4m^2+t}\left(\rho(s',t)-\int_{4m^2}^{t}\frac{dt'\rho(s',t')}{(s'-4m^2+t)^2}\right)\,. \tag{27}$$

Hence, for $4m^2\leq t\leq 16m^2$, the integral in Eq. (27) does not contribute and a knowledge of $dg(t)/dt$ (assumed known) completely determines $\sigma(t)$ there. As discussed previously we can then use Eq. (12) to generate $\rho(s,t)$ everywhere in the region $b_1(s)\leq t\leq b_2(s)$ [that is, $\rho(s,t)$ is certainly known for $t\leq 36m^2$]. Next, this knowledge of $\rho(s,t)$, and that of $dg(t)/dt$ as given, will yield $\sigma(t)$, $t\leq 36m^2$, from Eq. (27). This in turn generates, via Eq. (12), $\rho(s,t)$ in the region $b_2(s)\leq t\leq b_3(s)$ and so on. More specifically, $\sigma(t)$ known for $t\leq n^2s_0$, $n\geq 1$, from Eq. (27) determines $\rho(s,t)$ from Eq. (12) for $t\leq(n+1)^2s_0$ [cf. Eq. (18)]. Hence, an exact knowledge of $A_0(s)$ determines $\sigma(t)$ and

$\rho(s,t)$ everywhere. These in turn allow one to calculate $A(s,t)$ from Eq. (1) and all other partial-wave amplitudes from Eqs. (5) or (9). It is clear from Eq. (26) that the analog of Eq. (27) for $l \neq 0$ will be considerably more complicated since not only $\sigma(t)$, but also its derivative or a weighted integral of $\sigma(t)$ appear. The simple iterative procedure for determining $\rho(s,t)$ does not then go through.

Since our discussion thus far has been based on the Mandelstam representation of Eq. (1), we have been assuming that $\rho(s,t)$ does not require any subtractions in t.[8] We shall treat the case with substractions next. However, there remains a mathematical question concerning the convergence of the iterative scheme defined by Eq. (12) which gives $\rho(s,t)$ in terms of $\sigma(t)$. It is clear from the iterative structure of Eq. (12) that only *one* $\rho(s,t)$ corresponds to a given $\sigma(t)$. Therefore, the uniqueness of the solution to this nonlinear problem presents no difficulty. Nor does the existence of the solution $\rho(s,t)$, except possibly as $t\to\infty$, since for any value of s and all finite values of t only a finite number of integrals over finite t_1-t_2 regions contribute to $\rho(s,t)$. If we are willing to restrict $\sigma(t)$ suitably (say, to the class of Hölder continuous functions), then all of these integrals will themselves be finite. The outstanding question, then, is whether or not $\rho(s,t)$ is polynomial bounded as $t\to\infty$. A straightforward, but tedious, approach to this problem would be via fixed-point theorems applied to Eq. (12).[9] However, since we are assuming the existence of a Mandelstam representation for $A(s,t)$ and since this is sensible only if $A(s,t)$ is polynomial bounded, we shall simply assume that $\rho(s,t)$ is itself polynomial bounded. Nevertheless, it would certainly be more satisfying to be able to prove directly that, once $A_0(s)$ is suitably restricted (say, Hölder continuous), then Eqs. (12) and (27) necessarily yield a $\rho(s,t)$ which is polynomial bounded.

Let us consider now the nature of the inversion problem when bound-state poles are present and when subtractions in t are required. The representation for $A(s,t)$ becomes [cf. Eq. (1)]

$$A(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt'\sigma(t')}{t'-t}+\sum_{j=1}^{m}\frac{\Gamma_j(t)}{s-\bar{s}_j}+\frac{t^{n+1}}{\pi^2}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\,\frac{\rho(s',t')}{t'^{n+1}(t'-t)(s'-s)}+\sum_{j=0}^{n}\frac{t^j}{\pi}\int_{4m^2}^{\infty}\frac{ds'h_j(s')}{s'-s}\,, \tag{28}$$

where

$$h_j(s)=\frac{d^j}{dt^j}\,[\operatorname{Im}A(s,t)]\bigg|_{t=0},$$

and where the $\Gamma_j(t)$ are polynomials in t corresponding to those partial waves in which the bound-state poles appear at energies $\{\bar{s}_j\}$. A straightforward calculation shows that the iterative equation for $\rho(s,t)$, namely Eq. (12), still obtains no matter what the integer value of $n(\geq 0)$.

