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Toward quantization of a "three-string"*

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The linearized classical equations of motion for a system of three relativistic strings, which are coupled at a junction, are solved. The system is quantized covariantly and general features of the spectrum of states are explored.

I. INTRODUCTION

After the successful interpretation of dual models for mesons in terms of open relativistic strings,[1] there were attempts to take over the quark picture and to put masses at the ends of the strings.[2] Although, owing to nonlinearities, the equations of motion are not solvable, this quark interpretation of dual strings is sometimes carried over to other topologies, such as, for instance, the "three-string" (three strings coupled at a junction), which could represent baryons. Normally fermions are incorporated in the open-string models by introducing extra degrees of freedom "by hand." They also can be described by starting from a classical Lagrangian,[3] but this has a lack of geometrical interpretation, which is characteristic of the simple model.

Therefore it is not unreasonable to study the three-string (or "Y string"). Goldstone[4] may have been the first to study this system. Artru[5] discussed it together with other topologies and looked for general properties of the classical solution. The most general action for the Y string has been studied by Collins, Hopkinson, and Tucker,[6] but again nonlinearities of the classical equation do not allow one to write down the general solution, which would be the starting point of quantization. To see whether quantization is possible at all, we only look for those classical solutions which come from linearized equations of motion.

In Sec. II we write down the action and the equations of motion, boundary conditions, and constraints which follow from it. The general solution is given in terms of a real function, which is nothing other than the solution for the open string (m-string) and a complex function, whose normal modes take half-integer values only. The Hamiltonian formalism is set up, following the methods of Dirac for systems with primary constraints.[7] The classical system is quantized in Sec. III. A basic in the Fock space of states is realized by acting with three types of creation operators on a ground state. The constraints restrict the allowed (physical) states to only a subspace of this space in the form of matrix conditions. The problems arising from the interpretation of these conditions as conditions on the states directly are discussed in Sec. IV. Although it leads to a nonclosed algebra of certain operators, physical states can consistently be defined. The gauge conditions are very restrictive, nevertheless physical states are shown to exist. Because of the nonclosed algebra, we did not succeed in proving a "no-ghost" theorem to get the intercept of the model. Therefore in the second part of Sec. IV only general features of the spectrum are discussed. The last section contains a conclusion and final remarks.

Two appendixes are added. In Appendix A relations between the commutators of gauge operators are listed. Although in general our three-string model cannot be described in terms of transverse variables, we found the result of the quantization of a special class of classical solutions in the transverse gauge[8] interesting enough to present it in Appendix B.

II. CLASSICAL THEORY OF A Y STRING

We describe the Y string by functions $x^{\mu}_{(i)}(\sigma,\tau)$ ($i=1,2,3$), where $0\leq\sigma\leq\pi$. At the junction they have to fulfill

$$x^{\mu}_{(1)}(\pi,\tau)=x^{\mu}_{(2)}(\pi,\tau)=x^{\mu}_{(3)}(\pi,\tau)\,. \tag{2.1}$$

As for the m-string the action is taken to be proportional to the surface spanned in space-time by the evolution of the strings

$$S=-\gamma\int_{\tau_i}^{\tau_f}d\tau\int_0^{\pi}d\sigma\sum_i L_i\,, \tag{2.2}$$

where

$$L_i = [(\dot{x}_{(i)} \cdot x'_{(i)})^2 - \dot{x}_{(i)}{}^2 x'_{(i)}{}^2]^{1/2}, \quad \gamma = \frac{1}{2\pi\alpha'},$$
$$\dot{x}^\mu_{(i)} \equiv \frac{\partial}{\partial\tau} x^\mu_{(i)}, \quad x'^\mu_{(i)} \equiv \frac{\partial}{\partial\sigma} x^\mu_{(i)}. \tag{2.3}$$

This action is equivalent to the one written down by the authors of Ref. 6, save for coordinate transformation. In order to find the equations of motion and boundary conditions, $x^\mu_{(i)}$ is varied by a small amount $x^\mu_{(i)} \to x^\mu_{(i)} + \delta x^\mu_{(i)}$, such that

$$\delta x^\mu_{(i)}(\sigma, \tau_i) = 0 = \delta x^\mu_{(i)}(\sigma, \tau_f),$$
$$\delta x^\mu_{(i)}(\pi, \tau) = \epsilon^\mu \text{ arbitrary}, \tag{2.4}$$
$$\delta x^\mu_{(i)}(0, \tau) \text{ arbitrary}.$$

The principle of least action yields the Euler equations

$$\frac{\partial}{\partial\tau} P^\mu_{(i)} + \frac{\partial}{\partial\sigma} \Pi^\mu_{(i)} = 0, \tag{2.5}$$

where we defined

$$P^\mu_{(i)} \equiv -\frac{\partial L_i}{\partial \dot{x}^\mu_{(i)}}, \quad \Pi^\mu_{(i)} \equiv -\frac{\partial L_i}{\partial x'^\mu_{(i)}}, \tag{2.6}$$

and the edge conditions

$$\Pi^\mu_{(i)} = 0 \quad \text{for } \sigma = 0, \tag{2.7a}$$

$$\sum_i \Pi^\mu_{(i)} = 0 \quad \text{for } \sigma = \pi. \tag{2.7b}$$

The meaning of these conditions can be understood by defining the energy-momentum current to be

$$P^\mu = \sum_i \int_{(c)} (d\sigma P^\mu_{(i)} + d\tau\, \Pi^\mu_{(i)}), \tag{2.8}$$

where (c) is a curve on the surface. Then Eq. (2.5) expresses conservation of energy-momentum. Owing to Eq. (2.7a) no energy-momentum flows out of the ends of the wings and Eq. (2.7b) shows that the sum of energy-momentum at the junction vanishes. Notice that energy-momentum is not conserved for each wing separately. Calculating $P^\mu_{(i)}$ and $\Pi^\mu_{(i)}$ one finds the following identities:

$$P^\mu_{(i)} x'_{(i)\mu} = 0, \quad P_{(i)}{}^2 + \gamma^2 x'_{(i)}{}^2 = 0,$$
$$\Pi^\mu_{(i)} \dot{x}_{(i)\mu} = 0, \quad \Pi_{(i)}{}^2 + \gamma^2 \dot{x}_{(i)}{}^2 = 0. \tag{2.9}$$

To solve the equations of motion (2.5) we follow the usual line of reasoning for the m-strings and choose orthonormal coordinates defined by

$$\dot{x}^\mu_{(i)} x'_{(i)\mu} = 0, \quad \dot{x}_{(i)}{}^2 = -x'_{(i)}{}^2. \tag{2.10}$$

This choice is not possible in general for the Y string, but includes a large class of solutions whose quantization can be studied. In these coordinates (2.5) and (2.7) reduce to

$$\ddot{x}^\mu_{(i)} - x''^\mu_{(i)} = 0, \tag{2.11}$$

$$x'^\mu_{(i)}(0, \tau) = 0, \quad \sum_i x'^\mu_{(i)}(\pi, \tau) = 0. \tag{2.12}$$

For the following, we prefer to work with an alternative set of functions defined by

$$x^\mu_{(i)} = \psi^\mu + \lambda_i \phi^\mu + \lambda_i^* \phi^{*\mu}, \tag{2.13}$$

with $\lambda_1 = 1$, $\lambda_2 = e^{i\Delta}$, $\lambda_3 = e^{-i\Delta} = \lambda_2^*$, and $\Delta = \frac{2}{3}\pi$ is the "natural" angle of the three-string. These functions have the advantage that their boundary conditions

$$\psi'_\mu(0, \tau) = 0 = \psi'_\mu(\pi, \tau),$$
$$\phi'_\mu(0, \tau) = 0 \tag{2.14}$$

are more easily incorporated into a normal-mode expansion and that the junction condition (2.1) simply takes the form

$$\phi_\mu(\pi, \tau) = 0. \tag{2.15}$$

The (real) functions $\psi^\mu(\sigma, \tau)$ and the (complex) functions $\phi^\mu(\sigma, \tau)$ also obey the d'Alembert equation (2.11), and the most general solution with the boundary conditions (2.14) and (2.15) is

$$\psi^\mu(\sigma, \tau) = q^\mu + \rho^2 p^\mu \tau + \rho \sum_{n=1}^\infty \frac{a_n^{*\mu} e^{in\tau} + a_n^\mu e^{-in\tau}}{\sqrt{n}} \cos n\sigma \quad (n \text{ integer}), \tag{2.16}$$

$$\phi^\mu(\sigma, \tau) = \rho \sum_{r=1/2}^\infty \frac{b_r^{*\mu} e^{ir\tau} + c_r^\mu e^{-ir\tau}}{\sqrt{r}} \cos r\sigma, \quad (r \text{ half-integer}), \tag{2.17}$$

with $\rho = (2\alpha'/3)^{1/2}$.

