Quantum spin model for Reggeon field theory*

Richard C. Brower,† Miguel A. Furman,‡ and Krishnappa Subbarao§
Division of Natural Sciences, Physics, University of California, Santa Cruz, California 95064
(Received 9 August 1976)

We study the Hamiltonian form of Reggeon field theory on a lattice in the two-dimensional ($D = 2$) transverse or impact-parameter space. The Hamiltonian formalism allows naturally for the privileged character of the longitudinal variable or rapidity, which is kept continuous. Based on recent results for the one-dimensional theory, we argue that we may truncate the single-site basis of the Hamiltonian by retaining the lowest two states only, and we arrive at a lattice spin model. In terms of Pauli spin matrices $\sigma_k^{\vec{n}}$ at each site $\vec{n} = (n_1, n_2)$, our Reggeon quantum spin model has the Hamiltonian $H = \Sigma_{\vec{n}}(\delta/2)(1 - \sigma_x^{\vec{n}})$ $+ \Sigma_{\vec{n},\hat{i}} A[(1 - 2\sigma_+^{\vec{n}+\hat{i}})(1 - 2\sigma_+^{\vec{n}}) - \sigma_z^{\vec{n}+\hat{i}}\sigma_z^{\vec{n}}]$, where $\hat{i}$ represents the lattice unit vectors (1,0) and (0,1). The parameters δ and A are related to those of the original field theory. All the approximations are valid for small Regge slope α_0' in the region of the phase transition at output Regge intercept $\alpha(0) = 1$. The critical exponents of the original system can be determined by the properties of the low-energy states of H in strict analogy with the established relation between the ϕ^4 theory in $d = 2$ and the ground state of the quantum Ising model with a transverse magnetic field in $D = d - 1 = 1$ dimension. The general properties of the Reggeon quantum spin model are exhibited, and several new approximation schemes for Reggeon field-theoretic calculations are suggested. While the above Hamiltonian is adequate, by universality, to describe the critical Pomeron, systematic improvements can be made to study the theory away from criticality by retaining more states in the single-site basis of the Hamiltonian.

I. INTRODUCTION

To account for rising or even constant cross sections a Regge theory must include Regge cuts due to multi-Regge exchanges.[1] Gribov's Reggeon field theory provides a systematic way to include cuts with the correct t-channel discontinuities.[2] In the Reggeon field theory, the leading Regge pole, that is, the Pomeron, is treated as a quasiparticle with a field $\psi(y, \vec{x})$ in one "time" variable[3] y, which is the rapidity, and two transverse space coordinates $\vec{x}$, which are the impact parameters.

It has been shown, by means of an expansion in ϵ, in a transverse space of dimensionality $D = 4 - \epsilon$, that the Reggeon field theory undergoes a second-order phase transition as the bare "mass gap" Δ_0 becomes large and negative.[4] ($\Delta_0 \equiv 1 - \alpha_0$, where α_0 is the bare Regge intercept.) At the critical value Δ_0^* of Δ_0 the renormalized Regge intercept is unity and, owing to the many coincident Regge cuts, cross sections increase logarithmically: $\sigma_{\text{tot}} \sim (\ln s)^{\eta}$. Although considerable understanding of the phase transition has been achieved by the ϵ expansion, many questions remain.[5] At the physical dimension where $\epsilon = 4 - D = 2$, calculations for η to second order in ϵ do not indicate convergence. Indeed, for $D = 2$ some doubt as to the existence of a phase transition remains.[6] Also we feel that the nature of a physically meaningful theory[7,8] in the supercritical[9] region $\Delta_0 < \Delta_0^*$ is an open question in spite of the very interesting recent work by Amati *et al.*[10] Finally, there are no accurate calculations of the critical exponents, or Green's functions for large rapidity, let alone rapidities below the scaling region.

Several attempts to answer these questions have been made by replacing the Reggeon field theory by a lattice spin model.[11-13] Just as the Ising model is in the same universality class[14] as the Ginzburg-Landau theory, that is, a Euclidean ϕ^4 theory, we might find a lattice model in the universality class of Reggeon field theory. Then the critical exponents and scaling functions at $D = 2$ could be calculated by statistical-mechanical methods.[15,16]

However, two fundamental dissimilarities between Reggeon field theory and the ϕ^4 theory have impeded progress in this direction: (1) After Wick rotation, the Reggeon field theory, unlike ϕ^4 theory, is not symmetric in the "time" y and the space coordinates $\vec{x}$. (2) The interaction Hamiltonian is not Hermitian. In addition, discretization of the rapidity axis induces new unphysical interaction terms of the form $(\psi^n + \psi^{\dagger n})$, whose effects are difficult to isolate in the critical region.[13]

We present in this paper a lattice spin model which naturally allows for the privileged character of rapidity and avoids the problems arising from its discretization. We discretize only the $\vec{x}$ space by introducing a lattice. This implements a transverse-momentum cutoff. Since it is generally believed that large transverse momentum will be suppressed in any dynamical theory underlying the Reggeon field theory, the lattice may in fact be an improvement. We keep y as a conti-

nuous variable, so that we can imagine a continuous chain at each lattice site. This is achieved by working in the Hamiltonian formalism rather than the path-integral formalism. Whereas the action is a sum over $\vec{x}$ space *and* rapidity y, the Hamiltonian is a sum over $\vec{x}$ space only. Also, we shall invoke Gribov's "heavy Pomeron" approximation,[17] namely, that the bare Regge slope α_0' is small. As emphasized by Gribov and the CERN collaborators,[8,10] an expansion in α_0' is a useful one allowing for the treatment of the coupling to the external particles nonperturbatively. This approximation of small α_0' means that the chains are weakly coupled.

In the extreme limit $\alpha_0'=0$, the chains are decoupled. The spectrum of the Hamiltonian of a single chain has been recently studied by Bronzan, Shapiro, and Sugar.[18] The direct product of the eigenstates at every site provides a convenient basis to represent the fully coupled Hamiltonian. Then, following the same truncation procedure used earlier for ϕ^4 theory,[19] we arrive at the following effective Hamiltonian for the case of weakly coupled chains:

$$H=\sum_{\vec{n}}\frac{\delta}{2}(1-\sigma_x^{\vec{n}}) + \sum_{\vec{n},\hat{i}} A[(1-2\sigma_+^{\vec{n}+\hat{i}})(1-2\sigma_+^{\vec{n}})-\sigma_z^{\vec{n}+\hat{i}}\sigma_z^{\vec{n}}]. \quad (1.1)$$

This Hamiltonian defines our Reggeon quantum spin (RQS) model in terms of the Pauli matrices $\sigma_x^{\vec{n}}$, $\sigma_z^{\vec{n}}$, and $\sigma_+^{\vec{n}}\equiv(\sigma_x^{\vec{n}}+i\sigma_y^{\vec{n}})/2$ at the lattice sites $\vec{n}=(n_1,n_2)$, while δ and A are related to the parameters of the Reggeon field-theory Lagrangian, and $\hat{i}$ represents the lattice unit vectors (1, 0) and (0, 1). In the ϕ^4 case this procedure gives, in two dimensions (and, as generally believed, in higher dimensions too), critical exponents identical to those of the Ising model which is in the same universality class as ϕ^4 theory.[14] We can similarly expect that our effective Hamiltonian gives the same critical behavior as the original Reggeon field theory, independent of the fact that we have used a weak-coupling approximation.

In Sec. II we set up the quantum Hamiltonian formalism and summarize the general features of the problem, and introduce the lattice in impact-parameter space. We do a crude mean-field (or Hartree-Fock) approximation to exhibit a phase transition, and calculate the critical value of Δ_0 for small α_0'.

In Sec. III we first summarize the spin-model construction[19] for the ϕ^4 theory and review the relevant features of the $D=0$ Reggeon field theory.[18] Then we truncate the basis of the one-site Hamiltonian by keeping only the two lowest states, and obtain our effective Hamiltonian (1.1). We also discuss in this section the symmetries of the model and sketch, as an example, a calculation of the total cross section in the eikonal approximation.

In Sec. IV we discuss the possible connection between our D-dimensional RQS model and the transfer-matrix formulation of the $(D+1)$-dimensional lattice model of Cardy and Sugar.[11] Although the models are closely related, they are not identical. They share the same symmetry properties, but only the Reggeon quantum spin model strictly preserves the normal-ordering property of the original Hamiltonian that accounts for the persistence of the perturbative ground state $|0\rangle$ ($H|0\rangle=0$) for all values of the parameters. We believe this special property to be crucial for the understanding of the phase transition of Reggeon field theory.

Section V concludes with a discussion and suggests further calculations and two appendixes contain some calculational details referred to in the text.

II. HAMILTONIAN FORMULATION AND THE TRANSVERSE LATTICE

A. Hamiltonian and path-integral formulations

We begin by considering the standard Reggeon field theory defined by the Lagrangian

$$\mathcal{L}=-\bar{\psi}\frac{\partial\psi}{\partial y}-\alpha_0'\vec{\nabla}\bar{\psi}\cdot\vec{\nabla}\psi-\Delta_0\bar{\psi}\psi-\frac{ir_0}{2}\bar{\psi}(\psi+\bar{\psi})\psi, \quad (2.1)$$

in which $\psi=\psi(y,\vec{x})$ is the Reggeon field (the relation of $\bar{\psi}$ to ψ is discussed later), y the rapidity, and $\vec{x}=(x_1,x_2,\ldots,x_D)$ the D-dimensional impact-parameter space ($D=2$ is the physical dimension). The parameters α_0', $\alpha_0\equiv1-\Delta_0$, and r_0 are, respectively, the bare slope of the trajectory, its intercept, and the triple-Reggeon coupling.

We are interested in applying the theory to the Pomeron, for which we assume that the renormalized intercept is exactly unity so that the renormalized Δ_0 is zero. We see from (2.1) that Δ_0 plays a role similar to a bare "mass gap" of a nonrelativistic theory, and the renormalized propagator is expected to exhibit a power-law behavior at large y and $|\vec{x}|$ characteristic of a second-order phase transition.

Quantization in Heisenberg's picture is achieved in the standard way from the Lagrangian (2.1). The quantum field ψ and its canonical conjugate $(i\bar{\psi})$ have the equal-rapidity commutation relation

$$[\psi(y,\vec{x}),\bar{\psi}(y,\vec{x}')]=\delta^{(D)}(\vec{x}-\vec{x}'), \quad (2.2)$$

while $\psi(y,\vec{x})$ commutes with $\psi(y,\vec{x}')$ and similarly

for $\bar{\psi}$. The (normal-ordered) Hamiltonian is

$$H=\int d^D\vec{x}\left[\Delta_0\bar{\psi}\psi+\frac{ir_0}{2}\bar{\psi}(\psi+\bar{\psi})\psi+\alpha_0'\vec{\nabla}\bar{\psi}\cdot\vec{\nabla}\psi\right]. \tag{2.3}$$

The formal solution to the Heisenberg equations of motion[20] are

$$\begin{aligned}\psi(y,\vec{x})&=e^{yH}\psi(0,\vec{x})e^{-yH},\\ \bar{\psi}(y,\vec{x})&=e^{yH}\bar{\psi}(0,\vec{x})e^{-yH}.\end{aligned} \tag{2.4}$$

We note that in general, even for $r_0=0$, the operator $\bar{\psi}(y,\vec{x})$ is not the Hermitian conjugate of $\psi(y,\vec{x})$. However, we can choose one particular y, which we take to be $y=0$, at which $\bar{\psi}(0,\vec{x})=\psi^\dagger(0,\vec{x})$. The Hermiticity properties of the fields for $y\neq 0$ are then completely determined by Eqs. (2.4).

The remainder of this subsection as well as Sec. II C are not necessary for the derivation of our quantum spin model, and the interested reader may proceed directly to Sec. II B and then Sec. III for the derivation. However, we wish to discuss first the connection between the Hamiltonian and path-integral formulations of the theory and review briefly a general description of a second-order phase transition in the Hamiltonian formalism.

