High-energy color-singlet–color-singlet scattering in non-Abelian gauge theories*

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In the quark model, hadrons are thought to be singlet states with respect to "color" SU(3). Therefore, in this paper, we study singlet-singlet scattering in a non-Abelian gauge theory in which color SU(3) is the gauge symmetry. We avoid the difficult bound-state problem by representing the hadron as a scalar field ϕ, a color singlet, which interacts with the quarks through an effective coupling: $G\phi\Sigma_{i=1}^{3}\bar{\psi}_i\psi_i$. We calculate the high-energy ($s\to\infty, t\le 0$ fixed) $\phi\phi$ scattering amplitude to sixth order in the quark-gluon coupling constant g. The calculation is done by using the "infinite-momentum technique" as developed by Chang and Ma. To justify this technique we also calculate high-energy fermion-fermion scattering in a non-Abelian gauge theory using a more rigorous method. We compare our result with an "infinite-momentum technique" calculation done by McCoy and Wu and a similar calculation done by Tyburski. The $\phi\phi$ scattering amplitude is infrared finite. The total cross section for high-energy $\phi\phi$ scattering is found to be $\sigma_{\phi\phi}=64g^4G^4(2\pi)^{-12}\pi^7[N_1(b)-(3g^2/8\pi^2)\ln sN_2(b)]$, where $N_1(b)$ and $N_2(b)$ are positive functions depending only on $b=(\mu^2-4m^2)^{1/2}/\mu$, where μ is the hadron mass and m is the quark mass.

I. INTRODUCTION

A considerable amount of theoretical work has been done on the behavior of high-energy hadron-hadron scattering amplitudes. Qualitative results have been obtained using the Regge model[1] and the eikonal or diffraction model.[2] However, calculations based on a relativistic field theory are of more interest, since quantum field theory is the only well-defined theory which incorporates all the general principles: relativistic invariance, unitarity, analyticity, and crossing symmetry. The question is which field theory do we use? Model field theories which have been extensively studied in this context include ϕ^3 theory[3] and QED.[4] However, these theories cannot be expected to be realistic models for the hadron since they ignore its composite structure. Because of the success of the quark model, it is thought that quarks are the basic constituents of the hadron. Therefore, the more realistic field theory is one in which the quark fields interact through a gluon field—a non-Abelian gauge theory in which "color" SU(3) is the gauge symmetry. With this in mind, several authors have studied fermion-fermion scattering in non-Abelian gauge theories.[5,6,7]

Color was introduced into the usual quark model to explain the apparent conflict with the requirement of antisymmetry for a state of spin-$\frac{1}{2}$ fermions, and the assumption that three quarks bind in a totally symmetric state in space, spin, and SU(3) coordinates to form a baryon. Each quark comes in three colors[8] and the hadrons are then assumed to be singlet states with respect to color SU(3). Therefore, it is more interesting from a physical standpoint to study singlet-singlet scattering in non-Abelian gauge theories.

Of course, in order to correctly calculate hadron-hadron scattering amplitudes, we would first have to solve for the bound states in this theory. However, this is a difficult low-energy problem which we have avoided by representing the hadron as a scalar field ϕ, a color singlet, which interacts with the quarks through an effective coupling. The new term in the Lagrangian is

$$\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}\mu^2\phi^2+G\phi\sum_{i=1}^{3}\bar{\psi}_i\psi_i\,. \tag{1.1}$$

In this model, $\phi\phi$ scattering occurs through the interaction of the quarks and the gluon field. We have calculated, to sixth order in the quark-gluon coupling constant g, the $\phi\phi$ scattering amplitude in the high-energy limit ($s\to\infty$, $t\le 0$ fixed), through the usual procedure of keeping only the leading logarithm at each order of perturbation theory. Because we are scattering two singlet states, the $\phi\phi$ scattering amplitude is proportional to the nonleading, isospin-nonflip piece of the fermion-fermion scattering amplitude. Also, the $\phi\phi$ scattering amplitude is infrared finite, unlike the fermion-fermion scattering amplitude.