Because we want to quantize this classical system, we have to find its Hamiltonian. We assume canonical Poisson brackets and define them at the junction and at the ends of the strings in a way compatible with the conditions (2.1) and (2.12),

$$\{x^\mu_{(i)}(\sigma), x^\nu_{(k)}(\sigma')\} = 0 = \{P^\mu_{(i)}(\sigma), P^\nu_{(k)}(\sigma')\}, \tag{2.18}$$

$$\{x^\mu_{(i)}(\sigma), P^\nu_{(j)}(\sigma')\} = -g^{\mu\nu}[\Delta_1(\sigma, \sigma') + (3\delta_{ij} - 1)\Delta_2(\sigma, \sigma')], \tag{2.19}$$

where

$$\Delta_1(\sigma, \sigma') = \tfrac{1}{3} \sum_{n=-\infty}^\infty [\delta(\sigma - \sigma' + 2n\pi) + \delta(\sigma + \sigma' + 2n\pi)],$$
$$\Delta_2(\sigma, \sigma') = \tfrac{1}{3} \sum_{n=-\infty}^\infty (-1)^n [\delta(\sigma - \sigma' + 2n\pi) + \delta(\sigma + \sigma' + 2n\pi)]. \tag{2.20}$$

Alternatively for ψ_μ, ϕ_μ,

$$P_\psi^\mu \equiv -\sum_i \frac{\partial L_i}{\partial \dot\psi^\mu} = \tfrac{1}{3}\sum_i P_{(i)}^\mu ,$$
$$P_\phi^\mu \equiv -\sum_i \frac{\partial L_i}{\partial \dot\phi^\mu} = \tfrac{1}{3}\sum_i \lambda_i P_{(i)}^\mu , \tag{2.21}$$

one finds

$$\{\psi^\mu(\sigma), P_\psi^\nu(\sigma')\} = -g^{\mu\nu}\Delta_1(\sigma,\sigma') ,$$
$$\{\phi^\mu(\sigma), P_\phi^\nu(\sigma')\} = -g^{\mu\nu}\Delta_2(\sigma,\sigma') , \tag{2.22}$$

all others vanishing.

Similar to the theory of m-strings, the definition of σ can be extended to the interval $[-\pi, \pi]$ by

$$P_{(i)}^\mu(-\sigma,\tau) = P_{(i)}^\mu(\sigma,\tau) ,$$
$$x_{(i)}^\mu(-\sigma,\tau) = -x_{(i)}^\mu(\sigma,\tau) . \tag{2.23}$$

To translate (2.9) into constraints for ψ^μ and ϕ^μ, we introduce

$$\Gamma^\mu = P_\psi^\mu + \gamma\psi'^\mu ,$$
$$\chi^\mu = P_\phi^\mu + \gamma\phi^{*\prime\mu} . \tag{2.24}$$

Then the constraints (2.9) are equivalent to

$$\Gamma_\mu\Gamma^\mu + 2\chi_\mu\chi^{*\mu} = 0 ,$$
$$\chi_\mu\chi^\mu + 2\Gamma_\mu\chi^{*\mu} = 0 , \tag{2.25}$$
$$\chi_\mu^*\chi^{*\mu} + 2\Gamma_\mu\chi^\mu = 0 .$$

The constraint functionals

$$A_f = -\frac{3}{4\gamma}\int_{-\pi}^{\pi} d\sigma f(\sigma)(\Gamma^2 + 2\chi\cdot\chi^*) ,$$
$$B_f = -\frac{3}{4\gamma}\int_{-\pi}^{\pi} d\sigma f(\sigma)(\chi^2 + 2\Gamma\cdot\chi^*), \tag{2.26}$$
$$C_f = B_{f^*}^*$$

commute trivially with the canonical Hamiltonian

$$H_0 = \sum_i \int_0^\pi d\sigma (P_{(i)}\cdot \dot x_{(i)} - L_i) \tag{2.27}$$

since it vanishes identically. Furthermore, they form a closed Poisson bracket algebra:

$$\{A_f, A_g\} = A_h ,$$
$$\{A_f, B_g\} = B_h ,$$
$$\{A_f, C_g\} = C_h ,$$
$$\{B_f, B_g\} = C_h , \tag{2.28}$$
$$\{B_f, C_g\} = A_h ,$$
$$\{C_f, C_g\} = B_h ,$$

with $h = fg' - f'g$.

In deriving this algebra, the definition of the canonical Poisson brackets at $\sigma = 0, \pi$ by Eqs. (2.19) and (2.20) guarantees the vanishing of possible boundary terms due to partial integration. In the following we choose the functions $f_n(\sigma) = e^{in\sigma}$ (n integer) and relabel $A_{f_n} = A_n$, etc. Because of the existence of the (primary) constraints (2.25) [or (2.26), respectively] according to the work of Dirac on singular Lagrange functions,[7] the Hamiltonian which generates the equation of motion is given by

$$H = H_0 + \sum_n (\mu_n^{(1)}A_n + \mu_n^{(2)}B_n + \mu_n^{(3)}C_n) . \tag{2.29}$$

The coefficients $\mu_n^{(i)}$ are arbitrary owing to the gauge freedom coming from the constraints. From

$$\{\psi^\mu, A_0\} = \frac{1}{\gamma} P_\psi^\mu ,$$
$$\{\phi^\mu, A_0\} = \frac{1}{\gamma} P_\phi^{*\mu} \tag{2.30}$$

one can see that A_0 generates a dynamical evolution of the system. This allows fixing the gauge by

$$H = A_0 \tag{2.31}$$

with the resulting equations of motion

$$\ddot\psi_\mu - \psi_\mu'' = 0 , \tag{2.32}$$
$$\ddot\phi_\mu - \phi_\mu'' = 0 . \tag{2.33}$$

These are again the d'Alembert equations, which have been derived in the Lagrangian formalism.

For the coefficients of the solution (2.16) and (2.17) the nonvanishing Poisson brackets can be calculated to be

$$\{a_n^\mu, a_m^{*\nu}\} = ig^{\mu\nu}\delta_{n,m} ,$$
$$\{b_r^\mu, b_s^{*\nu}\} = ig^{\mu\nu}\delta_{r,s} ,$$
$$\{c_r^\mu, c_s^{*\nu}\} = ig^{\mu\nu}\delta_{r,s} , \tag{2.34}$$
$$\{q^\mu, p^\nu\} = -g^{\mu\nu} .$$

III. COVARIANT QUANTIZATION

The classical system is quantized by replacing the c numbers $q^\mu, p^\mu, a_n^\mu, b_r^\mu, c_r^\mu$ by operators and Poisson brackets by commutators, such that

$$[a_n^\mu, a_m^{\dagger\nu}] = -g^{\mu\nu}\delta_{n,m}, \quad [q^\mu, p^\nu] = -ig^{\mu\nu} ,$$
$$[b_r^\mu, b_s^{\dagger\nu}] = -g^{\mu\nu}\delta_{r,s}, \quad [c_r^\mu, c_s^{\dagger\nu}] = -g^{\mu\nu}\delta_{r,s} , \tag{3.1}$$

all others vanishing.