We are interested in the calculation of the Green's functions of the theory, which in the path-integral formalism may be obtained from the generating functional

$$Z[S,\bar{S};Y]=\int\delta\psi\delta\bar{\psi}\exp\left[\int_0^Y dy\int d^D\vec{x}(\mathcal{L}+\bar{\psi}S+\bar{S}\psi)\right] \tag{2.5}$$

in which $S=S(y,\vec{x})$ and $\bar{S}=\bar{S}(y,\vec{x})$ are independent complex c-number sources. The fields ψ and $\bar{\psi}$ are here independent c-number functions defined at the point $(y,\vec{x})$, and the meaning of the functional integral is given in Appendix A. We have kept a fictitious rapidity "volume" Y as a variable in order to define the "thermodynamic" limit $Y\to\infty$ to obtain the expected phase transition. Also, as explained in Appendix A, we take periodic boundary conditions at $y=0$, Y, so that $\psi(0,\vec{x})=\psi(Y,\vec{x})$ and $\bar{\psi}(0,\vec{x})=\bar{\psi}(Y,\vec{x})$.

For example, the two-point function is given by

$$\begin{aligned}G^{(1,1)}(y,\vec{x};Y)&\equiv\langle\psi(y,\vec{x})\bar{\psi}(0,\vec{0})\rangle\\ &\equiv\frac{1}{Z[0,0;Y]}\frac{\delta}{\delta\bar{S}(y,\vec{x})\delta S(0,\vec{0})}\\ &\quad\times Z[S,\bar{S};Y]\Big|_{S=\bar{S}=0}\\ &=\frac{\int\delta\psi\delta\bar{\psi}\psi(y,\vec{x})\bar{\psi}(0,\vec{0})\exp[\int_0^Y dy\int d^D\vec{x}\,\mathcal{L}]}{\int\delta\psi\delta\bar{\psi}\exp[\int_0^Y dy\int d^D\vec{x}\,\mathcal{L}]}\end{aligned} \tag{2.6}$$

and its contribution to the total cross section is proportional to its Fourier transform at zero momentum transfer,

$$\sigma_{\rm tot}(s)\propto\int d^D\vec{x}\,G^{(1,1)}(y,\vec{x};Y), \tag{2.7}$$

in the limit $Y\to\infty$ for fixed $y=\ln s$.

In the quantum-operator formalism the two-point Green's function is given by[20]

$$G^{(1,1)}(y,\vec{x};Y)=\frac{\mathrm{tr}[e^{-YH}T\psi(y,\vec{x})\bar{\psi}(0,\vec{0})]}{\mathrm{tr}e^{-YH}}, \tag{2.8}$$

in which the symbol T implies a rapidity-ordered product of the fields as appropriate for bosons.

In the limit $Y\to\infty$ only the ground state $|0\rangle$ of the Hamiltonian survives in Eq. (2.8) and we obtain, for the region above the critical surface, $\Delta_0>\Delta_0^*$ (where Δ_0^* is the critical value of Δ_0),

$$\begin{aligned}\lim_{Y\to\infty}G^{(1,1)}(y,\vec{x};Y)&\equiv G^{(1,1)}(y,\vec{x})\\ &=\langle 0|T\psi(y,\vec{x})\bar{\psi}(0,\vec{0})|0\rangle\\ &=\theta(y)\langle 0|\psi(0,\vec{x})e^{-yH}\bar{\psi}(0,\vec{0})|0\rangle,\end{aligned} \tag{2.9}$$

where we have used $\psi(0,\vec{x})|0\rangle=0$ and $H|0\rangle=0$ due to normal-ordering.

The asymptotic behavior for large y is obtained from Eq. (2.9),

$$G^{(1,1)}(y,\vec{x})\sim\langle 0|\psi(0,\vec{x})|1\rangle\langle 1|\bar{\psi}(0,\vec{0})|0\rangle e^{-yE_1} \tag{2.10}$$

and is controlled by the first excited state, $H|1\rangle=E_1|1\rangle$. The phase transition occurs as the "correlation length" in rapidity, E_1^{-1}, becomes infinite as Δ_0 approaches Δ_0^* from above and the lowest level of H becomes degenerate. When $\Delta_0=\Delta_0^*$ a scaling analysis[4,5] says that the behavior of $G^{(1,1)}(y,\vec{x})$ is

$$G^{(1,1)}(y,\vec{x})\sim y^{\eta-\nu D/2}F\left(\frac{\vec{x}^2}{\alpha' y^\nu}\right) \tag{2.11}$$

for large y and $|\vec{x}|$ with $\vec{x}^2/y^\nu$ fixed. Here η and ν are critical exponents which we would eventually like to compute and F is a scaling function. It is easily seen that η and ν control the behavior of the total cross section and the shrinkage of the elastic forward peak, respectively,

$$\sigma_{\rm tot}\sim(\ln s)^\eta,\quad\frac{d\sigma_{\rm el}}{dt}\sim(\ln s)^{2\eta}|\tilde{F}(\alpha' t(\ln s)^\nu)|^2, \tag{2.12}$$

where $\tilde{F}$ is the Fourier transform of F and t is the square of the four-momentum transfer.

For $\Delta_0<\Delta_0^*$ there is expected to be a new ground state which is no longer annihilated by $\psi(0,\vec{x})$ and

the fields ψ and $\bar{\psi}$ develop nonzero ground-state expectation values. It may then be more convenient to work in terms of a "shifted" or connected Green's function defined by

$$G_c^{(1,1)}(y,\vec{x};Y)=\frac{\delta}{\delta\bar{S}(y,\vec{x})}\frac{\delta}{\delta S(0,\vec{0})}\ln Z[S,\bar{S};Y]\Big|_{S=\bar{S}=0}$$

even if the physical particles still couple to the original fields. However, as we said in the Introduction, we feel that the nature of a physical theory for $\Delta_0<\Delta_0^*$ is an open question.[7-10]

Of particular interest is the partition function of the theory because much information about the phase transition can be learned from it and it is generally simpler to compute than Green's functions. It is defined by

$$Z\equiv Z[0,0;Y]=\int\delta\psi\delta\bar{\psi}\exp\left(\int_0^Y dy\int d^D\vec{x}\,\mathcal{L}\right)$$
$$=\mathrm{tr}e^{-YH}\,. \tag{2.13}$$

We see that the total rapidity Y plays the role of inverse temperature in statistical mechanics.[20] In the limit $Y\to\infty$ only the ground state of H survives in Eq. (2.13), and our problem is then that of the statistical mechanics of a D-dimensional quantum system at zero temperature.

In Appendix A we prove the equivalence of the operator and path-integral forms of the theory, and thereby give a precise meaning to the functional integrals.

B. Transverse lattice

A square lattice in $\vec{x}$ space is defined by the points

$$x_i=ln_i,\quad n_i=0,\pm1,\pm2,\ldots,\quad i=1,2,\ldots,D\,, \tag{2.14}$$

where l is the lattice unit spacing, and the field at the lattice point $\vec{n}$ is

$$\psi_{\vec{n}}(y)=l^{D/2}\psi(y,\vec{x}=l\vec{n}) \tag{2.15}$$

and similarly for $\bar{\psi}$. We also make the replacements

$$\int d^D\vec{x}\to l^D\sum_{\vec{n}}\,,$$
$$\vec{\nabla}\bar{\psi}\cdot\vec{\nabla}\psi\to l^{-2-D}\sum_{\hat{i}}(\bar{\psi}_{\vec{n}+\hat{i}}-\bar{\psi}_{\vec{n}})(\psi_{\vec{n}+\hat{i}}-\psi_{\vec{n}})\,, \tag{2.16}$$

where the $\hat{i}$ are the D independent unit vectors, so that the gradient term is replaced by the sum of nearest-neighbor couplings.[21]

The commutation relation, Eq. (2.2), becomes

$$[\psi_{\vec{n}}(y),\bar{\psi}_{\vec{m}}(y)]=\delta_{\vec{n},\vec{m}}\,, \tag{2.17}$$

where the $\delta_{\vec{n},\vec{m}}$ is the ordinary Kronecker δ. Since $\bar{\psi}_{\vec{n}}(y=0)=\psi^\dagger_{\vec{n}}(0)$, we identify $\bar{\psi}_{\vec{n}}(0)$ and $\psi_{\vec{n}}(0)$ with the ordinary creation and annihilation operators $a^\dagger_{\vec{n}}$ and $a_{\vec{n}}$, respectively. Then, since H is independent of y, we obtain, by setting $y=0$ in Eq. (2.3),

$$H=\sum_{\vec{n}}\Big[\Delta_0 a^\dagger_{\vec{n}}a_{\vec{n}}+\frac{ig_0}{2}a^\dagger_{\vec{n}}(a_{\vec{n}}+a^\dagger_{\vec{n}})a_{\vec{n}}$$
$$+\frac{\alpha_0'}{l^2}\sum_{\hat{i}}(a^\dagger_{\vec{n}+\hat{i}}-a^\dagger_{\vec{n}})(a_{\vec{n}+\hat{i}}-a_{\vec{n}})\Big]\,, \tag{2.18}$$

where $g_0=l^{-D/2}\gamma_0$.

C. Mean-field approximation for $\alpha_0'\to0$

We define the two-point function in $(E,\vec{k})$ space by

$$G^{(1,1)}(E,\vec{k})=\int d^D\vec{x}\int_0^\infty dy\,e^{Ey-i\vec{k}\cdot\vec{x}}G^{(1,1)}(y,\vec{x}) \tag{2.19}$$

and a second-order phase transition occurs when $G^{(1,1)}(E=0,\vec{k}=\vec{0})$ becomes infinite.

The generating functional, Eq. (2.5), expressed in $(E,\vec{k})$ space for $Y\to\infty$ is

$$Z[\bar{S},S]=\int\delta\bar{\psi}(E,\vec{k})\delta\psi(E,\vec{k})$$
$$\times\exp\left(A_0+\int\frac{dE\,d^D\vec{k}}{(2\pi)^{D+1}}\{[\bar{S}(E,\vec{k})+J(\vec{k})\bar{\psi}(E,\vec{k})]\psi(E,\vec{k})+\bar{\psi}(E,\vec{k})[S(E,\vec{k})+J(\vec{k})\psi(E,\vec{k})]\}\right)\,, \tag{2.20}$$

where the integrals over k_i are cutoff at $\Lambda=\pi/l$ and where A_0 is the action neglecting all intersite couplings, which are all contained in the $J\bar{\psi}\psi$ terms. It follows from (2.16) that

$$J(\vec{k})=\frac{\alpha_0'}{l^2}\prod_{i=1}^D\cos(lk_i)$$
$$\simeq\frac{\alpha_0'}{l^2}\left(1-\frac{l^2}{2}\vec{k}^2\right) \tag{2.21}$$

for small $|\vec{k}|$. The connected two-point Green's function is given by

$$G_c^{(1,1)}(E,\vec{k})=\frac{\delta}{\delta\bar{S}(E,\vec{k})\delta S(0,\vec{0})}\ln Z[\bar{S},S]\Big|_{S=\bar{S}=0}\,. \tag{2.22}$$

We can determine the critical value Δ_0^* for small α_0' by a mean-field approximation,[22] defined by the replacement

$$\sum_{\hat{i}} \bar{\psi}_{\vec{n}+\hat{i}} \psi_{\vec{n}} \to \frac{D}{2} \left(\langle\bar{\psi}\rangle \psi_{\vec{n}} + \bar{\psi}_{\vec{n}} \langle\psi\rangle\right) \tag{2.23}$$

in $Z[\bar{S}, S]$. Then we get

$$Z[\bar{S}, S] = \text{const} \times [\, Z_1(\bar{S} + DJ\langle\bar{\psi}\rangle, S + DJ\langle\psi\rangle)]^N , \tag{2.24}$$

where Z_1 is the single-site ($D=0$) generating functional and N is the total number of lattice sites. The expectation values $\langle\psi\rangle, \langle\bar{\psi}\rangle$ are determined by the self-consistency conditions

$$\langle\psi\rangle = \frac{\delta}{\delta \bar{S}(E, \vec{k})} \ln Z \Big|_{S=\bar{S}=0} , \qquad \langle\bar{\psi}\rangle = \frac{\delta}{\delta S(E, \vec{k})} \ln Z \Big|_{S=\bar{S}=0} . \tag{2.25}$$

It is then straightforward to find the expression for the Green's function for small α_0',

$$G^{(1,1)}(E, \vec{k}) = \frac{G_1(E)}{1 - 2DJ(\vec{k}) G_1(E)} , \tag{2.26}$$

in terms of the single-site Green's function $G_1(E)$. The critical surface is therefore given in this approximation by

$$G_1(0) = \frac{1}{2J(\vec{0})D} = \frac{l^2}{2D\alpha_0'} . \tag{2.27}$$

The single-site Green's function is given by

$$G_1(E) = \int_0^\infty dy \langle 0 | a e^{y(E-H_1)} a^\dagger | 0 \rangle , \tag{2.28}$$

where H_1 is the single-site Hamiltonian

$$H_1 = \Delta_0 a^\dagger a + \frac{i g_0}{2} a^\dagger (a + a^\dagger) a . \tag{2.29}$$

In Ref. 18 (see also Appendix B) it is shown that, for $\Delta_0 \to -\infty$,

$$G_1(0) = \int_0^\infty dy \exp\left(-y\Delta_0 - y^2 \frac{g_0^2}{8}\right) \to \frac{2(2\pi)^{1/2}}{|g_0|} \exp\left(\frac{2\Delta_0^2}{g_0^2}\right) , \tag{2.30}$$

and therefore Eq. (2.25) tells us that

$$\Delta_0^* = -\frac{|g_0|}{\sqrt{2}} \left(\ln \frac{|g_0| l^2}{4D(2\pi)^{1/2} \alpha_0'}\right)^{1/2} \tag{2.31}$$

in the mean-field approximation for α_0' small.