The calculation was done using the "infinite-momentum technique" as developed by Chang and Ma.[9,10] This method is extremely simple and therefore well suited for carrying out higher-order calculations. The infinite-momentum technique involves taking the limit $s\to\infty$ before performing momentum-space integrations and renormalization. It is not clear that this is justified when the integrals are divergent. McCoy and Wu[5] calculated the high-energy fermion-fermion scattering amplitude using the infinite-momentum technique. Nieh and Yao[6] performed an independent calculation using the standard Feynman parameter technique and their result differs from that of McCoy and Wu. However, the difficulty of the Feynman parameter technique multiplies with each succeeding order of perturbation theory. Tyburski[7] used a combination of the infinite-momentum technique and the Feynman parameter technique to arrive at a result in agreement with that of McCoy and Wu. Therefore, in order to justify the use of the infinite-momentum technique, we have calculated to sixth order the high-energy fermion-fermion scattering amplitude using a method which is easier than that of Nieh and Yao, but just as rigorous.

The calculation of the fermion-fermion scattering amplitude at sixth order involves two momentum-space integrations. Whenever the infinite-momentum technique yields a spurious divergence, or whenever a particular Feynman graph needs to be renormalized, we perform one of the momentum-space integrations using the Feynman parameter technique. The remaining convergent integral is calculated using what Tyburski calls the "++-- approximation" in order to systematically keep track of the leading terms. The details of this calculation are outlined in Sec. II of this paper. Our result agrees with the calculations done by McCoy and Wu and Tyburski. Therefore, we are free to use the infinite-momentum technique to calculate the high-energy $\phi\phi$ scattering amplitude. The important steps in this calculation are outlined in Sec. III. Section IV is devoted to our conclusions and summary.

II. FERMION-FERMION SCATTERING

In order to compare our result to that of McCoy and Wu, we restrict our considerations to the case of SU(2) gauge symmetry. The generalization to SU(3) symmetry is trivial.[7] To avoid the infrared problem, we introduce a complex scalar doublet and invoke the Higgs mechanism to give a mass λ to the vector gluons. The Feynman rules are those of the usual Yang-Mills theory with the addition of the vertex and propagator associated with the Higgs scalar, as illustrated in Fig. 2 of Ref. 5. The Faddeev-Popov ghost and the Higgs ghost do not appear in the leading logarithm approximation. We work exclusively in the 't Hooft-Feynman gauge.

A. Notation and low-order results

The components of a four-vector A_μ are written $(A_+, A_-, \vec{A})$, where $A_\pm = A_0 \pm A_3$ and $\vec{A} = (A_1, A_2)$. The invariant product takes the form $A\cdot B = \frac{1}{2}A_+B_- + \frac{1}{2}A_-B_+ - \vec{A}\cdot\vec{B}$. The Dirac matrices in this representation have the following properties:

$$\{\gamma_+, \gamma_-\} = 4\,, \quad \{\gamma_\pm, \vec{\gamma}\} = 0\,, \quad \gamma_\pm^{\,2} = 0\,.$$

We consider the scattering of two on-mass-shell fermions $p_1 + p_2 \to p_3 + p_4$ in the limit $s\to\infty$, $t \le 0$ fixed, where $s = (p_1+p_2)^2$ and $t = (p_1 - p_3)^2 = q^2$. For convenience, we choose

$$p_1 = p - \tfrac{1}{2}q\,, \quad p_2 = p' + \tfrac{1}{2}q\,,$$
$$p_3 = p + \tfrac{1}{2}q\,, \quad p_4 = p' - \tfrac{1}{2}q\,,$$

where the momentum transfer is purely transverse, $q = (0, 0, \vec{q})$,

$$p = (s^{1/2}, (\tfrac{1}{4}\vec{q}^{\,2} + m^2)/s^{1/2}, 0, 0)\,,$$
$$p' = ((\tfrac{1}{4}\vec{q}^{\,2} + m^2)/s^{1/2}, s^{1/2}, 0, 0)\,,$$

and m is the fermion mass. Let the mass of the Higgs scalar be M.