The quantum Hamiltonian has to be normal-ordered:

$$H = :H:$$
$$= -\tfrac{1}{3}\alpha' p^2 - \sum_{n=1}^\infty n a_n^\dagger\cdot a_n - \sum_{r=1/2}^\infty r(b_r^\dagger\cdot b_r + c_r^\dagger\cdot c_r) . \tag{3.2}$$

Together with the commutators (3.1) it allows one to interpret the $a_n^{\mu\dagger}, b_r^{\mu\dagger}, c_r^{\mu\dagger}$ as creation operators, and their Hermitian conjugate as annihilation operators, acting on a ground state which satisfies

$$a_n^\mu|0\rangle = b_r^\mu|0\rangle = c_r^\mu|0\rangle = p^\mu|0\rangle = 0 . \tag{3.3}$$

The Fock space is spanned by the vectors

$$|\lambda^{(1)}_{n,\mu_n},\lambda^{(2)}_{r,\mu_r},\lambda^{(3)}_{r,\mu_r}\rangle=\prod_{n=1}^{\infty}\prod_{r=1/2}^{\infty}\prod_{\mu_n=0}^{D-1}\prod_{\mu_r=0}^{D-1}(a_n^{\dagger\mu_n})^{\lambda^{(1)}_{n,\mu_n}}(b_r^{\dagger\mu_r})^{\lambda^{(2)}_{r,\mu_r}}(c_r^{\dagger\mu_r})^{\lambda^{(3)}_{r,\mu_r}}e^{ikq}|0\rangle\,, \tag{3.4}$$

where k^μ is an eigenvalue of the momentum operator, i.e.,

$$p^\mu|k^\mu\rangle=k^\mu|k^\mu\rangle\,, \tag{3.5}$$

and D is the space-time dimension of the system. The constraint functionals A_n, B_n, C_n [compare their definition in Eq. (2.26)] are now operators.

For later purposes we split them into different contributions, namely,

$$\begin{aligned}&A_n=S_n+2T_n\quad (A_{-n}=A_n^\dagger)\,,\\&B_n=G_n+2H_n\quad (B_{-n}=C_n^\dagger)\,,\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}S_n&\equiv-\frac{3}{4\gamma}\int_{-\pi}^{\pi}d\sigma\,e^{in\sigma}:\Gamma_\mu(\sigma)\Gamma^\mu(\sigma):\,,\\T_n&\equiv-\frac{3}{4\gamma}\int_{-\pi}^{\pi}d\sigma\,e^{in\sigma}:\chi_\mu(\sigma)\chi^{\dagger\mu}(\sigma):\,,\\G_n&\equiv-\frac{3}{4\gamma}\int_{-\pi}^{\pi}d\sigma\,e^{in\sigma}:\chi_\mu(\sigma)\chi^{\mu}(\sigma):\,,\\H_n&\equiv-\frac{3}{4\gamma}\int_{-\pi}^{\pi}d\sigma\,e^{in\sigma}:\Gamma_\mu(\sigma)\chi^{\dagger\mu}(\sigma):\,.\end{aligned} \tag{3.7}$$

Owing to normal-ordering, some of the commutators between the gauge operators A_n, B_n, C_n pick up c-number terms. To calculate them it is convenient to rewrite the expressions (3.7) as integrals in the complex plane in terms of the generalized Fubini-Veneziano fields

$$\begin{aligned}F^\mu(z)&=\rho^2p^\mu+i\rho\sum_{n=1}^{\infty}n^{1/2}(a_n^{\dagger\mu}z^n-a_n^\mu z^{-n})\,,\\V^\mu(z)&=i\rho\sum_{r=1/2}^{\infty}(rz)^{1/2}(c_r^{\dagger\mu}z^r-b_r^\mu z^{-r})\,,\\\overline{V}^\mu(z)&=i\rho\sum_{r}(rz)^{1/2}(b_r^{\dagger\mu}z^r-c_r^\mu z^{-r})\,.\end{aligned} \tag{3.8}$$

They correspond to the field which in dual theories is usually called $Q^\mu(z)$. In terms of the functions Γ^μ, χ^μ they are

$$\begin{aligned}&\Gamma_\mu(\sigma,\tau)=\gamma F_\mu(z),\quad z=\exp[i(\sigma+\tau)]\\&\chi_\mu(\sigma,\tau)=\gamma z^{-1/2}V_\mu(z)\,,\\&\chi_\mu^\dagger(\sigma,\tau)=\gamma z^{-1/2}\overline{V}_\mu(z)\,.\end{aligned} \tag{3.9}$$

We then find

$$\begin{aligned}S_n&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^n:F^\mu(z)F_\mu(z):\,,\\T_n&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^n:V^\mu(z)\overline{V}_\mu(z):\,,\\G_n&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^{n-1}:V^\mu(z)V_\mu(z):\,,\\G_{-n}^\dagger&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^{n+1}:\overline{V}^\mu(z)\overline{V}_\mu(z):\,,\\H_n&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^{1/2}z^n:F^\mu(z)\overline{V}_\mu(z):\,.\end{aligned} \tag{3.10}$$

Since the integrand of H_n has a branch cut in z we make the following reformulation. With the help of the scalar product

$$\langle f,g\rangle\equiv\int_{-\pi}^{\pi}d\sigma\,f^*(\sigma)g(\sigma)\,,$$

we have

$$\begin{aligned}H_n&=-\frac{1}{4\gamma}\langle e^{-in\sigma},:\Gamma_\mu(\sigma)\chi^{\dagger\mu}(\sigma):\rangle\\&=-\frac{1}{4\gamma}\frac{1}{2\pi}\sum_r\langle e^{-in\sigma},e^{-ir\sigma}\rangle\langle e^{-ir\sigma},:\Gamma_\mu(\sigma)\chi^{\dagger\mu}(\sigma):\rangle\,,\end{aligned} \tag{3.11}$$

where we have used the fact that the functions $f_k=(2\pi)^{-1/2}e^{ik\sigma}$ are complete and orthogonal in $-\pi<\sigma<\pi$ for either k integral or half-integral. With

$$\begin{aligned}W_r&\equiv-\frac{1}{4\gamma}\langle e^{-ir\sigma},:\Gamma_\mu(\sigma)\chi^{\dagger\mu}(\sigma):\rangle\\&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^{r+1/2}:F_\mu(z)\overline{V}^\mu(z):\,,\\W_{-r}^\dagger&=-\frac{1}{2\rho^2}\oint\frac{dz}{2i\pi z}\,z^{r-1/2}:F_\mu(z)V^\mu(z):\,,\end{aligned} \tag{3.12}$$

we finally get

$$H_n=\sum_{-\infty}^{\infty}a_{nr}W_r\,, \tag{3.13}$$

$$a_{nr}=\frac{1}{\pi}(-1)^{n+r+1/2}\frac{1}{n-r}\,. \tag{3.14}$$

Noticing that for $|x|>|y|$

$$\begin{aligned}F_\mu(x)F_\nu(y)&=:F_\mu(x)F_\nu(y):-\rho^2\frac{xy}{(x-y)^2}g_{\mu\nu}\,,\\V_\mu(x)\overline{V}_\nu(y)&=:V_\mu(x)\overline{V}_\nu(y):-\tfrac{1}{2}\rho^2\frac{x(x+y)}{(x-y)^2}g_{\mu\nu}\,,\end{aligned} \tag{3.15}$$

whereas for all remaining bilinears in $F, V, \overline{V}$ normal-ordering is irrelevant, the method of Ref. 9 can be used to get the algebra of the gauge operators, including the c numbers. The final result is

$$\begin{aligned}
&[A_n, A_m] = (n-m)A_{n+m} + \tfrac{1}{4}Dn^3\delta_{n,-m}\,,\\
&[A_n, B_m] = (n-m)B_{n+m}\,,\\
&[A_n, C_m] = (n-m)C_{n+m}\,,\\
&[B_n, B_m] = (n-m)C_{n+m}\,,\\
&[B_n, C_m] = (n-m)A_{n+m} + \tfrac{1}{4}Dn^3\delta_{n,-m}\\
&[C_n, C_m] = (n-m)B_{n+m}\,.
\end{aligned} \tag{3.16}$$