The point of this exercise is to show that the phase transition *can* occur for weakly coupled chains and to suggest that it may be studied by approximating the spectrum of the single-site Hamiltonian for $\Delta_0 \to -\infty$. Of course, the critical exponents obtained in mean field theory are the trivial ones.

III. THE REGGEON QUANTUM SPIN MODEL

A. Summary of the construction for the Ginzburg-Landau theory

A knowledge of the spectrum and eigenstates of the Hamiltonian (2.18) enables one to compute all the properties of the cutoff Reggeon field theory; for example, the partition function is determined by the ground-state energy. The two-point function is given, according to (2.9), by

$$G^{(1,1)}(y, \vec{n}) = \theta(y) \sum_m \langle 0 | \psi_{\vec{n}}(0) | m \rangle \times e^{-yE_m} \langle m | \bar{\psi}_{\vec{0}}(0) | 0 \rangle , \tag{3.1}$$

where the $|m\rangle$ are the eigenstates of the Hamiltonian corresponding to eigenvalues E_m.

It is very difficult to compute exactly the spectrum and eigenstates of H, and therefore one has to pursue approximation schemes. We shall follow a simple approximation analogous to that developed[19] for the Ginzburg-Landau theory, which we now summarize.

The Hamiltonian for this theory on a transverse D-dimensional lattice $\vec{x} = \vec{n}l$ is

$$H = \sum_{\vec{n}} : \left[\tfrac{1}{2}\pi_{\vec{n}}^2 + V(\phi_{\vec{n}}) + c^2 \sum_{\hat{i}} (\phi_{\vec{n}+\hat{i}} - \phi_{\vec{n}})^2\right] : , \tag{3.2}$$

with

$$\pi_{\vec{n}}^2 = \left(\frac{d\phi_{\vec{n}}}{dy}\right)^2 \to -\frac{\partial^2}{\partial \phi_{\vec{n}}^2} , \tag{3.3}$$

where $V(\phi_{\vec{n}})$ is the Ginzburg-Landau potential

$$V(\phi_{\vec{n}}) = \lambda_0 \left(\phi_{\vec{n}}^2 + \frac{m_0^2}{4\lambda_0}\right)^2 = \tfrac{1}{2} m_0^2 \phi_{\vec{n}}^2 + \lambda_0 \phi_{\vec{n}}^4 + \text{const} , \tag{3.4}$$

with $m_0^2 < 0$ and $\lambda_0 > 0$ (see Fig. 1). We are particularly interested in solving the Hamiltonian (3.2) near the phase transtion which occurs when its ground level becomes degenerate.

Let us first consider the problem for a single site, which amounts to solving the theory for no intersite coupling, $c^2 = 0$. This corresponds to the familiar Schrödinger problem for the one-dimensional anharmonic-oscillator potential (3.4). This has an infinite, discrete, positive-definite spectrum whose levels alternate in parity. However, when m_0^2 is large and negative, it has been shown[23] that the splitting owing to tunnelling of the two lowest levels vanishes exponentially, so the ground level becomes degenerate in the limit $m_0^2 \to -\infty$. At the same time, all other excited levels go to

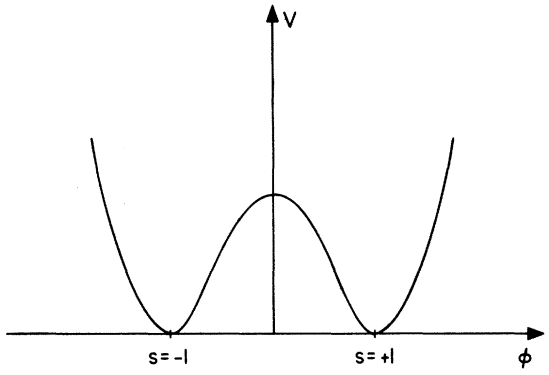

FIG. 1. The Ginzburg-Landau potential $V=\lambda_0(\phi^2+m_0^2/4\lambda_0)^2$ for $m_0^2<0$ and $\lambda_0>0$. The stable points are at $\phi=\pm|m_0|/2\sqrt{\lambda}_0$.

infinity.[23] The two lowest states $|\epsilon_0\rangle$ and $|\epsilon_1\rangle$ are essentially the even- and odd-parity combinations of states whose wave functions are localized in the right or left well of the potential, and as $m_0^2\to-\infty$, the height of the barrier increases, the tunnelling is reduced, and the ground level becomes degenerate.

If we now reintroduce a *small* intersite coupling, $c^2\gtrsim 0$, we expect that the ground level of the *full* Hamiltonian (3.2) will become degenerate when $-m_0^2$ is large but finite. Thus the two lowest states at each site play the key role in the phase transition of the full Hamiltonian. This suggests *truncating* the basis of single-site states by retaining only the two lowest ones, so that H is replaced by an effective Hamiltonian which is expressed in terms of 2×2 Pauli matrices, and which is simpler to solve. Critical phenomena being rather insensitive to the details of the Hamiltonian (but only depending on the dimensionality and symmetries of the system) can be expected to be described well by this kind of approximation.

The sum of single-site piece of H, Eq. (3.2), is replaced by

$$\sum_{\vec{n}} :[\tfrac{1}{2}\pi_{\vec{n}}^2+V(\phi_{\vec{n}})]:\to \sum_{\vec{n}}\left[\frac{\epsilon}{2}(1+\sigma_z^{\vec{n}})+\epsilon_0\right], \quad (3.5)$$

where $\epsilon\equiv\epsilon_1-\epsilon_0$ is the energy splitting of the two lowest levels at each site; the states in this representation are

$$|\epsilon_0\rangle_{\vec{n}}\to\begin{pmatrix}0\\1\end{pmatrix}_{\vec{n}}, \quad |\epsilon_1\rangle_{\vec{n}}\to\begin{pmatrix}1\\0\end{pmatrix}_{\vec{n}}. \quad (3.6)$$

For the intersite coupling we need the 2×2 matrix which is the representation of $\phi_{\vec{n}}$ in the truncated basis, given by

$$(\sigma_\phi^{\vec{n}})_{ij}=\langle\epsilon_i|\phi_{\vec{n}}|\epsilon_j\rangle \quad (3.7)$$

for $i,j=0,1$. Since $|\epsilon_0\rangle$ and $|\epsilon_1\rangle$ have, respectively, even and odd parity, it is evident that $\sigma_\phi^{\vec{n}}$ is proportional to $\sigma_x^{\vec{n}}$. The effective Hamiltonian is then

$$H=\sum_{\vec{n}}\left[\frac{\epsilon}{2}(1+\sigma_z^{\vec{n}})+\sum_{\hat{i}}\Delta(1-\sigma_x^{\vec{n}+\hat{i}}\sigma_x^{\vec{n}})\right], \quad (3.8)$$

where $\Delta=2c^2|\langle\epsilon_0|\phi_{\vec{n}}|\epsilon_1\rangle|^2$. It is convenient to perform a rotation which takes $(\sigma_x,\sigma_y,\sigma_z)$ into $(\sigma_z,\sigma_y,-\sigma_x)$ so that H is

$$H=\sum_{\vec{n}}\left[\frac{\epsilon}{2}(1-\sigma_x^{\vec{n}})+\sum_{\hat{i}}\Delta(1-\sigma_z^{\vec{n}+\hat{i}}\sigma_z^{\vec{n}})\right], \quad (3.9)$$

which corresponds to a (quantum) D-dimensional Ising model with a transverse magentic field.

For $D=1$ this quantum spin model has been rigorously shown[24] to be equivalent to the two-dimensional classical Ising model with asymmetric couplings,

$$H=\sum_{n,m}(K_{\|}S_{n,m+1}S_{n,m}+K_{\perp}S_{n+1,m}S_{n,m}) \quad (3.10)$$

and has also been solved exactly,[25] giving critical exponents identical to those of Onsager's solution of the two-dimensional classical Ising model. This demonstrates that the two-dimensional ϕ^4 theory and the two-dimensional classical Ising model are in the same universality class, and it is generally accepted that this is true of any dimensionality.[14] However, they are *not* identical models for short-range correlations. In Sec. IV the possibility of a similar relationship between the Cardy-Sugar[11] model for d dimensions and our $(d-1)$-dimensional Reggeon quantum spin model is considered.

B. The $D=0$ (single-site) Reggeon field theory

There is a close similarity between the $D=0$ Reggeon field theory in the limit $\Delta_0\to-\infty$ and the $D=0$ Ginzburg-Landau theory in the limit $m_0^2\to-\infty$, as we shall see in this subsection.

The only novel feature of the Reggeon field theory is the lack of Hermiticity of the single-site Hamiltonian, Eq. (2.29). Consequently the right-hand eigenstates $|\epsilon_i\rangle$ and left-hand eigenstates $\langle\epsilon^i|$, defined by

$$H_1|\epsilon_i\rangle=\epsilon_i|\epsilon_i\rangle, \quad \langle\epsilon^i|H=\langle\epsilon^i|\epsilon_i \quad (3.11)$$

are not conjugates of each other, $|\epsilon^i\rangle\equiv(\langle\epsilon^i|)^\dagger\neq|\epsilon_i\rangle$, and the eigenvalues ϵ_i are not necessarily real. However, since there is the "parity" operator $p=(-1)^{a^\dagger a}$ such that $PH_1=H_1^\dagger P$, the eigenvalues must occur in complex-conjugate pairs (corresponding to $|\epsilon_i\rangle$ and $|\epsilon_i^*\rangle=P|\epsilon_i\rangle$),

$$H_1|\epsilon_i^*\rangle=PH_1^\dagger|\epsilon_i\rangle=\epsilon_i^*|\epsilon_i\rangle, \quad (3.12)$$

and the Green's function $G_1(E)$ must be real ana-

lytic, $G_1^*(E) = G_1(E^*)$. Similar remarks can be made for the spectrum and Green's functions for $\alpha_0' \neq 0$.

As usual the left-hand (or dual) states are orthogonal to the right-hand states for different eigenvalues, and by multiplying $\langle \epsilon^i |$ by a suitable constant we can construct an orthonormal dual basis from the left eigenstates,

$$\langle \epsilon^i | \epsilon_j \rangle = \delta_{ij} \tag{3.13}$$

Finally Bronzan, Shapiro, and Sugar[18] have demonstrated the remarkable properties that, for $g_0 \neq 0$ and $\Delta_0 < 0$, all the eigenvalues are *real, discrete,* and *non-negative,* that the nondegenerate ground state $|\epsilon_0\rangle$ is the normal vacuum $|0\rangle$ (defined by $a|0\rangle = 0$) with $\epsilon_0 = 0$, and that, in the limit $\Delta_0 \to -\infty$, the first excited level ϵ_1 vanishes like

$$\epsilon_1 \simeq \left(\frac{2}{\pi}\right)^{1/2} \frac{\Delta_0^2}{|g_0|} \exp\left(-\frac{2\Delta_0^2}{g_0^2}\right), \tag{3.14}$$

while all other excited levels go up in energy (see also Appendix B).