We define the invariant transition amplitude T by

$$\langle p_3 p_4 | (S-1) | p_1 p_2 \rangle = -iN(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) T\,,$$

where N is the wave-function normalization.[11] We decompose T into an isospin-flip amplitude T^{f} and an isospin-nonflip amplitude T^{nf},

$$T = T^{\mathrm{f}} (\tau_a)_{i_1 i_3} (\tau_a)_{i_2 i_4} + T^{\mathrm{nf}} \delta_{i_1 i_3} \delta_{i_2 i_4}\,,$$

where i_j is the isospin index of the particle with momentum p_j and τ_a is a Pauli matrix which satisfies the relation

$$\tau_a \tau_b = i\epsilon_{abc}\tau_c + \delta_{ab}\,.$$

In lowest order we have the Born term, diagram 1 in Fig. 1. In the limit $s\to\infty$, the amplitude is

$$T_1 = \frac{g^2}{8}\frac{1}{-t+\lambda^2}\frac{s}{m^2}(\tau_a)_{i_1 i_3}(\tau_a)_{i_2 i_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\,, \qquad (2.1)$$

where λ_j is the helicity of the particle with momentum p_j. Helicity conservation is a consequence of taking the limit $s\to\infty$.[10]

In order g^4, the leading diagrams are those numbered 2 and 3 in Fig. 1. The amplitudes can be calculated using the infinite-momentum technique. They are

$$T_2 = \frac{g^4}{16}(2\pi)^{-4}\pi^2\frac{s}{m^2}(\tau_a\tau_b)_{i_1 i_3}(\tau_a\tau_b)_{i_2 i_4} \times \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)(\ln s - i\pi)\,, \qquad (2.2)$$

$$T_3 = -\frac{g^4}{16}(2\pi)^{-4}\pi^2\frac{s}{m^2}(\tau_a\tau_b)_{i_1 i_3}(\tau_b\tau_a)_{i_2 i_4} \times \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln s\,,$$

where

$$K(t) = \int \frac{d^2k}{\pi}[(\vec{k} + \tfrac{1}{2}\vec{q})^2 + \lambda^2]^{-1}[(\vec{k} - \tfrac{1}{2}\vec{q})^2 + \lambda^2]^{-1}\,. \qquad (2.3)$$

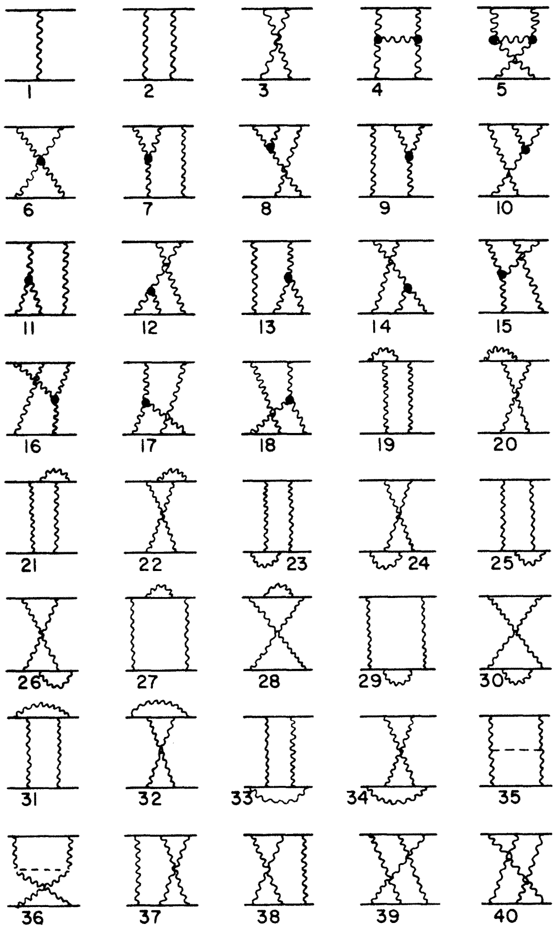

FIG. 1. Feynman diagrams which contribute to the leading logarithm of fermion-fermion scattering up to sixth order in the coupling constant.

We decompose the amplitudes (2.2) into isospin-flip and isospin-nonflip parts. Then we add the amplitudes together and get

$$T^{f}_{2,3}=-\frac{g^4}{4}(2\pi)^{-4}\pi^2\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln s\,,$$
$$T^{nf}_{2,3}=-i\frac{3g^4}{16}(2\pi)^{-4}\pi^3\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\,. \qquad (2.4)$$

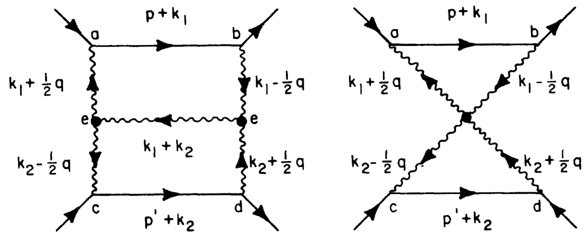

FIG. 2. Internal-momentum labels for diagrams 4 and 6.