The meaning of the gauge operators is clear: The states defined by (3.4) have to fulfill the conditions

$$\langle\psi_1|[A_n - \alpha(0)\delta_{n,0}]|\psi_2\rangle = 0\,, \tag{3.17a}$$

$$\langle\psi_1|B_n|\psi_2\rangle = 0\,, \tag{3.17b}$$

$$\langle\psi_1|C_n|\psi_2\rangle = 0\,, \tag{3.17c}$$

where $\alpha(0)$ is an arbitrary c number, arising from the normal-ordering of the Hamiltonian. Usually these conditions are weakened to have conditions on the states. Before discussing this point in the next section, we make some remarks on the algebra (3.15).

A central point in dual models is the conformal group O(2, 1). Defining

$$\overline{A}_0 = A_0 + \tfrac{1}{8}D\,, \tag{3.18}$$

then

$$[A_n, A_{-n}] = 2n\overline{A}_0 + \tfrac{1}{4}Dn(n^2-1) \tag{3.19}$$

and $\overline{A}_0, A_1, A_{-1}$ are a realization of the group O(2, 1). The linear combinations

$$L_n^{(i)} = \tfrac{1}{3}(A_n + \lambda_i B_n + \lambda_i^* C_n) \tag{3.20}$$

fulfill the algebra

$$[L_n^{(i)}, L_m^{(j)}] = \delta_{ij}[(n-m)L_{n+m}^{(i)} + \tfrac{1}{4}Dn^3\delta_{n,-m}]\,. \tag{3.21}$$

With a shift similar to (3.18), we therefore get three commuting conformal groups.

IV. THE GAUGE CONDITIONS AND THE SPECTRUM

To get a feeling how to handle the matrix conditions (3.17) we write down the explicit expressions for the gauge operators in terms of creation and annihilation operators.

For A_0 this is simply the Hamiltonian (3.2). Furthermore, for $n>0$

$$S_n = i\rho\sqrt{n}\,p\cdot a_n - \sum_{m=1}^{\infty}[m(n+m)]^{1/2}a_m^\dagger\cdot a_{n+m} + \tfrac{1}{2}\sum_{m=1}^{n-1}[m(n-m)]^{1/2}a_m\cdot a_{n-m}\,, \tag{4.1a}$$

for $n\geq 0$

$$T_n = -\tfrac{1}{2}\sum_{r=1/2}^{\infty}[r(n+r)]^{1/2}(b_r^\dagger\cdot b_{r+n} + c_n^\dagger\cdot c_{r+n}) + \tfrac{1}{2}\sum_{r=1/2}^{n-1/2}[r(n-r)]^{1/2}b_r\cdot c_{n-r}\,, \tag{4.1b}$$

$$G_n = -\sum_{r=1/2}^{\infty}[r(n+r)]^{1/2}c_r^\dagger\cdot b_{r+n} + \tfrac{1}{2}\sum_{r=1/2}^{n-1/2}[r(n-r)]^{1/2}b_r\cdot b_{n-r}\,, \tag{4.1c}$$

$$G_{-n}^\dagger = -\sum_{r=1/2}^{\infty}[r(n+r)]^{1/2}b_r^\dagger\cdot c_{r+n} + \tfrac{1}{2}\sum_{r=1/2}^{n-1/2}[r(n-r)]^{1/2}c_r\cdot c_{n-r}\,, \tag{4.1d}$$

for $r>0$

$$W_r = \tfrac{1}{2}i\rho\sqrt{r}\,p\cdot c_r - \tfrac{1}{2}\sum_{s=1/2}^{\infty}[s(s+r)]^{1/2}b_s^\dagger\cdot a_{r+s} - \tfrac{1}{2}\sum_{n=1}^{\infty}[n(n+r)]^{1/2}a_n^\dagger\cdot c_{r+n} + \tfrac{1}{2}\sum_{n=1}^{r-1/2}[n(r-n)]^{1/2}a_n\cdot c_{r-n}\,, \tag{4.1e}$$

$$W_{-r}^\dagger = \tfrac{1}{2}i\rho\sqrt{r}\,p\cdot b_r - \tfrac{1}{2}\sum_{s=1/2}^{\infty}[s(s+r)]^{1/2}c_s^\dagger\cdot a_{r+s} - \tfrac{1}{2}\sum_{n=1}^{\infty}[n(n+r)]^{1/2}a_n^\dagger\cdot b_{r+n} + \tfrac{1}{2}\sum_{n=1}^{r-1/2}[n(r-n)]^{1/2}a_n\cdot b_{r-n}\,. \tag{4.1f}$$

Since S_n is nothing other than the well-known Virasoro operator L_n, and T_n has a similar structure, the constraints (3.17a) may be weakened by

$$\begin{aligned}
&A_0|\psi\rangle = \alpha(0)|\psi\rangle\,,\\
&A_n|\psi\rangle = 0, \text{ for } n\geq 1\,.
\end{aligned} \tag{4.2}$$

It is not possible to write similar conditions for B_n and C_n, since each of them contains an infinite sum of the operators W_r or $W_r^\dagger$, respectively. Applying these operators to the vacuum, one finds

$$W_r|0\rangle = W_{-r}^\dagger|0\rangle = 0, \text{ for } r\geq\tfrac{1}{2} \tag{4.3}$$

whereas for $r<\frac{1}{2}$ these expressions do not vanish. Let us generalize it to

$$W_r|\psi\rangle = W_{-r}^\dagger|\psi\rangle = 0, \text{ for } r\geq\tfrac{1}{2}\,. \tag{4.4}$$

Similarly one can try to impose as further conditions on the states.

$$G_n|\psi\rangle = G_{-n}^\dagger|\psi\rangle = 0, \text{ for } n\geq 0\,. \tag{4.5}$$

These last two equations imply (3.17b) and (3.17c). Although the set of operators $L = \{S_n, T_n, G_n, G_{-n}^\dagger, W_r,$

$W^\dagger_{-r}\}$ does not form a closed commutator algebra, as discussed in Appendix A, "physical" states can consistently be defined by (4.2)–(4.5). To see this, we switch to the equivalent set of operators $\hat{L}=\{A_n, L_n, K_n, V_n, X_r, Y_r\}$ with

$$A_n=S_n+2T_n,\quad V_n=\tfrac{1}{2}(G_n-G^\dagger_{-n}),$$
$$L_n=T_n+\tfrac{1}{2}(G_n+G^\dagger_{-n}),\quad K_n=T_n-\tfrac{1}{2}(G_n+G^\dagger_{-n}),\quad (4.6)$$
$$X_r=\sqrt{2}\,(W_r+W^\dagger_{-r}),\quad Y_r=\sqrt{2}\,(W_r-W^\dagger_{-r}).$$

The commutators of these operators (as far as they are again in $\hat{L}$) together with identities among them are also listed in Appendix A. These relations allow us to conclude the following: If $Z_i\in\hat{L}$, with $Z_{i,j}|\psi\rangle=0$ and $i,j>0$ (integer and half-integer)

$$[Z_i,Z_j]=\sum_{k>0} v_k Z_k+\sum_{\substack{k,\,l\neq i,\,j\\ k,\,l>0}} u_{kl}[Z_k,Z_l]\,,\qquad (4.7)$$

such that

$$[Z_i,Z_j]|\psi\rangle=0\,.\qquad (4.8)$$

Therefore the interpretation of the original matrix conditions by Eqs. (4.2)–(4.5) does not lead to contradictions. One would like to show that they imply positive-definite norm for physical states. For the m-string models this is achieved by the "no-ghost" theorem,[10] fixing the "critical" dimension D and the number $\alpha(0)$. But here we have to pay the price for not having a closed commutator algebra between the gauge operators, and we were not able to carry over the methods of the usual proof of the "no-ghost" theorem. Interestingly enough, we find for a special Y-string system that it is ghost-free for $D=26$, $\alpha(0)=0$ by quantizing it in a transverse gauge (see Appendix B).