An explanation similar to tunnelling is possible for Reggeon field theory but it is less easy to visualize for the complex potential

$$V(\phi, \chi) \equiv H(\alpha_0' = 0) = \Delta_0(\phi^2 + \chi^2) + i g_0 \phi(\phi^2 + \chi^2) \tag{3.15}$$

in terms of the "coordinates" $\phi = (\bar{\psi} + \psi)/2$, $\chi = i(\bar{\psi} - \psi)/2$.

For $\Delta_0 < 0$ the two extrema at $\chi = \pm |\Delta_0/g_0|$ and $\phi = i\Delta_0/g_0$ (see Fig. 2) are stable against *real* fluctuations in ϕ, χ. Now we look at the expectation value in the even and odd combination $|\pm\rangle = (|\epsilon_0\rangle \pm |\epsilon_1\rangle)/\sqrt{2}$ of our two lowest states $|\epsilon_0\rangle$, $|\epsilon_1\rangle$ of the operators $\chi = i(a^\dagger - a)/2$, $\phi = (a^\dagger + a)/2$. Again as $\Delta_0 \to -\infty$, we find that the expectation values lie exactly at the "classical" extrema (see Appendix B)

$$\langle \pm | \chi | \pm \rangle \to \pm \left| \frac{\Delta_0}{g_0} \right|, \quad \langle \pm | \phi | \pm \rangle \to \frac{i\Delta_0}{g_0}. \tag{3.16}$$

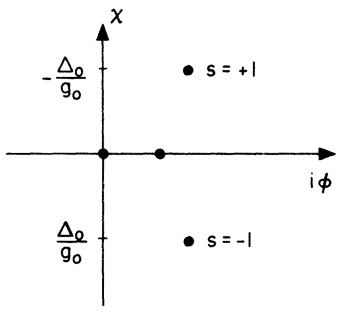

FIG. 2. Extrema for the Reggeon field-theory "potential" $V = (\Delta_0 + i g_0 \phi)(\phi^2 + \chi^2)$. For $\Delta_0 < 0$ and $g_0 > 0$ the stable points are $\chi = \pm \Delta_0/g_0$, $\phi = i\Delta_0/g_0$.

However, there is a subtlety in that ϕ is *not* diagonal in this basis. Nonetheless, by taking the proper matrix elements we construct in the next subsection the analog model for Reggeon field theory by the same procedure as was used for the Ginzburg-Landau theory.[19]

C. Derivation of the Reggeon quantum spin model

If we return to the coupled-site problem $\alpha_0' \neq 0$, we can represent the *exact* Hamiltonian in a basis of states constructed from the direct product of single-site states,

$$|i_1, i_2, \ldots\rangle = |\epsilon_{i_1}\rangle_1 \otimes |\epsilon_{i_2}\rangle_2 \otimes \cdots . \tag{3.17}$$

For $\Delta_0 = O(\ln \alpha_0')$ and α_0' small, we can approximate the Hamiltonian by restricting the basis to the lowest *two* states $|\epsilon_0\rangle$ and $|\epsilon_1\rangle$ at each site. The single-site part of the Hamiltonian then is written as a sum over 2×2 Pauli matrices $\sigma_z^{\vec{n}}$,

$$\sum_{\vec{n}} \left[\Delta_0 a_{\vec{n}}^\dagger a_{\vec{n}} + \frac{i g_0}{2} a_{\vec{n}}^\dagger (a_{\vec{n}} + a_{\vec{n}}^\dagger) a_{\vec{n}} \right] \to \sum \frac{\delta}{2}(1 - \sigma_z^{\vec{n}}), \tag{3.18}$$

where $\delta = \epsilon_1 - \epsilon_0$ is the energy splitting of the lowest two states.

To calculate the coupling between sites, we need the 2×2 matrices which are the representations for $\psi_{\vec{n}}$ and $\bar{\psi}_{\vec{n}}$ (or $a_{\vec{n}}, a_{\vec{n}}^\dagger$) in the truncated basis at the $\vec{n}$th site. These are given, respectively, by

$$\langle \epsilon^i | a | \epsilon_j \rangle \quad \text{and} \quad \langle \epsilon^i | a^\dagger | \epsilon_j \rangle \tag{3.19}$$

for $i, j = 0, 1$.

Since we already know that the lowest state is $|\epsilon_0\rangle = |0\rangle$, we need only calculate the first excited state $|\epsilon_1\rangle$. Following the elegant methods of Bronzan, Shapiro, and Sugar,[18] we show in Appendix B that, in the limit $\Delta_0 \to -\infty$, $|\epsilon_1\rangle$ is given by

$$|\epsilon_1\rangle = c_1 a^\dagger (a^\dagger a + 1)^{-1} |x\rangle \big|_{x = i\sqrt{2}\Delta_0/g_0}, \tag{3.20}$$

where c_1 is a normalization constant and $|x\rangle$ is the plane-wave eigenstate of the operator $(a^\dagger + a)/\sqrt{2}$. Then we calculate the matrix elements (3.19) and the result is [see Eqs. (B26), (B27)]

$$\langle \epsilon^i | a | \epsilon_j \rangle = 2 \left| \frac{\Delta_0}{g_0} \right| \left[\sigma_+ + \frac{i}{2}(1 - \sigma_z) \right]_{ij}, \tag{3.21}$$

$$\langle \epsilon^i | a^\dagger | \epsilon_j \rangle = 2 \left| \frac{\Delta_0}{g_0} \right| \left[\sigma_- + \frac{i}{2}(1 - \sigma_z) \right]_{ij}, \tag{3.22}$$

where we have taken g_0 to be positive.

We find it convenient to make a change of representation by performing a rotation which takes $(\sigma_x, \sigma_y, \sigma_z)$ into $(\sigma_y, \sigma_z, \sigma_x)$, so that the matrix representing χ is diagonal. In this new basis the fields ψ and $\bar{\psi}$ are represented by

$$\sigma_\psi = 2i\left|\frac{\Delta_0}{g_0}\right|\left[\tfrac{1}{2}(1+\sigma_z)-\sigma_+\right]\equiv\sigma_\phi+i\sigma_\chi, \tag{3.23}$$

$$\sigma_{\bar\psi} = 2i\left|\frac{\Delta_0}{g_0}\right|\left[\tfrac{1}{2}(1-\sigma_z)-\sigma_+\right]\equiv\sigma_\phi-i\sigma_\chi. \tag{3.24}$$

We note that the (unperturbed) ground state at the $\vec{n}$th site, defined at the eigenstate of $(1-\sigma_x)^{\vec{n}}$ with zero eigenvalue,

$$|0\rangle_{\vec{n}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_{\vec{n}} \tag{3.25}$$

is annihilated by $\sigma_\psi^{\vec{n}}$, as it should be.

Substituting (3.24) in (2.18) we get our effective Hamiltonian

$$H = \sum_{\vec{n}}\frac{\delta}{2}(1-\sigma_x)^{\vec{n}} - \frac{\alpha_0'}{l^2}\sum_{\vec{n},\hat{i}}[\sigma_{\bar\psi}^{\vec{n}+\hat{i}}\sigma_\psi^{\vec{n}}+\sigma_{\bar\psi}^{\vec{n}}\sigma_\psi^{\vec{n}+\hat{i}}],$$

$$H = \sum_{\vec{n}}\frac{\delta}{2}(1-\sigma_x)^{\vec{n}} + A\sum_{\vec{n},\hat{i}}[(1-2\sigma_+)^{\vec{n}+\hat{i}}(1-2\sigma_+)^{\vec{n}} - \sigma_z^{\vec{n}+\hat{i}}\sigma_z^{\vec{n}}], \tag{3.26}$$

where we have used the identity $\sigma_{\bar\psi}^{\vec{n}}\sigma_\psi^{\vec{n}}=0$.

In terms of the original constants g_0, Δ_0, and α_0'/l^2, the parameters δ and A are

$$\delta = \left(\frac{2}{\pi}\right)^{1/2}\frac{\Delta_0^2}{|g_0|}\exp\left(-\frac{2\Delta_0^2}{g_0^2}\right), \quad A = \frac{2\alpha_0'}{l^2}\left(\frac{\Delta_0}{g_0}\right)^2. \tag{3.27}$$

The lack of Hermiticity of H, Eq. (3.26), is a manifestation of this same property of the original Hamiltonian, Eq. (2.18). The σ_+ is appropriate for $g_0>0$; for $g_0\to -g_0$, $H\to H^\dagger$ and $\sigma_+\to\sigma_-$ so the original non-Hermiticity symmetry is preserved.

D. Symmetries

The original Hamiltonian, Eq. (2.3), is invariant under the transformations

$$H\to H^\dagger, \quad \psi(0,\vec{x})\to -\psi(0,\vec{x}), \quad \bar\psi(0,\vec{x})\to -\bar\psi(0,\vec{x}) \tag{3.28}$$

[recall that $\bar\psi(0,\vec{x})=\psi(0,\vec{x})^\dagger$]. This symmetry is implemented by an operator

$$U = \exp\left[-i\pi\int d^D\vec{x}\,\bar\psi(0,\vec{x})\psi(0,\vec{x})\right] = U^{-1} = U^\dagger, \tag{3.29}$$

which has the properties

$$UHU^{-1}=H^\dagger,$$
$$U\psi(y,\vec{x})U^{-1} = -\bar\psi(-y,\vec{x})^\dagger, \tag{3.30}$$
$$U\bar\psi(y,\vec{x})U^{-1} = -\psi(-y,\vec{x})^\dagger.$$

This symmetry is essential since it has the consequence of making the m-to-n Reggeon amplitude $G^{(n,m)}$ equivalent to the n-to-m amplitude $G^{(m,n)}$. We can find a symmetry operator similar to U for our spin model, with the properties

$$PHP^{-1}=H^\dagger,$$
$$P\sigma_\psi^{\vec{n}}P^{-1} = -\sigma_{\bar\psi}^{\vec{n}\dagger}, \quad P\sigma_{\bar\psi}^{\vec{n}}P^{-1} = -\sigma_\psi^{\vec{n}\dagger}, \tag{3.31}$$

where here H is our spin Hamiltonian (3.26) [for $y\neq 0$, the relations similar to (3.30) follow from this]. It is easily seen that, in our representation (3.23)–(3.24), P is given by

$$P = \prod_{\vec{n}}\sigma_x^{\vec{n}} = P^{-1} = P^\dagger, \tag{3.32}$$

which has the property of flipping the sign of $\sigma_\chi^{\vec{n}}$.

As we now show, if $\langle 0|\sigma_\psi|0\rangle = \langle 0|\sigma_{\bar\psi}|0\rangle = 0$, then the symmetry under the interchange of n with m for the Green's functions $G^{(n,m)}$ is guaranteed. Now $G^{(n,m)}(y_1,\vec{n}_1,\ldots,y_n,\vec{n}_n;y_1',\vec{n}_1',\ldots y_m',\vec{n}_m')$ is given above the critical surface, $\Delta_0\geq\Delta_0^*$ or $\alpha(0)\leq 1$, by

$$G^{(n,m)} = \langle 0|T\sigma_\psi^{\vec{n}_1}(y_1)\cdots\sigma_\psi^{\vec{n}_n}(y_n)\sigma_{\bar\psi}^{\vec{n}_1'}(y_1')\cdots \times\sigma_{\bar\psi}^{\vec{n}_m'}(y_m')|0\rangle. \tag{3.33}$$

For $\Delta_0<\Delta_0^*$, as already discussed in Sec. IA, the Green's function is

$$G^{(n,m)} = \lim_{Y\to\infty}\frac{\mathrm{tr}\left(e^{-YH}T\prod_{i=1}^{n}\sigma_\psi^{\vec{n}_i}(y_i)\prod_{j=1}^{m}\sigma_{\bar\psi}^{\vec{n}_j'}(y_j')\right)}{\mathrm{tr}(e^{-YH})} \tag{3.34}$$

in which the limiting procedure automatically selects the correct ground state. In the first case, it follows from $P|0\rangle = |0\rangle$ that

$$G^{(n,m)}(y_1,\vec{n}_1,\ldots,y_n,\vec{n}_n;y_1',\vec{n}_1',\ldots,y_m',\vec{n}_m') = (-1)^{n+m}G^{(m,n)}(-y_m',\vec{n}_m',\ldots,-y_1',\vec{n}_1';-y_n,\vec{n}_n,\ldots,-y_1,\vec{n}_1)^*. \tag{3.35}$$

For the second case, if $\langle\sigma_\psi\rangle = \langle\sigma_{\bar\psi}\rangle\neq 0$, it is easy to obtain the same result using the new ground state and replacing $\sigma_\psi\to\sigma_\psi-\langle\sigma_\psi\rangle$, $\sigma_{\bar\psi}\to\sigma_{\bar\psi}-\langle\sigma_{\bar\psi}\rangle$ in Eq. (3.33), or by using Eq. (3.34) directly.