The isospin-nonflip amplitude is always smaller than the isospin-flip amplitude by a factor of $\ln s$.

B. Calculation of Fermion-Fermion scattering in order g^6

The leading diagrams are those numbered 4–40 in Fig. 1. Of these, only diagrams 4, 6, 7, 15, 19, 27, 31, 35, and 37 need to be calculated explicitly. The remaining diagrams are obtained from these by symmetry considerations. If we use the infinite-momentum technique to calculate diagrams 4–6, 15–18, and 31–34 the result will contain a spurious ultraviolet divergence. Also, diagrams 7–14 and 19–30 need to be renormalized. Therefore, we must use our method to calculate the amplitudes associated with diagrams 4–34. To illustrate the method, we will calculate diagrams 4–6 in detail.

The labeling of the internal momenta for diagrams 4 and 6 is shown in Fig. 2. Individually, the diagrams 4–6 are of order s^2. However, if we add these diagrams together the s^2 dependence cancels. The 4-gluon vertex of diagram 6 contains two terms which contribute in the $s\to\infty$ limit. One term has the same isospin structure as that of diagram 4 and the other has the same isospin structure as that of diagram 5. We add these terms to diagrams 4 and 5, respectively, and cancel the s^2 dependence algebraically. Then, the amplitude for diagram 4 is

$$T_4=-\frac{g^6}{16}(2\pi)^{-8}\epsilon_{ace}\epsilon_{bde}(\tau_a\tau_b)_{i_1i_3}(\tau_c\tau_d)_{i_2i_4}$$
$$\times\int d^4k_1d^4k_2N_4[(k_1+\tfrac{1}{2}q)^2-\lambda^2+i\epsilon]^{-1}[(k_1-\tfrac{1}{2}q)^2-\lambda^2+i\epsilon]^{-1}[(k_1+k_2)^2-\lambda^2+i\epsilon]^{-1}$$
$$\times[(p+k_1)^2-m^2+i\epsilon]^{-1}[(k_2+\tfrac{1}{2}q)^2-\lambda^2+i\epsilon]^{-1}[(k_2-\tfrac{1}{2}q)^2-\lambda^2+i\epsilon]^{-1}$$
$$\times[(p'+k_2)^2-m^2+i\epsilon]^{-1}\,. \qquad (2.5)$$

Since diagram 4 consists of two 3-gluon vertices, the numerator N_4 has nine terms

$$N_4 = (q^2 - 2k_1{}^2 - 2k_2{}^2 + \lambda^2)\bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)\gamma_\nu u_1 \bar{u}_4\gamma^\mu(\not{p}' + \not{k}_2)\gamma^\nu u_2 \tag{2.6a}$$

$$+ \bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)(2\not{k}_2 + \not{k}_1 - \tfrac{1}{2}\not{q})u_1\bar{u}_4\gamma^\mu(\not{p}' + \not{k}_2)(\not{k}_2 - \not{k}_1 + \not{q})u_2 \tag{2.6b}$$

$$+ \bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)(\not{k}_2 - \not{k}_1 + \not{q})u_1\bar{u}_4\gamma^\mu(\not{p}' + \not{k}_2)(-2\not{k}_1 - \not{k}_2 - \tfrac{1}{2}\not{q})u_2 \tag{2.6c}$$

$$+ \bar{u}_3(-2\not{k}_2 - \not{k}_1 - \tfrac{1}{2}\not{q})(\not{p} + \not{k}_1)\gamma_\mu u_1\bar{u}_4(\not{k}_1 - \not{k}_2 + \not{q})(\not{p}' + \not{k}_2)\gamma^\mu u_2 \tag{2.6d}$$

$$+ \bar{u}_3(-2\not{k}_2 - \not{k}_1 - \tfrac{1}{2}\not{q})(\not{p} + \not{k}_1)(2\not{k}_2 + \not{k}_1 - \tfrac{1}{2}\not{q})u_1\bar{u}_4\gamma_\mu(\not{p}' + \not{k}_2)\gamma^\mu u_2 \tag{2.6e}$$