In the remainder of this section we will study the question of whether the gauge conditions allow for the existence of physical states at all.

First we remark that physical states $|\psi\rangle$ are sufficiently defined by

$$A_0|\psi\rangle=\alpha(0)|\psi\rangle\,,\qquad (4.9)$$
$$A_1|\psi\rangle=0=A_2|\psi\rangle\,,\qquad (4.10)$$
$$(L_0-K_0)|\psi\rangle=0\,,\qquad (4.11)$$
$$V_0|\psi\rangle=0\,,\qquad (4.12)$$
$$X_{1/2}|\psi\rangle=0=Y_{1/2}|\psi\rangle\,.\qquad (4.13)$$

The commutators and identities of Appendix A can be used to express all other gauge operators in terms of these.

It is more convenient to build up the Fock space with the operators

$$\beta^{\dagger\mu}_r=\frac{1}{\sqrt{2}}(b^{\dagger\mu}_r+c^{\dagger\mu}_r),\quad \gamma^{\dagger\mu}_r=\frac{1}{\sqrt{2}}(b^{\dagger\mu}_r-c^{\dagger\mu}_r)\qquad (4.14)$$

instead of $b^\dagger_r, c^\dagger_r$.

Defining the number operators

$$R_a=-\sum_{n=1}^{\infty} n a^\dagger_n\cdot a_n\,,$$
$$R_\beta=-\sum_{r=1/2}^{\infty} r\,\beta^\dagger_r\cdot\beta_r\,,\qquad (4.15)$$
$$R_\gamma=-\sum_{r=1/2}^{\infty} r\gamma^\dagger_r\cdot\gamma_r,\quad R=R_a+R_\beta+R_\gamma$$

we can write the mass-shell condition (4.9) as

$$R|\psi\rangle=[\alpha(0)+\tfrac{1}{3}\alpha' p^2]|\psi\rangle\,,\qquad (4.16)$$

and (4.11) is equivalent to

$$(R_\beta-R_\gamma)|\psi\rangle=0\,.\qquad (4.17)$$

This last equation has very drastic consequences. To see them, let us split the total Fock space F into disjoint (and orthogonal) spaces F_i:

$$F=F_1\oplus F_2\oplus F_3\oplus F_4\,,\qquad (4.18)$$

where the subspaces are defined through

$$F_1=\{|a\rangle; R_\beta|\psi\rangle=0, R_\gamma|\psi\rangle=0\}\,,$$
$$F_2=\{|\beta\rangle; R_\beta|\psi\rangle\neq 0, R_\gamma|\psi\rangle=0\}\,,\qquad (4.19)$$
$$F_3=\{|\gamma\rangle; R_\beta|\psi\rangle=0, R_\gamma|\psi\rangle\neq 0\}\,,$$
$$F_4=\{|\beta\gamma\rangle; R_\beta|\psi\rangle\neq 0, R_\gamma|\psi\rangle\neq 0\}\,.$$

Then from (4.17) we conclude that there are no physical states in F_2 and F_3, and there may be projections of physical states only in a subset $\hat{F}_4$ of F_4. In particular, on all mass levels characterized by half-integer eigenvalues of the total number operator R no physical state is present.

The condition (4.12) even strengthens the relation between (β)- and (γ)-type oscillators. Because of

$$[V_0,\beta^{\dagger\mu}_r]=\tfrac{1}{2}r\gamma^{\dagger\mu}_r\,,$$
$$[V_0,\gamma^{\dagger\mu}_r]=-\tfrac{1}{2}r\beta^{\dagger\mu}_r\,,\qquad (4.20)$$

only those states of $\hat{F}_4$ have a chance to survive, which are created by combinations of the form

$$f^{\dagger\mu\nu}_{2r}\equiv\beta^{\dagger\mu}_r\gamma^{\dagger\nu}_r-\beta^{\dagger\nu}_r\gamma^{\dagger\mu}_r\,.\qquad (4.21)$$

Another interesting consequence can be read off from (4.13). Since

$$[X_r,a^{\dagger\mu}_m]=\tfrac{1}{2}[m(m-r)]^{1/2}\times\begin{cases}\beta^{\dagger\mu}_{m-r} & \text{if } m-r>0\,,\\ -\beta_{r-m} & \text{if } m-r<0\,,\end{cases}\qquad (4.22)$$

for instance, for a state

$$|a_{nm}\rangle=(a_n^{\dagger\mu})(a_m^{\dagger\nu})|0,k\rangle\,,$$

we get

$$X_{1/2}|a_{nm}\rangle\sim\{[n(n-\tfrac{1}{2})]^{1/2}a_m^{\dagger\nu}\beta_{n-1/2}^{\dagger\mu}+[m(m-\tfrac{1}{2})]^{1/2}a_n^{\dagger\mu}\beta_{m-1/2}^{\dagger\nu}\}|0,k\rangle\,, \tag{4.23}$$

and no linear combination of those states vanishes. This argument can be taken over to all states $|a\rangle$, such that there is only one physical state, namely $|0\rangle$ in F_1. In other words, the "meson" sector of the model is empty, and as

$$X_r|a\rangle\in F_2 \text{ but } X_r|\beta_r\rangle\notin F_2 \tag{4.24}$$

a physical state cannot be a superposition of states in F_1 and F_4.

Let us now explore the states for the lowest eigenvalues M of the number operator R. For $M=0$ there is the ground state $|0,k\rangle$, which is a tachyon for $\alpha(0)=1$, or has mass zero for $\alpha(0)=0$. On the level $M=1$ the state

$$|\psi_1\rangle=\epsilon_{\mu\nu}f_1^{\mu\nu}|0,k\rangle\quad(\epsilon_{\mu\nu}=-\epsilon_{\nu\mu},k_\mu\epsilon^{\mu\nu}=0) \tag{4.25}$$

is physical and has positive norm, independent of the value of $\alpha(0)$. Because of the antisymmetry of the polarization tensor, it is a vector state. For $\alpha(0)=1$ $(k^2=0)$ there is another spin-1 state,

$$|\psi_2\rangle=\epsilon_\mu k_\nu f_1^{\mu\nu}|0,k\rangle\quad(k_\mu\epsilon^\mu=0)\,, \tag{4.26}$$

which turns out to have zero norm. No scalar state appears on this level.