The algebra involved for our quantum spin model is quite elegant. The Hamiltonian, Eq. (3.26), is real, though not Hermitian. Also, if we absorb a factor $2i|\Delta_0/g_0|$ in the definition of σ_ψ and $\sigma_{\bar\psi}$, we get from Eqs. (3.23)–(3.24) and suppressing the index $\vec{n}$,

$$\hat{\sigma}_{\psi} = \tfrac{1}{2}(1+\sigma_z) - \sigma_+ = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$
$$\hat{\sigma}_{\bar{\psi}} = \tfrac{1}{2}(1-\sigma_z) - \sigma_+ = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}. \tag{3.36}$$

These matrices have the properties

$$\hat{\sigma}_{\psi}^2 = \hat{\sigma}_{\psi},\ \hat{\sigma}_{\psi}|0\rangle = 0,\ \hat{\sigma}_{\bar{\psi}}^2 = \hat{\sigma}_{\bar{\psi}},\ \langle 0|\hat{\sigma}_{\bar{\psi}} = 0,$$
$$\hat{\sigma}_{\bar{\psi}}\hat{\sigma}_{\psi} = 0,\ \hat{\sigma}_{\psi}\hat{\sigma}_{\bar{\psi}} = \hat{\sigma}_{\psi} + \hat{\sigma}_{\bar{\psi}} - 1. \tag{3.37}$$

These properties are preserved by the replacements

$$\tfrac{1}{2}(1-\sigma_x)^{\vec{n}} \to c_{\vec{n}}^{\dagger} c_{\vec{n}},$$
$$\hat{\sigma}_{\psi}^{\vec{n}} \to c_{\vec{n}}^{\dagger} c_{\vec{n}} + c_{\vec{n}}, \tag{3.38}$$
$$\hat{\sigma}_{\bar{\psi}}^{\vec{n}} \to c_{\vec{n}}^{\dagger} c_{\vec{n}} - c_{\vec{n}}^{\dagger},$$

where the operators $c_{\vec{n}}, c_{\vec{n}}^{\dagger}$ have *anti*commutation relations at the same site but *commute* at different sites.

E. Eikonal calculation

Let us now discuss how we may calculate the cross section in the simple eikonal approximation of Ref. 8. The total cross section for particles a and b (Fig. 3) is given, in the original theory, by

$$\sigma_{\text{tot}}^{ab}(y) = \int d^2\vec{x} \sum_{n,m=1}^{\infty} \frac{(i\beta_a)^n (i\beta_b)^m}{n!\,m!} \times \langle 0|[\psi(0,\vec{x})]^n e^{-yH} [\bar{\psi}(0,\vec{0})]^m |0\rangle, \tag{3.39}$$

where we assume flat form factors represented by the positive constants β_a, β_b. The assumptions involved in Eq. (3.39) are (1) all Pomerons are emitted and absorbed in an uncorrelated way, and (2) all (m) Pomerons are created at one point and all (n) Pomerons are absorbed at one other point in rapidity–impact-parameter space. Let us examine this in the context of our spin model.

First, we note that the matrix element inside

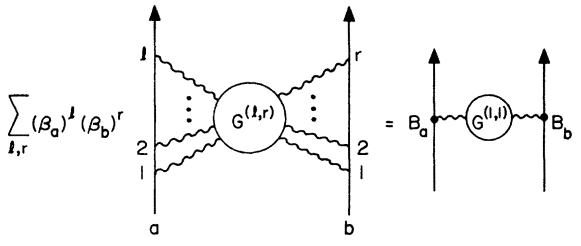

FIG. 3. The eikonal sum for $A_{ab}(y)$ reduces to the single- (renormalized) Pomeron propagator with (renormalized) couplings $B_i = |g_0/2\Delta_0|[1 - \exp(-2|\Delta_0/g_0|\beta_i)]$ replacing the bare Pomeron-particle coupling β_i.

the sum in (3.39) is nothing but the Green's function $G^{(n,m)}(y_i, \vec{x}_i; y_j, \vec{x}_j)$ in which all $(y_j, \vec{x}_j)$ have been set equal to $(0,0)$ and all $(y_i, \vec{x}_i)$ to $(y, \vec{x})$. Let us call it $G^{(n,m)}(y, \vec{x})$. If we replace the fields ψ and $\bar{\psi}$ by the matrices (3.23)–(3.24), we get from (3.37)

$$G^{(n,m)}(y,\vec{x}) = \left(2i\left|\frac{\Delta_0}{g_0}\right|\right)^{n+m-2} G^{(1,1)}(y,\vec{x}) \tag{3.40}$$

and then, performing the sum in (3.39),

$$\sigma_{\text{tot}}^{ab} = B_a B_b \sum_{\vec{n}} G^{(1,1)}(y,\vec{n}), \tag{3.41}$$

where $B_i = |g_0/2\Delta_0|[1 - \exp(-2\beta_i|\Delta_0/g_0|)]$. Thus the cross section factorizes exactly, and is proportional to a one-Pomeron-exchange contribution, but with "renormalized" couplings B_i instead of the bare one β_i. Furthermore, for $|\Delta_0|$ large, the cross section is independent of the incident particles.

The exact form of these results, however, must be taken with caution because they are derived from Eq. (3.40), which is only approximate. There are two reasons for this: (1) As pointed out by Cardy,[26] it is not true that $\langle \epsilon^i|(\psi)^n|\epsilon_j\rangle = (\sigma_\psi)^n$ except for $n \ll |\Delta_0/g_0|$, and (2) because the lattice model is in principle equivalent to the original theory only when *all* the points $\vec{x}_i, \vec{x}_j$ in $G^{(n,m)}$ are far apart from each other, i.e., the distances in impact parameter are much greater than a lattice spacing. Thus, when we set all $\vec{x}_i$'s equal to $\vec{x}$ in $G^{(n,m)}$, we are introducing a strong cutoff dependence which might not be justified. Furthermore, the scaling behavior[4,5] for $G^{(n,m)}$ in general depends on n and m, so Eq. (3.40) is not correct unless the scaling limit and the eikonal limit $\prod_{i=1}^{n}\psi(y_i,\vec{x}_i) \to [\psi(y,\vec{x})]^n$ cannot be interchanged.

There is, however, a case in which Eq. (3.40) *might* be correct. This happens when the critical exponents η and ν satisfy $\eta - D\nu/2 = 0$. In this case, all the Green's functions $G^{(n,m)}$ scale in the same way, independent of n, m. Indeed, this possibility is consistent with the scaling suggested by Amati *et al.*,[10] based on approximate calculations for the "deep" supercritical region $\Delta_0 \ll \Delta_0^*$. Here the spin model gives rise to an expanding disk that saturates the Froissart bound

$$\eta = D,\ \nu = 2 \tag{3.42}$$

In this case, the eikonal result of Eq. (3.41) may represent a beautiful self-consistency between s-channel and t-channel multi-Pomeron effects. The standard s-channel eikonalization absorption mechanism for saturating the Froissart bound $[\beta_i \neq 0,\ \alpha(0) > 1,\ g_0 = g_{PPP} = 0]$, is reexpressed in terms of the renormalized single-Pomeron ex-

change for nonzero triple-Pomeron coupling ($g_0 \neq 0$). This intriguing possibility clearly merits further study.

IV. POSSIBLE RELATIONSHIP TO THE CARDY-SUGAR MODEL

The Cardy-Sugar (CS) model[11] is a classical Ising-type model with spins $s_{m,n} = \pm 1$ at each site in a transverse ($\vec{x} = \vec{n}l$) as well as *rapidity* ($y = m\bar{l}$) lattice (here $\bar{l}$ is the lattice spacing along the rapidity axis and m is an integer).

We will study the relationship between this model and ours in terms of their transfer matrices. The transfer matrix[27] describes the couplings between spins (labeled s) in a plane at m and spins (labeled s') in a plane at $m+1$ (see Fig. 4), and is defined by

$$T_{CS}(s', s) \equiv \langle s' | T_{CS} | s \rangle \equiv e^{-H_{CS}(s', s)} , \tag{4.1}$$

where $H_{CS}(s', s)$ is that piece of the Hamiltonian which couples only spins lying in these two planes (including single-spin terms).

Let us consider for simplicity the case $D = 1$, on a $M \times N$ lattice ($m = 1, \ldots, M$ and $n = 1, \ldots, N$). Then T_{CS} is a $2^N \times 2^N$ matrix, and

$$-H_{CS}(s', s) = \sum_{n=1}^{N} \{ K(s_n s_{n+1} + s'_n s'_{n+1}) + L[s'_n s'_{n+1}(s_n + s_{n+1}) - (s'_n + s'_{n+1}) s_n s_{n+1}] \} . \tag{4.2}$$

We assume periodic boundaries in x space so that $s_{N+1} \equiv s_1$ and $s'_{N+1} \equiv s'_1$ (see Fig. 5).

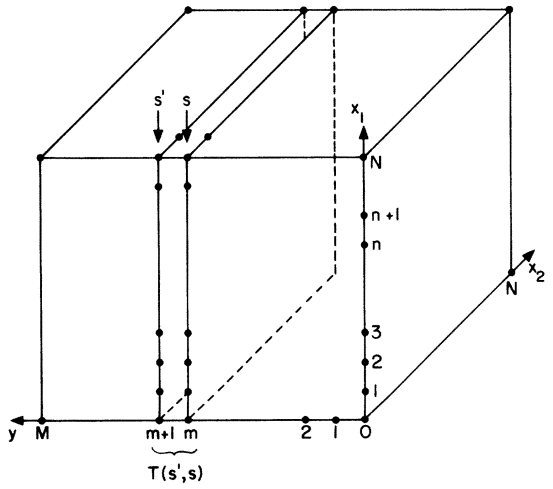

FIG. 4. The transfer matrix $T(s', s)$ translates the state of the plane at $y = m\bar{l}$ to the state of the plane at $y = (m+1)\bar{l}$ on a $M \times N \times N$ lattice for $D = 2$. The partition function with periodic boundary conditions in rapidity is $Z = \mathrm{tr}(T^M)$.

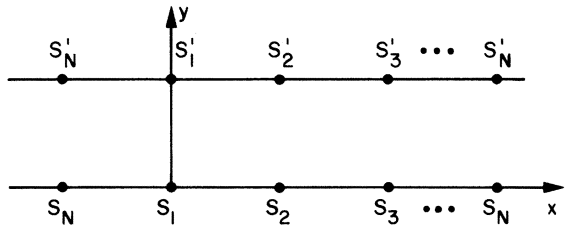

FIG. 5. Two rows of a lattice for $D = 1$ with periodic boundary conditions in the transverse dimension.

For our Reggeon quantum spin model the transfer matrix is

$$T(s', s) = \langle s' | e^{-yH(\sigma)} | s \rangle |_{y=\bar{l}} , \tag{4.3}$$

where

$$H(\sigma) \equiv H_{12} + H_{23} + \cdots + H_{N1} ,$$

$$H_{n,n+1} \equiv \frac{\delta}{4} (2 - \sigma_x^n - \sigma_x^{n+1}) + A[(1 - 2\sigma_+^n)(1 - 2\sigma_+^{n+1}) - \sigma_z^n \sigma_z^{n+1}] ,$$

$$|s\rangle \equiv |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle \tag{4.4}$$

and $|s_n\rangle = |\pm\rangle$ is the eigenstate of σ_z^n.