To be specific and for simplicity, let us examine the case in which there is just one bound state in the s wave and in which only one subtraction in t is required.

$$A(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt'\sigma(t')}{t'-t}+\frac{t}{\pi^2}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\,\frac{\rho(s',t')}{t'(t'-t)(s'-s)}+\frac{\Gamma}{s-\bar{s}_1}+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{h_1(s')}{s'-s}\,. \tag{29}$$

The partial-wave projections for $l\geq 1$ are still given by Eq. (9), while the $l=0$ case becomes

$$A_0(s)=\frac{2}{\pi(s-4m^2)}\int_{4m^2}^{\infty}dt'\sigma(t')Q_0\left(1+\frac{t'}{2k^2}\right)+\frac{\Gamma}{s-\bar{s}_1}+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{h_1(s')}{s'-s}+\frac{2}{\pi^2(s-4m^2)}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\,\frac{\rho(s',t')}{s'-s}\left[Q_0\left(1+\frac{t'}{2k^2}\right)-\frac{2k^2}{t'}\right]. \tag{30}$$

The expression for $dg(t)/dt$, again for $t\geq 4m^2$, remains that given by Eq. (27).

Therefore, given $A_0(s)$ for all s, we first use Eqs. (27) and (12) to determine $\sigma(t)$ and $\rho(s,t)$. The residue of $A_0(s)$ at $s=\bar{s}_1$ yields Γ. Finally, if we use Eq. (30) to compute $\operatorname{Im}A_0(s)$ for $s>4m^2$, we obtain $h_1(s)$ directly and hence the full scattering amplitude via Eq. (29).

IV. DISCUSSION

We have shown that if $A(s,t)$ requires no more than one subtraction in t and has bound states only in the s wave (or, of course, none at all), then a knowledge of $A_0(s)$ uniquely determines the full scattering amplitude, $A(s,t)$. At first sight this may appear to be too strong a result since the

Gel'fand-Levitan theorem would lead us to expect a one-parameter family of potentials and, hence, of scattering amplitudes. However, when one requires exponential damping of the potential, then one can obtain a unique result from the Gel'fand-Levitan scheme.[10]

When there are more subtractions in t, Eqs. (12) and (27) still obtain so that $\sigma(t)$ and $\rho(s,t)$ are uniquely determined (aside from questions of convergence as $t\to\infty$). However, when more than one $h_j(s)$ term projects into $l=0$, then only some linear combination of these $\{h_j(s)\}$ is determined by the generalization of Eq. (30). If, however, we were also given those other $\{A_l(s)\}$ into which the subtraction terms $\{h_j(s)\}$ are projected, then we could obtain a set of linear algebraic equations to determine uniquely the $\{h_j(s)\}$ and hence the full scattering amplitude, $A(s,t)$.

Finally, the relativistic case has several complications, the most serious of which is that inelastic processes are required for any nontrival amplitude.[11] As a result, the elastic unitarity condition, Eq. (4), no longer holds everywhere nor will Eq. (12).

One method of coping with this difficulty may be the following. In the present nonrelativistic case Eq. (12) can be expressed as an operator equation in the form of a mapping

$$\rho = O_1(\sigma,\rho)\,, \tag{31}$$

where $O_1(\sigma,\rho)$ is defined by the right-hand side of Eq. (12) itself. The question of the uniqueness of the solution ρ to Eq. (31), given σ, depends on the existence of nontrivial solutions to the linear Fréchet derivative equation of the form

$$\delta\rho = M_1(\sigma,\rho;\delta\sigma,\delta\rho)\,. \tag{32}$$

Similarly, Eq. (26) can be written as

$$O_2(\sigma,\rho)=0 \tag{33}$$

with the corresponding Fréchet form [for a *given* $A_0(s)$ which is not varied]

$$\delta\sigma = M_2(\delta\rho)\,. \tag{34}$$

Equations (32) and (34) together can be used to pose the uniqueness in the linear form

$$\delta\rho = M_3(\sigma,\rho;\delta\rho)\,. \tag{35}$$

The construction procedure of Sec. III implies that the solution for ρ is unique [i.e., $\delta\rho=0$ is the only solution to Eq. (35)].

Now for the relativistic, crossing-symmetric case the relation among $A_0(s)$, $\sigma(s)$, and $\rho(s,t)$ is still linear [and very similar to Eq. (9)] so that one will again obtain Eq. (34). One might then examine Atkinson's fixed-point proofs[9] subject to the constraint of Eq. (34) as a means of deciding the uniqueness question, assuming that solutions do exist since $A_0(s)$ is given.

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[1]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Sec. 20.2 and references given therein.

[2]M. Coz and C. Coudray, J. Math. Phys. 17, 888 (1976).

[3]J. T. Cushing, Phys. Rev. D 4, 1177 (1971).

[4]R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N.Y.) 10, 62 (1960).

[5]J. S. Frederiksen, P. W. Johnson, and R. L. Warnock, J. Math. Phys. 16, 1886 (1975).

[6]J. T. Cushing, Nuovo Cimento 28, 818 (1963); 31, 688 (1964).

[7]J. T. Cushing, *Applied Analytical Mathematics for Physical Scientists* (Wiley, New York, 1975), p. 377.

[8]We assume here, just as in the nonrelativistic Schrödinger case, that subtractions in s are not required (cf. Ref. 4, Sec. 4).

[9]D. Atkinson, Nucl. Phys. B7, 375 (1968).

[10]Ref. 4, p. 619.

[11]S. Ø. Aks, J. Math. Phys. 6, 516 (1965).