$$+ \bar{u}_3(-2\not{k}_2 - \not{k}_1 - \tfrac{1}{2}\not{q})(\not{p} + \not{k}_1)\gamma^\mu u_1\bar{u}_4\gamma^\mu(\not{p}' + \not{k}_2)(-2\not{k}_1 - \not{k}_2 - \tfrac{1}{2}\not{q})u_2 \tag{2.6f}$$

$$+ \bar{u}_3(\not{k}_1 - \not{k}_2 + \not{q})(\not{p} + \not{k}_1)\gamma_\mu u_1\bar{u}_4(2\not{k}_1 + \not{k}_2 - \tfrac{1}{2}\not{q})(\not{p}' + \not{k}_2)\gamma^\mu u_2 \tag{2.6g}$$

$$+ \bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)(2\not{k}_2 + \not{k}_1 - \tfrac{1}{2}\not{q})u_1\bar{u}_4(2\not{k}_1 + \not{k}_2 - \tfrac{1}{2}\not{q})(\not{p}' + \not{k}_2)\gamma^\mu u_2 \tag{2.6h}$$

$$+ \bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)\gamma^\mu u_1\bar{u}_4(2\not{k}_1 + \not{k}_2 - \tfrac{1}{2}\not{q})(\not{p}' + \not{k}_2)(-2\not{k}_1 - \not{k}_2 - \tfrac{1}{2}\not{q})u_2\,, \tag{2.6i}$$

where u_j is shorthand for the spinor $u(p_j, \lambda_j)$ and $\not{a} = a_\mu\gamma^\mu$. The mass factors in the numerator have been dropped because they do not contribute in the leading logarithm approximation.

Let us for the moment calculate the amplitude (2.5) using only the last eight terms of the numerator (2.6b)–(2.6i). We combine the denominators which contain momentum k_1 through the introduction of Feynman parameters α_1, α_2, α_3, and α_4. Then we shift the origin and evaluate the integral over d^4k_1, in the usual way. Next, we use the "++−− approximation" to simplify the amplitude (2.5). The "++−− approximation" corresponds to having fermion-gluon couplings proportional to γ_+ along the upper fermion line through which flows a large p_+ and fermion-gluon couplings proportional to γ_- along the lower fermion line through which flows a large p'_-. Keeping only the leading logarithms we write the amplitudes as

$$T_4(b-i) = -i\frac{g^6}{16}(2\pi)^{-8}\pi^2\frac{s}{m^2}\epsilon_{ace}\epsilon_{bde}(\tau_a\tau_b)_{i_1i_3}(\tau_c\tau_d)_{i_2i_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}$$
$$\times\int\tfrac{1}{2}d^2k_2\,dk_{2+}\,dk_{2-}\,s^{1/2}$$
$$\times\int_0^1 d\alpha_1\,d\alpha_2\,d\alpha_3\,d\alpha_4\;\delta(1-\textstyle\sum\alpha_i)[k_{2+}k_{2-} - (\vec{k}_2 + \tfrac{1}{2}\vec{q})^2 - \lambda^2 + i\epsilon]^{-1}[k_{2+}k_{2-} - (\vec{k}_2 - \tfrac{1}{2}\vec{q})^2 - \lambda^2 + i\epsilon]^{-1}$$
$$\times\left[k_{2+} - \frac{\vec{k}_2{}^2 - i\epsilon}{s^{1/2}}\right]^{-1}\left\{\frac{N_4{}^{(1)}}{D_4^2} - \frac{2N_4{}^{(2)}}{D_4}\right\}, \tag{2.7}$$

where

$$N_4{}^{(1)} = 3s^{1/2}k_{2-}\alpha_1\alpha_2(1-\alpha_1)\left(1 + \frac{k_{2-}}{s^{1/2}}\right) - 4s^{1/2}k_{2-}\alpha_1\alpha_2\alpha_1\,, \tag{2.8a}$$

$$N_4{}^{(2)} = \tfrac{3}{2}(1-\alpha_1)\left(1 + \frac{k_{2-}}{s^{1/2}}\right) - 7\alpha_1\,, \tag{2.8b}$$

$$D_4 = s^{1/2}k_{2-}\alpha_1\alpha_2 + (-t\alpha_3\alpha_4 + \lambda^2) - i\epsilon\,. \tag{2.8c}$$