To compute the matrix element (4.3) is in general very complicated because of the noncommutativity of the different terms in $H(\sigma)$, but one can see that T_{CS} and T are not identical. However, we might still hope to see their relationship. Let us consider only the couplings [arising from (4.3)] within the square defined by the spins s_1, s_2, s'_2, s'_1. These couplings are given by

$$\langle s'_1 s'_2 | e^{-\bar{l} H_{12}} | s_1 s_2 \rangle \equiv e^{-\mathcal{K}(s'_1 s'_2; s_1 s_2)} . \tag{4.5}$$

Consider a more general form for H_{12}

$$H_{12} = \sum_i \kappa_i \Sigma_i(\sigma^1, \sigma^2) \tag{4.6}$$

where the κ_i are real constants and the Σ_i's are the operators which satisfy the basic properties of our Hamiltonian (3.26), namely

$$\Sigma_i(\sigma^1, \sigma^2) = \Sigma_i(\sigma^2, \sigma^1) , \tag{4.7a}$$

$$\sigma_x^1 \sigma_x^2 \Sigma_i(\sigma^1, \sigma^2) \sigma_x^1 \sigma_x^2 = \Sigma_i(\sigma^1, \sigma^2)^\dagger , \tag{4.7b}$$

$$\Sigma_i(\sigma^1, \sigma^2)^* = \Sigma_i(\sigma^1, \sigma^2) , \tag{4.7c}$$

$$\Sigma_i(\sigma^1, \sigma^2) |0\rangle_1 |0\rangle_2 = 0 , \tag{4.7d}$$

with $|0\rangle_n$ given by Eq. (3.25). It is easily seen that the operators which satisfy the first three of these requirements are

$$\Sigma_1 = (1-\sigma_x)^1 + (1-\sigma_x)^2, \quad \Sigma_2 = (1-\sigma_x)^1(1-\sigma_x)^2,$$
$$\Sigma_3 = i\sigma_y^1(1-\sigma_x)^2 + i\sigma_y^2(1-\sigma_x)^1,$$
$$\Sigma_4 = \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2, \tag{4.8}$$
$$\Sigma_5 = i(\sigma_y^1 + \sigma_y^2), \quad \Sigma_6 = \sigma_z^1\sigma_z^2, \quad \Sigma_7 = 1,$$

while only the first *four* Σ's satisfy the additional restriction (4.7d). Then we can prove that $\mathcal{H}(s_1', s_2'; s_1, s_2)$, defined by Eq. (4.5), has only the couplings allowed by the symmetries of the Cardy-Sugar model within the square, *provided* we neglect the restriction (4.7d).

The key point of our proof is the fact that the Σ_i's close algebraically under *anti*commutation, i.e., the anticommutator of any two is a linear combination of themselves. This is nontrivial because the Σ_i's do *not* form a complete set of matrices for any operator symmetric under the interchange $\sigma_k^1 \longleftrightarrow \sigma_k^2$. This closure property makes it possible to expand

$$e^{-\bar{t}H_{12}} = \sum_{i=1}^{7} \beta_i \Sigma_i(\sigma^1, \sigma^2) \tag{4.9}$$

since any power of H_{12} can be written as a linear combination of the Σ_i's by using the elementary recursion relation

$$H_{12}^{n+1} = \tfrac{1}{2}\{H_{12}, H_{12}^n\}. \tag{4.10}$$

Next, one can compute the matrix elements $\langle s_1's_2' | \Sigma_i | s_1 s_2 \rangle$ and show that they are linear combinations of the seven polynomials[28]

$$S_1 = s_1's_1 + s_2's_2, \quad S_2 = s_1's_2's_1s_2,$$
$$S_3 = s_1's_2'(s_1 + s_2) - (s_1' + s_2')s_1s_2,$$
$$S_4 = s_1's_2 + s_1s_2', \tag{4.11}$$
$$S_5 = s_1' + s_2' - s_1 - s_2, \quad S_6 = s_1's_2' + s_1s_2, \quad S_7 = 1,$$

which are the *complete* set of couplings allowed by the symmetries of the Cardy-Sugar model for the square. Since the S_i's also close algebraically under *multiplication*, as can be checked using $s_i^2 = s_i'^2 = 1$, we can take the logarithm of Eq. (4.5) to get

$$\mathcal{H}(s_1's_2'; s_1s_2) = \sum_{i=1}^{7} K_i S_i, \tag{4.12}$$

which shows the relationship between the models if Eq. (4.7d) is neglected and the couplings are restricted to a basic square.

We believe, however, that the normal-ordering property (4.7d), also satisfied by the exact Hamiltonian of the Reggeon field theory, is a crucial one and plays a key role in the phase transition of the model. Thus if we require (4.7d) to hold, we are restricted to the first four of the operators Σ_i in (4.8). One easily checks that these, too, close under anticommutation, and therefore an expansion like (4.9) still holds. However, the matrix elements $\langle s_1's_2' | \Sigma_i | s_1 s_2 \rangle$ for $i = 1, 2, 3, 4$ are linear combinations of the first four S_i only, and one can easily check that these do *not* close under multiplication: the full set of S_i's are needed to form an algebra.

The exact relationship between the two models is clearly very complex, and a more careful study is desirable without the limitation to a finite lattice. However, it is likely that the conclusion one would reach is the same as ours. Thus it seems that our model has a higher degree of symmetry than that of Cardy and Sugar, because the condition that the perturbative ground state is annihilated by the full Hamiltonian, Eq. (3.26), is satisfied automatically. It may be possible to impose a restriction on the different couplings of a generalized Cardy-Sugar model so as to satisfy this condition, but this seems to be impractical in actual calculations. Without restrictions, a renormalization-group approach for its solution will evidently involve a larger parameter space than our model. The problem is probably the same as the one dealt with in Ref. 13, that any rapidity discretization inevitably brings in new relevant operators corresponding to $(\psi)^n$ or $(\psi^\dagger)^m$. Consequently, we feel that the Cardy-Sugar model is really in a different (less "symmetric") universality class than Reggeon field theory, and it will be therefore more difficult or even impossible to find the appropriate phase transition in this model.

V. CONCLUSIONS

We have constructed a quantum spin model for Reggeon field theory by discretizing the impact-parameter space while keeping rapidity a continuous variable. Our Reggeon quantum spin model is expected to leave intact the behavior of the original theory in the critical region. We have also studied in what case the eikonal approximation for our spin model gives the entire cross section in terms of a single renormalized Pomeron exchange. Two interesting consequences follow in this case: (1) If the eikonal model develops a disk structure for $\alpha(0) > 1$ (Refs. 9, 10) this must be seen in the single-Pomeron sector. (2) The bare couplings to the external particles are completely "shielded" for $|\Delta_0/g_0| \to \infty$, so that cross sections approach a universal constant. This is reminiscent of results for the eikonal saturation of the Froissart bound of Cheng and Wu.[29] Clearly, better calculations are needed to substantiate these features. Several kinds of calculations can be

pursued in our approach.

Our spin model may be solved with the help of standard methods in statistical mechanics.[15] Although it is unlikely that exact solutions will be found even for $D=1$ (like the one given by Pfeuty[25] for the quantum Ising model in a transverse magnetic field), there are several techniques which may be employed.

Nevertheless, some rigorous results may be derived without an explicit solution. For example, the Froissart bound ($\eta \leq 2$) and other consistency conditions of s-channel unitarity probably follow, as argued by Cardy and Sugar,[11] from little more than the existence of a second-order phase transition. These important problems may well be settled within the context of our model.

More likely, approximate renormalization-group techniques using Kadanoff block constructions[16] will be better tools for the accurate description of the phase transition and, in particular, the calculation of critical exponents. We feel that the lack of success with these calculations to date is due to confusion on the special role played by the rapidity dimension. In our Hamiltonian formalism, however, we are led naturally to renormalization-group calculations on the two-dimensional *transverse* lattice only, in which these confusions are avoided.

Another tactic may be the use of variational approaches for the construction of the ground state along the lines of the ϕ^4 theory.[30] Although the lack of Hermiticity in our problem is a difficulty, it is probably not insurmountable.

There is also the issue raised by Abarbanel *et al.*[7] and Amati *et al.*[10] on the nature of the phase for $\Delta_0 < \Delta_0^*$ [or $\alpha(0)>1$][9] and whether one has a nonsensical model, or in the presence of sources, one saturates the Froissart bound. Particularly interesting is the question raised by Cardy[31] on whether classical kinks or solutions play a decisive role for $\Delta_0 \ll \Delta_0^*$. Perhaps the simplicity of our spin model will allow a study of these questions.

Another class of problems we could study are the approach to scaling and the region away from the phase transition. These may be dealt with the use of the finite-rapidity formalism[20] described in Sec. I A. However, this is a considerably more complicated problem because in principle all excited states of the Hamiltonian play a role. Nevertheless, we may still use our quantum spin approximation by keeping more than just two states at each site, with an effective Hamiltonian given in terms of spin matrices of higher dimensionality.

While completing the manuscript we have learned of similar results from Cardy and from the CERN collaborators.

ACKNOWLEDGMENTS

We would like to thank John Bronzan and Shimon Yankielowicz for many useful discussions on their work on one-dimensional Reggeon field theory and two-dimensional lattice ϕ^4 theory, respectively. We would also like to thank Michael Nauenberg for several discussions and for reading the manuscript.

APPENDIX A

We first wish to prove Eq. (2.13) for $D=0$ (or $\alpha_0'=0$) and give a precise meaning to the functional integral. That is, we want to prove

$$\int \delta\psi\delta\bar{\psi} \exp\left\{\int_0^Y dy\left[-\bar{\psi}\frac{\partial\psi}{\partial y} - \Delta_0\bar{\psi}\psi - \frac{ig_0}{2}\bar{\psi}(\psi+\bar{\psi})\psi\right]\right\} = \mathrm{tr}e^{-YH_1}\,, \quad \text{(A1)}$$

where H_1 is given by Eq. (2.29)

$$H_1 = \Delta_0 a^\dagger a + \frac{ig_0}{2} a^\dagger (a + a^\dagger) a \quad \text{(A2)}$$

and $a^\dagger, a$ are the ordinary creation and annihilation operators, satisfying

$$[a, a^\dagger] = 1, \quad a|0\rangle = \langle 0|a^\dagger = 0\,. \quad \text{(A3)}$$

Imagine now that we divide the inverval $(0, Y)$ into $M+1$ segments of length ϵ so that $\epsilon \to 0$, $M \to \infty$ but $\epsilon(M+1) = Y$ is kept fixed. We associate a complex variable z_i, $i = 0, 1, 2, \ldots, M+1$, to each of the points at the ends of the segments (see Fig. 6). Consider now the coherent state

$$|z\rangle \equiv \exp\left(-\frac{|z|^2}{2} + za^\dagger\right)|0\rangle \quad \text{(A4)}$$

which has the following properties (z is any complex number):

$$a|z\rangle = z|z\rangle, \quad \langle z|a^\dagger = \langle z|z^*\,,$$

$$\langle z_2|z_1\rangle = \exp\left(-\frac{|z_2|^2}{2} - \frac{|z_1|^2}{2} + z_2^* z_1\right)\,, \quad \text{(A5)}$$

$$\int \frac{d^2z}{\pi}\,|z\rangle\langle z| = 1$$

(the integral is over the *entire complex z plane*).

The key point of our proof is the fact that

$$\langle z_2|:F(a^\dagger, a):|z_1\rangle = \langle z_2|z_1\rangle F(z_2^*, z_1)\,, \quad \text{(A6)}$$

where $:F(a^\dagger, a):$ is any normal-ordered function of $a^\dagger, a$.