For $M=2$, physical states are superpositions of

$$\begin{aligned}|\psi\rangle&=a_1^{\dagger\lambda}f_1^{\dagger\mu\nu}|0,k\rangle\,,\\|\phi\rangle&=f_1^{\dagger\lambda\rho}f_1^{\dagger\mu\nu}|0,k\rangle\,.\end{aligned} \tag{4.27}$$

Rather than work out the states in detail, for which one would need the intercept of the model, let us point out another general feature of the spectrum. Because of the antisymmetry of $f_1^{\mu\nu}$, the highest spin appearing on this level is two, despite the fact that, in $|\phi\rangle$ for instance, four Lorentz indices can be contracted. For the same reason the maximal spin for a state on the level $M=N$,

$$f_1^{\dagger\mu_1\nu_1}\cdots f_1^{\dagger\mu_n\nu_n}a_1^{\dagger\lambda_1}\cdots a_1^{\dagger\lambda_m}|0,k\rangle,\qquad n+m=N\,, \tag{4.28}$$

is N. This is in contrast to the spectrum of the string version of the Virasoro-Shapiro model (the closed string),[11] where the gauge conditions force the states to be symmetric in two types of oscillators, and therefore already on the level next to the ground state a spin-2 state exists. This may be a clue to justify the value $\alpha(0)=0$ in our model in contrast to $\alpha(0)=2$ for the Virasoro-Shapiro model and $\alpha(0)=1$ for the open-string model.

V. CONCLUSIONS AND FINAL REMARKS

We showed that it is possible to follow the methods for quantizing the m-string for a Y-string configuration too. However, the quantization is not complete, because we were not able to prove that ghosts are absent in the spectrum. These difficulties arise at that point, where an assumption enters into the game, namely the interpretation of the matrix conditions (3.17). The reasons are the two different orthogonal systems of functions, in which the classical solution has to be expanded because of the boundary conditions, which are different from the m-string. They reflect themselves in the appearance of half-integer labeled operators b_r^μ, c_r^μ besides the well-known oscillators a_n^μ. This type of oscillators is not new in dual models. It appeared first in the Neveu-Schwarz model,[12] but there they are anticommuting operators. As Bose operators they were used in a model for off-shell states in dual resonance theory.[13] As a matter of fact, the gauge operators L_n^c of that model are identical to our operators K_n written in terms of γ_r^μ.

Since we want to keep the analogy to m-models as close as possible, we formulate physical states in parts of the original gauge operators, and then come to intriguing identities between commutators. The spectrum of states looks quite interesting, especially the strong connection between β- and γ-type operators, although it is no baryon spectrum at all. It is clear that we cannot get half-integer-spin states by naively quantizing a classical theory. Of course, one could introduce more degrees of freedom to make fermions out of the three-string configuration. It might well be that the resulting spectrum would agree better with the spectrum of baryon resonances than the Ramond model.[14] But one would gain nothing in the understanding of the classical Lagrangian one has to start with.

A nice possibility would be to reinterpret the model at an earlier stage, so as to overcome the problem of the nonclosing algebra and at the same time get anticommuting objects. But we do not see a real way to do this.

We did not show that scattering amplitudes for three-strings fulfill duality. Going through the program of Mandelstam[15] looks really tedious and may even be impossible, since he essentially uses the transverse gauge which in general is not applicable to our model. But one should try to construct vertices in the model which, as is known from the m model, are related to operators which map physical states onto physical states. For this construction the consideration on the transverse gauge of Appendix B could give valuable hints, and one might also use similar features of our model with the Virasoro-Shapiro model.

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APPENDIX A

In this appendix we want to list the relations between the operators that were introduced in Sec. IV. The fundamental operators are A_n, B_n, and C_n, and their algebra is given by (3.16). Then [compare Eqs. (3.6) and (3.13)]

$$A_n=S_n+2T_n \quad (A_{-n}=A_n^\dagger),$$
$$B_n=G_n+2H_n \quad (B_{-n}=C_n^\dagger), \tag{A1}$$
$$H_n=\frac{1}{\pi}\sum_r \frac{(-1)^{n+r+1/2}}{n-r}W_r.$$

To write down the commutators of these operators in a compact form, use

$$\alpha_0^\mu=\rho^2p^\mu,\quad \alpha_n^\mu=-i\rho\sqrt{n}\,a_n^\mu,\quad \alpha_{-n}^\mu=\alpha_n^{\dagger\mu},$$
$$d_r^\mu=-i\rho\sqrt{r}\,c_r^\mu,\quad d_{-r}^\mu=i\rho\sqrt{r}\,b_r^{\dagger\mu}. \tag{A2}$$

Then α_n^μ, d_r^μ, and $d_{-r}^{\mu\dagger}$ are annihilation (creation) operators for positive (negative) indices n and r. Their nonvanishing commutators are

$$[\alpha_n^\mu,\alpha_{-m}^\nu]=-n\rho^2g^{\mu\nu}\delta_{n,m},$$
$$[d_r^\mu,d_s^{\dagger\nu}]=-r\rho^2g^{\mu\nu}\delta_{r,s}. \tag{A3}$$

In terms of these operators

$$S_n=-\frac{1}{2\rho^2}\sum_{-\infty}^{\infty}:\alpha_m\cdot\alpha_{n-m}:,$$
$$T_n=-\frac{1}{2\rho^2}\sum_{-\infty}^{\infty}:d_r\cdot d_{r-n}^\dagger:,$$
$$G_n=-\frac{1}{2\rho^2}\sum_{-\infty}^{\infty}d_{-r}^\dagger\cdot d_{r-n}^\dagger, \tag{A4}$$
$$W_r=-\frac{1}{2\rho^2}\sum_{-\infty}^{\infty}\alpha_m\cdot d_{r-m}.$$

The algebra between these operators does not close, but we find

(i) $$[S_n,S_m]=(n-m)S_{n+m}+\alpha(n)\delta_{n+m,0},$$
$$[S_n,T_m]=[S_n,G_m]=0,$$
$${}^*[S_n,W_r]=-\frac{1}{2\rho^2}\sum_m (n-m)\alpha_m\cdot d_{(r+n)-m};$$

(ii) $$[T_n,T_m]=\tfrac{1}{2}(n-m)T_{n+m}+\tfrac{1}{4}\beta(n)\delta_{n+m,0},$$
$$[T_n,G_m]=\tfrac{1}{2}(n-m)G_{n+m},$$
$${}^*[T_n,W_r]=-\frac{1}{4\rho^2}\sum_m (m-r)\alpha_m\cdot d_{(r+n)-m}; \tag{A5}$$

(iii) $$[G_n,G_m]=[G_n,W_{-r}^\dagger]=0,$$
$${}^*[G_n,G_{-m}^\dagger]=-\frac{1}{\rho^2}\sum_r (n-r):d_{-r}^\dagger\cdot d_{(m+n)-r}:+\tfrac{1}{2}\beta(n)\delta_{n+m,0},$$
$${}^*[G_n,W_r]=-\frac{1}{2\rho^2}\sum_m (m-r)\alpha_m\cdot d_{m-(n+r)}^\dagger;$$

(iv) $$[W_r,W_s]=\tfrac{1}{4}(r-s)G_{-(r+s)}^\dagger,$$
$${}^*[W_r,W_{-s}^\dagger]=\tfrac{1}{4}(r-s)S_{r+s}-\frac{1}{4\rho^2}\sum_t (r-t):d_t\cdot d_{t-(r+s)}^\dagger:+\tfrac{1}{2}\gamma(r)\delta_{r+s,0},$$

with

$$\alpha(n)=\tfrac{1}{12}Dn(n^2-1),$$
$$\beta(n)=\tfrac{1}{12}Dn(2n^2+1), \tag{A6}$$
$$\gamma(r)=\tfrac{1}{12}Dr(r^2-\tfrac{1}{4}).$$