In writing the denominator D_4 we have used the fact that the integral over the Feynman parameters is dominated by the region $\alpha_1 \approx \alpha_2 \approx 0$. We can rewrite the expression inside the curly brackets in (2.7) by combining denominators. The terms which integrate to $\ln^3 s$ cancel, leaving

$$\left\{\frac{10s^{1/2}k_{2-}\alpha_1\alpha_2\alpha_1 - 3(-t\alpha_3\alpha_4 + \lambda^2)}{[s^{1/2}k_{2-}\alpha_1\alpha_2 + (-t\alpha_3\alpha_4 + \lambda^2) - i\epsilon]^2}\right\}. \tag{2.9}$$

Upon integration, both terms in (2.9) are of order $\ln^2 s$. It is a trivial matter to integrate these terms, and the result is

$$T_4(b-i) = \frac{g^6}{8}(2\pi)^{-8}\pi^4\frac{s}{m^2}\epsilon_{ace}\epsilon_{bde} \times(\tau_a\tau_b)_{i_1i_3}(\tau_c\tau_d)_{i_2i_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \times K(t)(\tfrac{1}{2}\ln^2 s - i\pi\ln s)\,. \tag{2.10}$$

When we add together diagrams 4 and 5 we get

$$T^{\text{f}}_{4,5}(b-i) = -\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln^2 s\,, \tag{2.11}$$

$$T^{\text{nf}}_{4,5}(b-i) = -i\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln s\,.$$

Now we consider the first term in the numerator (2.6a). We can write (2.6a) in a more suggestive form:

$$\{2q^2 - 3\lambda^2 - [(k_1 + \tfrac{1}{2}q)^2 - \lambda^2] - [(k_1 - \tfrac{1}{2}q)^2 - \lambda^2] - [(k_2 + \tfrac{1}{2}q)^2 - \lambda^2] - [(k_2 - \tfrac{1}{2}q)^2 - \lambda^2]\}$$
$$\times\bar{u}_3\gamma_\mu(\not{p} + \not{k}_1)\gamma_\nu u_1\bar{u}_4\gamma^\mu(\not{p}' + \not{k}_2)\gamma^\nu u_2 \tag{2.12}$$

Each of the last four terms in (2.12) cancels with one of the terms in the denominator of (2.5). The resultant amplitude has the same gluon denominator structure as that of diagrams 7–18. When we add this amplitude, together with the similar amplitude from diagram 5, to diagrams 7–18, the $\ln^3 s$ dependence cancels. To calculate diagrams 7–18 we use the same method as outlined above, except that diagrams 7–14 need to be renormalized. The renormalization is done by subtraction. This is the same renormalization procedure as that used by Tyburski and Nieh and Yao. As they have emphasized, the point of subtraction has no effect on the leading logarithm. The result is

$$T^{\mathrm{f}}_{7\text{-}18}=\frac{g^6}{2}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{7\text{-}18}=i\,\frac{3g^6}{2}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln s\,. \tag{2.13}$$

Now we calculate the amplitude associated with the numerator $(2q^2-3\lambda^2)\bar{u}_3\gamma_\mu(\not{p}+\not{k}_1)\gamma_\nu u_1\bar{u}_4\gamma^\mu(\not{p}'+\not{k}_2)\gamma^\nu u_2$ in (2.12). We can use the infinite-momentum technique with the result

$$T^{\mathrm{f}}_{4,5}(a)=\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-q^2+\tfrac{3}{2}\lambda^2)K^2(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{4,5}(a)=i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-q^2+\tfrac{3}{2}\lambda^2)K^2(t)\ln s\,. \tag{2.14}$$

Diagrams 19–34 are easier to calculate than diagrams 4–6 because we need only keep the leading term in the "++-- approximation." Diagrams 19–30 are also renormalized by subtraction. The result is

$$T^{\mathrm{f}}_{19\text{-}34}=-\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{19\text{-}34}=-i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}K(t)\ln s\,. \tag{2.15}$$