If we now write

$$e^{-YH_1} = e^{-\epsilon H_1}e^{-\epsilon H_1}\cdots e^{-\epsilon H_1} \quad (M+1 \text{ factors})\,, \quad \text{(A7)}$$

we have

$$\langle z_{M+1}|e^{-YH_1}|z_0\rangle = \int \frac{d^2z_M}{\pi}\frac{d^2z_{M-1}}{\pi}\cdots\frac{d^2z_1}{\pi}\langle z_{M+1}|e^{-\epsilon H_1}|z_M\rangle\cdots\langle z_1|e^{-\epsilon H_1}|z_0\rangle \ . \tag{A8}$$

Now for ϵ sufficiently small it is straightforward to prove using Eq. (A6), to order ϵ^2,

$$\langle z_{i+1}|e^{-\epsilon H_1}|z_i\rangle \simeq \exp\left[-\frac{|z_{i+1}|^2}{2}-\frac{|z_i|^2}{2}+z_{i+1}^*z_i-\epsilon\left(\Delta_0 z_{i+1}^*z_i+\frac{ig_0}{2}z_{i+1}^*(z_{i+1}^*+z_i)z_i\right)\right] \ , \tag{A9}$$

which becomes exact as $\epsilon\to 0$. If we impose periodic boundary conditions so that $z_0\equiv z_{M+1}$ and integrate over z_0 we get

$$\begin{aligned}\int\frac{d^2z_0}{\pi}\langle z_0|e^{-YH_1}|z_0\rangle &= \sum_{n,m=0}^{\infty}\int\frac{d^2z_0}{\pi}\langle m|z_0\rangle\langle z_0|n\rangle\langle n|e^{-YH_1}|m\rangle \\ &= \mathrm{tr}\,e^{-YH_1} \\ &\simeq \int\prod_{i=0}^{M}\left(\frac{d^2z_i}{\pi}\right)\exp\left\{\epsilon\sum_{i=0}^{M}\left[-z_{i+1}^*\frac{z_{i+1}-z_i}{\epsilon}-\Delta_0 z_{i+1}^*z_i-\frac{ig_0}{2}z_{i+1}^*(z_{i+1}+z_i)z_i\right]\right\} \ ,\end{aligned} \tag{A10}$$

which, in the limit $\epsilon\to 0$, $M\to\infty$ with $Y=\epsilon(M+1)$ fixed (if it exists), becomes Eq. (A1) and defines the functional integral.

We see that the integration region for $\delta\psi\delta\bar{\psi}$ is the entire complex ψ plane with real and imaginary coordinates $\phi=(\psi+\bar{\psi})/2$ and $\chi=i(\bar{\psi}-\psi)/2$. Only when $\psi,\bar{\psi}$ are considered as Heisenberg *operators* does the non-Hermiticity of H require $\bar{\psi}(y)$ to be distinguished from $\psi^\dagger(y)$.

Three comments are in order: (1) Crucial to our proof is the fact that H_1 is normal-ordered; if it was not, the effective Lagrangian we would obtain from Eq. (A10) would have tadpole terms in addition. (2) Each of the $M+1$ integrals in (A10) is a two-dimensional surface integral over the entire complex z_i plane, and is manifestly convergent for small enough ϵ. Another matter is the existence of the *limit* $M\to\infty$, $\epsilon\to 0$ in the $(M+1)$-fold integral (A9), which is a standard problem in path integrals; we assume that this limit exists.[32] (3) If the two-dimensional surface integral over z_i is written as a two-fold line integral by choosing some coordinate system on the z_i plane, the convergence of the integral may no longer be manifest, and appropriate distortion of contours may be needed. This problem has arisen in previous work[8,10,11] and has caused the need for unconventional redefinition of canonical momenta. We feel that our method of using coherent states in the path integral provides a clear starting point for the discussion of these points.

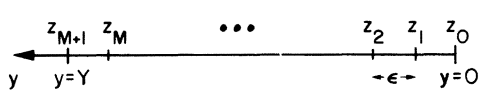

FIG. 6. One row of *rapidity* divided into $M+1$ segments of length $\epsilon=Y/(M+1)$. There are $M+1$ complex variables z_i defined at the ends of each segment.

The method outlined is easily applied to Hamiltonians of the "canonical" form,

$$H=:\left[\frac{\pi^2}{2}+V(x)\right]: \ , \tag{A11}$$

where $x\equiv(a^\dagger+a)/\sqrt{2}=x^\dagger$ and $\pi\equiv i(a^\dagger-a)/\sqrt{2}=\pi^\dagger$, and the standard results[33] are obtained in a rather straightforward way.

The extension to include D transverse dimensions in addition to rapidity presents no complications. We start from a lattice with N sites in $\vec{x}$ space, $\vec{x}=\vec{n}l$, with H given by (2.18), and divide the rapidity chains in $M+1$ intervals of length ϵ so that $Y=(M+1)\epsilon$. For each chain at lattice site $\vec{n}$ we define $M+1$ complex variables $z_{\vec{n}}^i$ in a similar way as for the $D=0$ case, and a coherent state

$$|z_{\vec{n}}^i\rangle\equiv\exp\left(-\frac{|z_{\vec{n}}^i|^2}{2}+z_{\vec{n}}^i a_{\vec{n}}^\dagger\right)|0\rangle \ , \tag{A12}$$

which satisfies the properties

$$\begin{aligned}&a_{\vec{n}'}|z_{\vec{n}}^i\rangle=\delta_{\vec{n}'\vec{n}}z_{\vec{n}}^i|z_{\vec{n}}^i\rangle \ , \\ &\langle z_{\vec{n}}^i|z_{\vec{n}}^j\rangle=\exp\left(-\frac{|z_{\vec{n}}^i|^2}{2}-\frac{|z_{\vec{n}}^j|^2}{2}+z_{\vec{n}}^{i*}z_{\vec{n}}^j\right) \ .\end{aligned} \tag{A13}$$

We now write e^{-YH} as in (A7) and use the completeness relation for the coherent states. The net effect is that the intersite coupling part of Hamiltonian (2.16),

$$V\equiv\frac{\alpha_0'}{l^2}\sum_{\vec{n},\hat{i}}(a_{\vec{n}+\hat{i}}^\dagger-a_{\vec{n}}^\dagger)(a_{\vec{n}+\hat{i}}-a_{\vec{n}}) \ , \tag{A14}$$

gets replaced by

$$V=\frac{\alpha_0'}{l^2}\sum_{\vec{n},\hat{i}}(z_{\vec{n}+\hat{i}}^{j+1*}-z_{\vec{n}}^{j+1*})(z_{\vec{n}+\hat{i}}^j-z_{\vec{n}}^j) \tag{A15}$$

between rapidity sites $j+1$ and j. In the limit $l\to 0$

and $N \to \infty$, where N is the number of lattice sites or rapidity chains, this gives back the integral of the gradient terms in the action,

$$\alpha_0' \int d^D\vec{x}\, \vec{\nabla}\bar{\psi} \cdot \vec{\nabla}\psi \ , \tag{A16}$$

and the proof is completed as in the $D=0$ case.

Similar methods apply for Green's functions.

APPENDIX B

We wish to calculate the first two eigenvalues ϵ_0, ϵ_1 and eigenstates $|\epsilon_0\rangle$, $|\epsilon_1\rangle$ for the Hamiltonian

$$H_1 = \Delta_0 a^\dagger a + \frac{ig_0}{2}\, a^\dagger (a + a^\dagger) a \tag{B1}$$

and the matrix elements $\langle \epsilon^i | a | \epsilon_j \rangle$, $\langle \epsilon^i | a^\dagger | \epsilon_j \rangle$ for $\Delta_0 \to -\infty$. Our techniques are similar to those of Ref. 18 to which we refer the reader.

We write $H_1 = a^\dagger H_0 a$, where

$$H_0 = \Delta_0 + \frac{ig_0}{\sqrt{2}}\, x, \quad x = \frac{1}{\sqrt{2}}\,(a^\dagger + a) . \tag{B2}$$

Equation (B1) tells us that $|0\rangle$ is an eigenstate of H_1 with eigenvalue zero. In Ref. 18 it is argued that for $\Delta_0 < 0$ and $g_0 \neq 0$, all other eigenvalues ϵ_i are real and positive, so $|\epsilon_0\rangle = |0\rangle$ is always the ground state.

Let us consider the two-point Green's function defined by

$$G_1(E) = \int_0^\infty dy\, e^{Ey} \langle 0 | a e^{-yH_1} a^\dagger | 0 \rangle$$

$$= -\left\langle 0 \left| a \, \frac{1}{E - H_1} \, a^\dagger \right| 0 \right\rangle . \tag{B3}$$

Using now the identity

$$aH_1 = aa^\dagger H_0 a$$

$$= (N+1) H_0 a , \tag{B4}$$

where $N \equiv a^\dagger a$ is the number occupation operator, we find

$$G_1(E) = -\langle 0 | [E - (N+1)H_0]^{-1} | 0 \rangle$$

$$= \left\langle 0 \left| \left(H_0 - \frac{E}{N+1} \right)^{-1} \right| 0 \right\rangle . \tag{B5}$$

At $E = 0$, we insert a complet set of eigenstates of the operator x and get

$$G_1(0) = \int_{-\infty}^{\infty} dx \left\langle 0 \left| \frac{1}{H_0} \right| x \right\rangle \langle x | 0 \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}}\, e^{-x^2} \, \frac{1}{\Delta_0 + (ig_0/\sqrt{2})x} . \tag{B6}$$

For $\Delta_0 > 0$, we write

$$\left(\Delta_0 + i \frac{g_0}{\sqrt{2}}\, x \right)^{-1} = \int_0^\infty dy \exp\left[-y \left(\Delta_0 + \frac{ig_0}{\sqrt{2}}\, x \right) \right] \tag{B7}$$

and then find

$$G_1(0) = \int_0^\infty dy \exp\left(-y\Delta_0 - \frac{g_0^2}{8}\, y^2 \right) , \tag{B8}$$

which can now be analytically continued to $\Delta_0 < 0$ provided $g_0 \neq 0$. For $\Delta_0 \to -\infty$ we use the method of steepest descent and get

$$G_1(0) \simeq \frac{2(2\pi)^{1/2}}{|g_0|} \exp\left(\frac{2\Delta_0^2}{g_0^2} \right) . \tag{B9}$$

Inserting now a complete set of energy states

$$\sum_{i=0}^{\infty} |\epsilon_i\rangle \langle \epsilon^i| = 1 \tag{B10}$$

in Eq. (B3) we get, for small E and $\Delta_0 \to -\infty$,

$$G_1(E) \simeq -\frac{\langle 0 | a | \epsilon_1 \rangle \langle \epsilon^1 | a^\dagger | 0 \rangle}{E - \epsilon_1} , \tag{B11}$$

where we are using the fact, proven in Ref. 18, that in this limit only the first excited level ϵ_1 goes to zero while all the others move up to infinity proportionally to $-\Delta_0$.

The divergence in $G_1(0)$ as $\Delta_0 \to -\infty$ indicates that $\epsilon_1 \sim \exp(-\Delta_0^2/g_0^2)$ in this limit. The leading contribution to $G_1(0)$ for $\Delta_0 \to -\infty$ comes from the pole at $x = i\rho \equiv i\sqrt{2}\,\Delta_0/g_0$ in Eq. (B6) as it moves through the contour in this limit.