The commutators which lead out of the set of operators $\{S_n,T_n,G_n,W_r\}$ fulfill

$$[S_n,W_r]-[S_{n-N},W_{r+N}]=NW_{n+r},$$
$$[T_n,W_r]-[T_{n-N},W_{r+N}]=\tfrac{1}{2}NW_{n+r},$$
$$[G_n,G_{-m}^\dagger]-[G_{n-N},G_{-(m+N)}^\dagger]=2NT_{n+m}+\beta(n)\delta_{n+m,0}, \tag{A7}$$
$$[G_n,W_r]-[G_{n-N},W_{r+N}]=NW_{-(r+n)}^\dagger,$$
$$[W_r,W_{-s}^\dagger]-[W_{r-N},W_{-(s+N)}^\dagger]=\tfrac{1}{2}N(S_{r+s}+T_{r+s})+\gamma(r)\delta_{r+s,0}.$$

Furthermore, the relations with an asterisk give the following identities:

$$[S_n,W_r]+2[T_{n-N},W_{r+N}]=(n-r-N)W_{n+r},$$
$$[S_n,W_{-r}^\dagger]+[G_{n-N},W_{r+N}]=(n-r-N)W_{-(n+r)}^\dagger, \tag{A8}$$
$$4[W_r,W_{-s}^\dagger]+[G_N,G_{-(s+r-N)}]=(r-s)S_{r+s}+2(N-s)T_{r+s}+2\gamma(r)\delta_{r+s,0}+\tfrac{1}{2}\beta(N)\delta_{r+s,0}.$$

To simplify things, use linear combinations of these operators:

$$U_n \equiv \tfrac{1}{2}(G_n + G^\dagger_{-n}), \quad V_n \equiv \tfrac{1}{2}(G_n - G^\dagger_{-n}),$$
$$X_r \equiv \sqrt{2}(W_r + W^\dagger_{-r}), \quad Y_r \equiv \sqrt{2}(W_r - W^\dagger_{-r}). \tag{A9}$$

They have the property

$$U_{-n} = U^\dagger_n, \quad V_{-n} = -V^\dagger_n,$$
$$X_{-r} = X^\dagger_r, \quad Y_{-r} = -Y^\dagger_r. \tag{A10}$$

To minimize the deviation of the above commutators from a closed algebra, we furthermore take

$$L_n \equiv T_n + U_n, \quad K_n = T_n - U_n. \tag{A11}$$

Then for the set $\{A_n = S_n + 2T_n, L_n, K_n, V_n, X_r, Y_r\}$ we find

$$\begin{aligned}
&\text{(I)}\ [A_n, A_m] = (n-m)A_{n+m} + [\alpha(n) + \beta(n)]\delta_{n+m,0},\\
&\quad [A_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{2}\beta(n)\delta_{n+m,0},\\
&\quad [A_n, K_m] = (n-m)K_{n+m} + \tfrac{1}{2}\beta(n)\delta_{n+m,0},\\
&\quad [A_n, V_m] = (n-m)V_{n+m},\\
&\quad [A_n, X_r] = (n-r)X_{n+r},\\
&\quad [A_n, Y_r] = (n-r)Y_{n+r};\\
&\text{(II)}\ [L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{2}\beta(n)\delta_{n+m,0},\\
&\quad [L_n, K_m] = 0,\\
&\quad [L_n, V_m] - [L_{n-N}, V_{m+N}] = \tfrac{1}{2}NV_{n+m},\\
&\quad [L_n, X_r] - [L_{n-N}, X_{r+N}] = NX_{n+r},\\
&\quad [L_n, Y_r] = 0;\\
&\text{(III)}\ [K_n, K_m] = (n-m)K_{n+m} + \tfrac{1}{2}\beta(n)\delta_{n+m,0},\\
&\quad [K_n, V_m] - [K_{n-N}, V_{m+N}] = \tfrac{1}{2}NV_{n+m},\\
&\quad [K_n, X_r] = 0,\\
&\quad [K_n, Y_r] - [K_{n-N}, Y_{r+N}] = NY_{n+r};\\
&\text{(IV)}\ [V_n, V_m] = -\tfrac{1}{4}(n-m)(L_{n+m} + K_{n+m}) - \tfrac{1}{4}\beta(n)\delta_{n+m,0},\\
&\quad [V_n, X_r] - [V_{n-N}, X_{r+N}] = -\tfrac{1}{2}NY_{n+r},\\
&\quad [V_n, Y_r] - [V_{n-N}, Y_{r+N}] = \tfrac{1}{2}NX_{n+r};\\
&\text{(V)}\ [X_r, X_s] = (r-s)(A_{r+s} - K_{r+s}) + 2\gamma(r)\delta_{r+s,0},\\
&\quad [X_r, Y_s] - [X_{r-N}, Y_{s+N}] = -\tfrac{1}{2}NV_{r+s};\\
&\text{(VI)}\ [Y_r, Y_s] = -(r-s)(A_{r+s} - L_{r+s}) + 2\gamma(r)\delta_{r+s,0},
\end{aligned} \tag{A12}$$

and the identities

$$\begin{aligned}
&\text{(a)}\ [L_n + K_n, V_m] = (n-m)V_{n+m},\\
&\text{(b)}\ [L_n, X_r] - 2[V_{n-N}, Y_{r+N}] = NX_{n+r},\\
&\text{(c)}\ [K_n, Y_r] + 2[V_{n-N}, X_{r+N}] = NY_{n+r},\\
&\text{(d)}\ 4[X_r, Y_s] + [L_N - K_N, V_{s+r-N}] = -(r-s)V_{r+s} + 4\gamma(r)\delta_{r+s,0} + \beta(N)\delta_{r+s,0}.
\end{aligned} \tag{A13}$$

APPENDIX B: QUANTIZATION IN THE TRANSVERSE GAUGE

For the m-string models, transverse oscillators play a crucial role. Goddard, Goldstone, Rebbi, and Thorn[8] quantized the model noncovariantly in terms of these operators and required Poincaré invariance at the end. This fixed the dimension D and the intercept $\alpha(0)$. The transverse gauge can also be consistently defined for a Y string,[6] but in general not in our special case with orthonormal coordinates on each wing. We will consider it in our model, too, with the hope that the transverse variables lead to a better understanding.

A system of orthonormal coordinates is specified by choosing a timelike vector n^μ ($n^2 \geq 0$) and

$$n_\mu x^\mu_{(i)} = \tfrac{2}{3}\alpha'(n\cdot P)\tau, \tag{B1}$$

$$n_\mu P^\mu_{(i)} = \frac{1}{3\pi}(n\cdot P), \tag{B2}$$

where P is the total momentum of the system. Equation (B1) defines a new τ variable, and Eq. (B2) defines the σ variable

$$\int_0^\sigma d\sigma' n\cdot P_{(i)}(\sigma',\tau) = \frac{\sigma}{3\pi}(n\cdot P). \tag{B3}$$

Notice that these definitions make sense only for a Y string in which no momentum in the direction of n is exchanged between the wings.