Combining Eqs. (2.11), (2.13), (2.14), and (2.15) we obtain the transition amplitude associated with diagrams 4–34,

$$T^{\mathrm{f}}_{4\text{-}34}=\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-t+\tfrac{3}{2}\lambda^2)K^2(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{4\text{-}34}=i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-t+\tfrac{3}{2}\lambda^2)K^2(t)\ln s\,. \tag{2.16}$$

The remaining diagrams 35–40 can be calculated using the infinite-momentum technique. The amplitude for diagrams 35 and 36 is

$$T^{\mathrm{f}}_{35,36}=\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-\tfrac{1}{2}\lambda^2)K^2(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{35,36}=i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-\tfrac{1}{4}\lambda^2)K^2(t)\ln s\,. \tag{2.17}$$

The amplitude for the QED-like diagrams, 37–40, is given by

$$T^{\mathrm{f}}_{37\text{-}40}=0\,,$$
$$T^{\mathrm{nf}}_{37\text{-}40}=-i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}J(t)\ln s\,, \tag{2.18}$$

where

$$J(t)=\int\frac{d^2k_1d^2k_2}{\pi^2}[\vec{k}_1^{\,2}+\lambda^2]^{-1}[\vec{k}_2^{\,2}+\lambda^2]^{-1}\times[(\vec{q}-\vec{k}_1-\vec{k}_2)^2+\lambda^2]^{-1}\,. \tag{2.19}$$

We combine Eqs. (2.16), (2.17), and (2.18) to obtain the leading fermion-fermion scattering amplitude to order g^6. The result is

$$T^{\mathrm{f}}_{4\text{-}40}=\frac{g^6}{4}(2\pi)^{-8}\pi^4\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times(-t+\lambda^2)K^2(t)\ln^2 s\,,$$
$$T^{\mathrm{nf}}_{4\text{-}40}=i\,\frac{3g^6}{4}(2\pi)^{-8}\pi^5\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\times[(-t+\tfrac{5}{4}\lambda^2)K^2(t)\ln s-J(t)\ln s]\,. \tag{2.20}$$

This result is in complete agreement with that of McCoy and Wu and Tyburski. If we combine Eq. (2.20) with the low-order results obtained earlier, in Eqs. (2.1) and (2.4), we find there is strong evidence that the isospin-flip amplitude exponentiates. The fermion-fermion scattering amplitude up to sixth order in the coupling constant is

$$T^{\mathrm{f}}=\frac{g^2}{8}\frac{1}{-t+\lambda^2}\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\left(1-\frac{g^2}{8\pi^2}(-t+\lambda^2)K(t)\ln s+\frac{1}{2}\frac{g^4}{64\pi^4}(-t+\lambda^2)^2K^2(t)\ln^2 s\right),$$
$$T^{\mathrm{nf}}=-i\,\frac{3g^4}{16}(2\pi)^{-4}\pi^3\frac{s}{m^2}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\left(K(t)-\frac{g^2}{4\pi^2}\ln s[(-t+\lambda^2)K^2(t)-J(t)]\right). \tag{2.21}$$

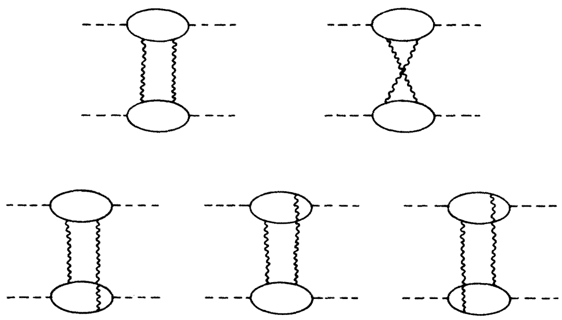

FIG. 3. Feynman diagrams which contribute to $\phi\phi$ scattering in fourth order in the quark-gluon coupling constant.

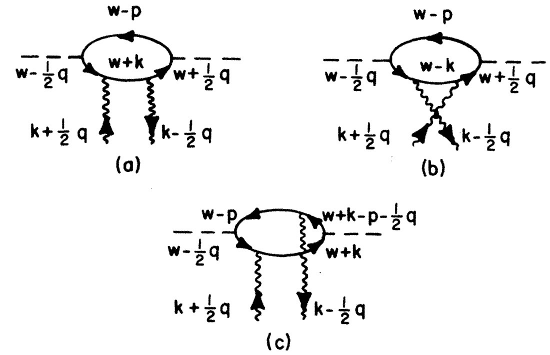

FIG. 4. Internal-momentum labels for the upper half of the diagrams in Fig. 3.