The rapid vanishing of ϵ_1 suggests approximating the eigencondition for $|\epsilon_1\rangle$ by

$$H_1 |\epsilon_1\rangle = 0, \quad \langle 0 | \epsilon_1 \rangle = 0 . \tag{B12}$$

By inspection we see that the solution is

$$|\epsilon_1\rangle = a^\dagger \, \frac{c_1}{N+1} \, |x\rangle \Big|_{x=i\rho} , \tag{B13}$$

where $|x\rangle$ is the plane-wave eigenstate of the operator $(a + a^\dagger)/\sqrt{2}$,

$$|x\rangle = \pi^{-1/4} \exp\left(-\frac{x^2}{2} - \frac{a^{\dagger 2}}{2} + x\sqrt{2}\, a^\dagger \right) |0\rangle \tag{B14}$$

(we have used $H = a^\dagger H_0 a$ and $H_0 |x = i\rho\rangle = 0$). Likewise, the left eigenvector defined by $\langle \epsilon^1 | H = 0$, $\langle \epsilon^1 | 0 \rangle = 0$ is

$$\langle \epsilon^1 | = \left\langle x \left| \, \frac{c^1}{N+1} \, a \right| \right._{x=i\rho} \tag{B15}$$

and we choose the normalization constants c_1, c^1, so that

$$\langle \epsilon^1 | \epsilon_1 \rangle = 1 . \tag{B16}$$

This implies

$$(c^1c_1)^{-1}\left\langle x\left|\frac{1}{N+1}\right|x\right\rangle\Big|_{x=i\rho}$$
$$=\int_0^1 dz\langle x|z^N|x\rangle\Big|_{x=i\rho}$$
$$=\int_0^1\frac{dz}{\sqrt{\pi}}\frac{1}{(1-z^2)^{1/2}}\exp\left(\rho^2\frac{1-z}{1+z}\right)$$
$$\xrightarrow[\Delta_0\to-\infty]{}\frac{e^{\rho^2}}{2\sqrt{\pi}\,\rho^2}\ , \tag{B17}$$

therefore, choosing $c^1=c_1$, we get

$$c^1=c_1=|\rho|(4\pi)^{1/4}e^{-\rho^2/2}. \tag{B18}$$

Now we compute the matrix elements

$$\langle 0|a|\epsilon_1\rangle=c_1\langle 0|x\rangle\Big|_{x=i\rho}$$
$$=c_1\pi^{-1/4}e^{\rho^2/2}$$
$$\underset{\Delta_0\to-\infty}{\sim}2|\rho|=2\left|\frac{\Delta_0}{g_0}\right|,$$

and similarly

$$\langle\epsilon^1|a|\epsilon_1\rangle\underset{\Delta_0\to-\infty}{\sim}2\left|\frac{\Delta_0}{g_0}\right|. \tag{B19}$$

Finally we compute

$$\langle\epsilon^1|a|\epsilon_1\rangle=c^1c_1\left\langle x\left|\frac{1}{N+1}a\right|x\right\rangle\Big|_{x=i\rho}$$
$$=c^1c_1\int_0^1 dz\left\langle x\left|z^N\int_{-\infty}^{\infty}dx'\right|x'\right\rangle$$
$$\times\langle x'|a|x\rangle\Big|_{x=i\rho}. \tag{B20}$$

Now recalling

$$\langle x'|a|x\rangle=\frac{1}{\sqrt{2}}\left(x+\frac{d}{dx'}\right)\delta(x'-x) \tag{B21}$$

and

$$\langle x|z^N|x'\rangle=\frac{1}{[\pi(1-z^2)]^{1/2}}$$
$$\times\exp\left[-\tfrac{1}{4}(x+x')^2\frac{1-z}{1+z}\right.$$
$$\left.-\tfrac{1}{4}(x-x')^2\frac{1+z}{1-z}\right], \tag{B22}$$

we obtain

$$\langle\epsilon^1|a|\epsilon_1\rangle=\frac{c^1c_1}{2}\int_0^1\frac{dz}{\sqrt{\pi}}\frac{1}{(1-z^2)^{1/2}}\left(x+x\frac{1-z}{1+z}\right)$$
$$\times\exp\left(-x^2\frac{1-z}{1+z}\right)\Big|_{x=i\rho} \tag{B23}$$
$$\underset{\Delta_0\to-\infty}{\sim}c^1c_1\frac{i\rho}{\sqrt{2}}\,2\frac{e^{\rho^2}}{2\rho^2\sqrt{\pi}}=2i\frac{\Delta_0}{g_0}\ . \tag{B24}$$

Likewise

$$\langle\epsilon^1|a^\dagger|\epsilon_1\rangle\sim 2i\frac{\Delta_0}{g_0}\ . \tag{B25}$$

Summarizing our results, $\langle\epsilon^i|a|\epsilon_j\rangle$ and $\langle\epsilon^i|a^\dagger|\epsilon_j\rangle$ are represented by two 2×2 matrices,

$$\langle\epsilon^i|a|\epsilon_j\rangle=2\left|\frac{\Delta_0}{g_0}\right|\begin{pmatrix}0&1\\0&\pm i\end{pmatrix}$$
$$=2\left|\frac{\Delta_0}{g_0}\right|\left[\sigma_+\pm\frac{i}{2}(1-\sigma_z)\right], \tag{B26}$$

$$\langle\epsilon^i|a^\dagger|\epsilon_j\rangle=2\left|\frac{\Delta_0}{g_0}\right|\begin{pmatrix}0&0\\1&\pm i\end{pmatrix}$$
$$=2\left|\frac{\Delta_0}{g_0}\right|\left[\sigma_-\pm\frac{i}{2}(1-\sigma_z)\right], \tag{B27}$$

where the $\pm$ sign is the sign of g_0. Note that they are the *transpose* of each other and not their Hermitian conjugates. This is a manifestation of the Hermiticity property of H_1: $H_1^\dagger(a,a^\dagger)=H_1(-a,-a^\dagger)$.

It is also useful to consider the matrices for $\phi=(a+a^\dagger)/2$ and $\chi=i(a^\dagger-a)/2$:

$$\langle\epsilon^i|\phi|\epsilon_j\rangle=\left|\frac{\Delta_0}{g_0}\right|\begin{pmatrix}0&1\\1&\pm2i\end{pmatrix}$$
$$=\frac{\Delta_0}{g_0}[\pm i(1-\sigma_z)+\sigma_x]\ ,$$
$$\langle\epsilon^i|\chi|\epsilon_j\rangle=\left|\frac{\Delta_0}{g_0}\right|\begin{pmatrix}0&i\\-i&0\end{pmatrix} \tag{B28}$$
$$=-\left|\frac{\Delta_0}{g_0}\right|\sigma_y\ .$$

As we may have expected, the eigenvalues of σ_χ and σ_ϕ are, respectively, $\pm\Delta_0/g_0$ and $i\Delta_0/g_0$, which are precisely the values of the classical fields χ,ϕ at which the "potential" $V(\phi,\chi)=(\Delta_0+ig_0\phi)(\phi^2+\chi^2)$ has its stable saddle points (Fig. 2).

Finally, from Eqs. (B9) and (B11) for $E=0$, we get

$$\frac{2(2\pi)^{1/2}}{|g_0|}\exp\left(\frac{2\Delta_0^2}{g_0^2}\right)=\frac{1}{\epsilon_1}\langle 0|a|\epsilon_1\rangle\langle\epsilon^1|a^\dagger|0\rangle$$

or

$$\epsilon_1=\left(\frac{2}{\pi}\right)^{1/2}\frac{\Delta_0^2}{|g_0|}\exp\left(-\frac{2\Delta_0^2}{g_0^2}\right). \tag{B29}$$

*Work supported by the National Science Foundation.
†Work supported in part by the Alfred P. Sloan Foundation.
‡Work supported in part by a UCSC fellowship.
§Work supported by the National Science Foundation.

[1]R. C. Brower and J. H. Weis, Phys. Lett. 41B, 631 (1972).

[2]V. N. Gribov, Zh. Eksp. Teor. Fiz. 53, 654 (1967) [Sov. Phys.–JETP 26, 414 (1968)]; V. N. Gribov and A. A. Migdal, Yad. Fiz. 8, 1002 (1968) [Sov. J. Nucl. Phys. 8, 583 (1969)]; *ibid.* 8, 1213 (1968) [*ibid.* 8, 703 (1969)].

[3]We will deal throughout with the Wick rotated "time" ($y=it$) which we also refer to as the *longitudinal* coordinate. The number of "time" plus "space" variables is d, and $D=d-1$ is the number of "space" or *transverse* coordinates.

[4]A. A. Migdal, A. M. Polyakov, and K. A. Ter-Martirosyan, Phys. Lett. 48B, 239 (1974); H. D. I. Abarbanel and J. B. Bronzan, *ibid.* 48B, 345 (1974). Actually $\Delta_0^* \to -\infty$ as the momentum cutoff $\Lambda \to \infty$; we shall always consider Λ large but finite.

[5]For a review see H. D. I. Abarbanel, J. B. Bronzan, R. L. Sugar, and A. R. White, Phys. Rep. 21C, 119 (1975).

[6]J. L. Cardy, Nucl. Phys. B109, 255 (1976), has addressed this problem.

[7]H. D. I. Abarbanel, J. B. Bronzan, A. Schwimmer, and R. L. Sugar, Phys. Rev. D 14, 632 (1976).

[8]D. Amati, L. Caneschi, and R. Jengo, Nucl. Phys. B101, 397 (1975); V. Alessandrini, D. Amati, and R. Jengo, *ibid.* B108, 425 (1976); M. Ciafaloni and G. Marchesini, *ibid.* B109, 261 (1976); R. Jengo, Trieste report, 1976 (unpublished).

[9]The parameter Δ_0 controls the approach to the phase transition, and plays the role of a "temperature." As Δ_0 decreases, $\alpha(0)$ increases to $\alpha(0)=1$ at the critical value Δ_0^*. Hence the supercritical domain $\Delta_0<\Delta_0^*$ is *referred to* as $\alpha(0)>1$, even though in any sensible extrapolation the output singularity *must* satisfy the Froissart bound.

[10]D. Amati, M. LeBellac, G. Marchesini and M. Ciafaloni, Nucl. Phys. B112, 107 (1976).

[11]J. L. Cardy and R. L. Sugar, Phys. Rev. D 12, 2514 (1975).

[12]R. C. Brower, J. Ellis, R. Savit, and W. Zakrzewski, Nucl. Phys. B94, 460 (1975); M. B. Green, Cambridge Report No. HEP 76/6 (unpublished).

[13]R. C. Brower, J. Ellis, R. Savit, and J. Zinn-Justin, Nucl. Phys. B112, 52 (1976).

[14]Two theories are in the same universality class if they have the same critical exponents, or the same power behavior at large distances at the phase transition. See for example K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974), and references therein, particularly J. Hubbard, Phys. Lett. 39A, 365 (1972).

[15]For a review of various expansions, see for example, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. 3.

[16]For a review of renormalization-group methods for classical Ising-type models, see for example, Th. Niemeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena* (Ref. 15), Vol. 11. The renormalization-group approach to the ground-state problem has been studied by Z. Friedman, Phys. Rev. Lett. 36, 1326 (1976), and K. Subbarao, Phys. Rev. Lett. 37, 1712 (1976).

[17]V. N. Gribov, Nucl. Phys. B106, 189 (1976).

[18]J. B. Bronzan, J. A. Shapiro, and R. L. Sugar, Phys. Rev. D 14, 618 (1976).

[19]J. Lajzerowicz and P. Pfeuty, J. Phys. 32, 193 (1971); B. Stoeckly and D. J. Scalapino, Phys. Rev. B 11, 205 (1975).

[20]Our formalism is that of "temperature Green's functions" for a quantized diffusion problem. See for example A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), Chap. 7.

[21]This prescription is not unique; however, we are interested in the behavior of the theory in the limit $|x| \gg l$ in which all prescriptions for the gradient are equivalent. See for example Ref. 30 for another prescription, which is *equivalent* to a momentum cutoff $\Lambda=\pi/l$.

[22]D. J. Scalapino, Y. Imry, and P. Pincus, Phys. Rev. B 11, 2042 (1975).

[23]D. J. Scalapino, M. Sears, and R. A. Ferrell, Phys. Rev. B 6, 3409 (1972).

[24]M. Suzuki, Phys. Lett. 34A, 94 (1971).

[25]P. Pfeuty, Ann. Phys. (N.Y.) 57, 79 (1970). Stoeckly and Scalapino (Ref. 19) also do a careful study of the development of the phase transition as the number of lattice sites goes to infinity.

[26]John L. Cardy, Nucl. Phys. B115, 141 (1976).

[27]T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964).

[28]In our representation, defined by Eqs. (3.23)–(3.24), the operator $\sigma_\chi^{\vec{n}} = |\Delta_0/g_0|\sigma_z^{\vec{n}}$ is diagonal. Only in this representation our discussion on the relationship to the Cardy-Sugar model is meaningful, because in the derivation of their model Cardy and Sugar pick up the leading contribution to the functional integral over the χ field at the two saddle points $\chi_{\vec{n}} = |\Delta_0/g_0| s_{\vec{n}}$, with $s_{\vec{n}} = \pm 1$. These are the values of $\sigma_z^{\vec{n}}$ on the basis (4.4). Indeed, in a representation in which $\sigma_\chi^{\vec{n}}$ is not diagonal, one can check that the matrix elements $\langle s_1' s_2' | \sum_i | s_1 s_2 \rangle$ are not expressible in terms of the S_i's, Eq. (4.11).

[29]H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970).

[30]S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 487 (1976).

[31]Ref. 6 and private communication.

[32]It *may* be possible to study this question numerically at least for the one-dimensional path integral; see M. Nauenberg, F. Kuttner, and M. Furman, Phys. Rev. A 13, 1185 (1976).

[33]See for example R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); and E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).