Making the GGRT choice[8] $n = (1, -1, 0, 0, \ldots, 0)$ and using light-cone coordinates

$$u_\pm = \frac{1}{\sqrt{2}}(u_0 \pm u_1) \text{ and } u_i \ (i = 2, \ldots, D-1),$$

the gauge (B1) and (B2) is equivalent to

$$\psi_+ = \tfrac{2}{3}\alpha' P_+\tau, \quad \phi_+ = 0,$$
$$P_{\psi+} = \frac{1}{3\pi}P_+, \quad P_{\phi_+} = 0. \tag{B4}$$

The primary constraints (2.9) relate the ($-$) components to the transverse ones ($\underline{u}\cdot\underline{v} \equiv u_i v^i$):

$$\psi'_- = \frac{3\pi}{P_+}(\underline{\psi}'\cdot\underline{P}_\psi + \underline{\phi}'\cdot\underline{P}_\phi + \underline{\phi}^{*\prime}\cdot\underline{P}_{\phi^*}),$$

$$\phi'_- = \frac{3\pi}{P_+}(\underline{\psi}'\cdot\underline{P}_{\phi^*} + \underline{\phi}'\cdot\underline{P}_\psi + \underline{\phi}^{*\prime}\cdot\underline{P}_\phi),$$

$$P_{\psi_-} = \frac{3\pi}{2P_+}(\underline{P}_\psi^2 + \gamma^2\underline{\psi}'^2 + 2\underline{P}_\phi\cdot\underline{P}_{\phi^*} + 2\gamma^2\underline{\phi}'\cdot\underline{\phi}^{*\prime}),$$

$$P_{\phi_-} = \frac{3\pi}{2P_+}(\underline{P}_{\phi^*}^2 + \gamma^2\underline{\phi}'^2 + 2\underline{P}_\psi\cdot\underline{P}_\phi + 2\gamma^2\underline{\psi}'\cdot\underline{\phi}^{*\prime}). \tag{B5}$$

The independent variables are then $(\psi_i, \phi_i, P_{\psi_i}, P_{\phi_i}, P_+, q_-)$, where q_- has been added to the set, since

ψ_- is given only up to a σ derivative. No such variable is needed for ϕ_- because of the junction condition $\phi_-(\pi,\tau)=0$. The Poisson brackets are

$$\begin{aligned}&\{\psi_i(\sigma),P_{\psi_j}(\sigma')\}=-\tfrac{1}{3}g_{ij}\,\delta(\sigma-\sigma')\,,\\&\{\phi_i(\sigma),P_{\phi_j}(\sigma')\}=-\tfrac{1}{3}g_{ij}\,\delta(\sigma-\sigma')\,,\\&\{q_-,P_+\}=-1\,,\end{aligned}\qquad\text{(B6)}$$

and the equations of motion follow from the Hamiltonian

$$H=2\alpha'P_+\int_0^\pi P_{\psi_-}(\sigma)d\sigma\,.\qquad\text{(B7)}$$

For the transverse components they are

$$\begin{aligned}&\dot\psi_i=\frac{1}{\gamma}\,P_{\psi_i},\quad \dot P_{\psi_i}=\gamma\,\psi_i''\,,\\&\dot\phi_i=(1/\gamma)\,P_{\phi_i^*},\quad \dot P_{\phi_i^*}=\gamma\phi_i''\,.\end{aligned}\qquad\text{(B8)}$$

For the (−) components, they are found by inserting these expressions into (B5)

$$\begin{aligned}&\dot\psi_-=(1/\gamma)\,P_{\psi_-},\quad \dot P_{\psi_-}=\gamma\psi_-''\,,\\&\dot\phi_-=(1/\gamma)\,P_{\phi_-^*},\quad \dot P_{\phi_-^*}=\gamma\phi_-''\,,\end{aligned}\qquad\text{(B9)}$$

and for the (+) components they follow directly from (B4). The solution of these equations is again given by (2.16) and (2.17).

Only the independent variables $(a_n^i, b_r^i, c_r^i, p^i, q^i, P_+, q_-)$ are quantized, assuming canonical commutators for them, which are analogous to (3.1). The nontransverse oscillators can be expressed in terms of the operators $\underline{A}_n, \underline{B}_r, \underline{C}_r$, where $\underline{A}_n$ is the transverse part of A_n,

$$\underline{B}_r=\tfrac{1}{2}\sum_r a_{nr}\underline{B}_n,\quad \underline{C}_r=\underline{B}_{-r}^\dagger\,,\qquad\text{(B10)}$$

and a_{nr} is again the matrix (3.14). The result is

$$\begin{aligned}&\alpha_n^{(-)}=\frac{1}{P_+}\,[\underline{A}_n-\alpha(0)\delta_{n,0}]\,,\\&d_r^{(-)}=\frac{2}{P_+}\,\underline{B}_r\,.\end{aligned}\qquad\text{(B11)}$$

If one looks at the covariance and takes as generators for the Poincaré algebra

$$\begin{aligned}P^\mu&=\sum_i\int_0^\pi d\sigma\,P_{(i)}^\mu=p^\mu\,,\\M^{\mu\nu}&=\tfrac{1}{2}\sum_i\int_0^\pi d\sigma\,(x_{(i)}^\mu P_{(i)}^\nu+P_{(i)}^\nu x_{(i)}^\mu-x_{(i)}^\nu P_{(i)}^\mu-P_{(i)}^\mu x_{(i)}^\nu)\\&=\frac{1}{\rho}\left[\tfrac{1}{2}(q^\mu\alpha_0^\nu+\alpha_0^\nu q^\mu)-\tfrac{1}{2}(q^\nu\alpha_0^\mu+\alpha_0^\mu q^\nu)-i\sum_{n>0}\frac{1}{n}\,A_n^{\mu\nu}-i\sum_{r>0}\frac{1}{r}\,D_r^{\mu\nu}\right],\end{aligned}\qquad\text{(B12)}$$

where

$$\begin{aligned}A_n^{\mu\nu}&=\alpha_{-n}^\mu\,\alpha_n^\nu-\alpha_{-n}^\nu\,\alpha_n^\mu\,,\\D_r^{\mu\nu}&=(d_r^{\dagger\mu}d_r^\nu-d_r^{\dagger\nu}d_r^\mu)+(d_{-r}^\mu d_{-r}^{\dagger\nu}-d_{-r}^\nu d_{-r}^{\dagger\mu})\,,\end{aligned}\qquad\text{(B13)}$$

the Poincaré algebra comes out in the right way, with the exception of

$$[M^{i-},M^{j-}]=\left(\frac{1}{\rho^2P_+}\right)^2\left[\sum_{n>0}\left([3-\tfrac{1}{8}(D-2)]\,n-\frac{\alpha(0)}{n}\right)A_n^{ij}+\sum_{r>0}\left([3-\tfrac{1}{8}(D-2)]r-\frac{\alpha(0)}{r}\right)D_r^{ij}\right],\qquad\text{(B14)}$$

which should be zero. From this we get $D=26$, the same critical dimension as for the m-model, and $\alpha(0)=0$.

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[1] *Dual Theory*, edited by M. Jacob (North-Holland, Amsterdam, 1974); P. H. Frampton, *Dual Resonance Models* (Benjamin, New York, 1974); J. Scherk, Rev. Mod. Phys. 47, 123 (1975).

[2] A. Chodos and C. B. Thorn, Nucl. Phys. B72, 509 (1974); I. Bars and A. J. Hanson, Phys. Rev. D 13, 1744 (1976).

[3] Y. Iwasaki and K. Kikkawa, Phys. Rev. D 8, 440 (1973).

[4] J. Goldstone (unpublished).

[5] X. Artru, Nucl. Phys. B85, 442 (1975).

[6] P. A. Collins, J. F. L. Hopkinson, and R. W. Tucker, Nucl. Phys. B100, 157 (1975).

[7] The reference to original papers and related papers can be found in A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale dei Lincei, Rome, Italy, 1976).

[8] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).

[9] L. Brink, D. Olive, and J. Scherk, Nucl. Phys. B61, 173 (1973).

[10] R. C. Brower, Phys. Rev. D 6, 1655 (1972); P. God-

dard and C. B. Thorn, Phys. Lett. 40B, 235 (1972).
[11]M. A. Virasoro, Phys. Rev. 177, 2309 (1969); J. A. Shapiro, Phys. Lett. 33B, 361 (1970).
[12]A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971).
[13]E. F. Corrigan and D. B. Fairlie, Nucl. Phys. B91, 527 (1975).
[14]P. Ramond, Phys. Rev. D 3, 2415 (1971).
[15]S. Mandelstam, Nucl. Phys. B64, 205 (1973).