III. CALCULATION OF $\phi\phi$ SCATTERING

Because we are scattering two singlet states, we no longer have an infrared problem. Therefore, we do not need to introduce the Higgs scalars. The Feynman rules are those of the usual Yang-Mills theory with the exception of the singlet-fermion-fermion vertex, given by $iG\delta_{jk}$, where j and k are the color indices of the two fermions. The Pauli matrix τ_a is replaced by the SU(3) matrix λ_a, which satisfies the relation

$$\lambda_a\lambda_b = if_{abc}\lambda_c + d_{abc}\lambda_c + \tfrac{2}{3}\delta_{ab}\,.$$

We consider the scattering of two on-mass-shell color singlets ϕ, $p_1+p_2 \to p_3+p_4$, in the limit $s\to\infty$, $t\le 0$ fixed. The kinematics are the same as those for fermion-fermion scattering, except

$$p = (s^{1/2},\ (\tfrac{1}{4}\vec{q}^{\,2}+\mu^2)/s^{1/2},\ 0,\ 0),$$

$$p' = ((\tfrac{1}{4}\vec{q}^{\,2}+\mu^2)/s^{1/2},\ s^{1/2},\ 0,\ 0)\,,$$

and μ is the ϕ mass.

In lowest order, which is order g^4, the leading diagrams are those shown in Fig. 3. These diagrams are very similar to the photon-photon scattering diagrams calculated by Chang and Ma.[10] In fact, we use their method—the infinite-momentum technique—to calculate the $\phi\phi$ scattering amplitudes. The invariant transition amplitude can be written as

$$T = -i\int \frac{d^4k}{(2\pi)^4} A_{\mu\nu}B^{\mu\nu}[(k+\tfrac{1}{2}q)^2]^{-1}[(k-\tfrac{1}{2}q)^2]^{-1}\,. \tag{3.1}$$

$A_{\mu\nu}$ represents the upper half of a particular diagram in Fig. 3; $B_{\mu\nu}$ represents the lower half. To calculate (3.1), we must first calculate $A_{\mu\nu}$. The labeling of the internal momenta for the upper half of the diagrams in Fig. 3 is shown in Fig. 4. $A_{\mu\nu}$ for diagram (a) of Fig. 4 is

$$A^{(a)}_{\mu\nu} = \frac{g^2}{4}G^2\,\mathrm{Tr}\lambda_a\lambda_b\int\frac{d^4w}{(2\pi)^4}\mathrm{Tr}[(\not{w}-\tfrac{1}{2}\not{q}+m)\gamma_\mu(\not{w}+\not{k})\gamma_\nu(\not{w}+\tfrac{1}{2}\not{q}+m)(\not{w}-\not{p}+m)]$$
$$\times[(w-\tfrac{1}{2}q)^2-m^2+i\epsilon]^{-1}[(w+\tfrac{1}{2}q)^2-m^2+i\epsilon]^{-1}[(w-p)^2-m^2+i\epsilon]^{-1}[(w+k)^2-m^2+i\epsilon]^{-1}\,. \tag{3.2}$$

The loop momentum w contains a large p_+. To be more explicit, we change variables to w', where

$$w = (s^{1/2}w'_+,\ w'_-/s^{1/2},\ \vec{w}')\,.$$

Then, the terms in the denominator of (3.2) are

$$[w'_+w'_- - (\vec{w}-\tfrac{1}{2}\vec{q})^2 - m^2 + i\epsilon]^{-1}[w'_+w'_- - (\vec{w}+\tfrac{1}{2}\vec{q})^2 - m^2 + i\epsilon]^{-1}$$
$$\times[(w'_+-1)(w'_- - \tfrac{1}{4}\vec{q}^{\,2} - \mu^2) - \vec{w}^2 - m^2 + i\epsilon]^{-1}[s^{1/2}w'_+k_- - O(1) + i\epsilon]^{-1}\,.$$

We integrate over w'_- and get a nonzero result only for $0<w'_+<1$. In the limit $s\to\infty$, we can replace the terms which are $O(1)$ by a constant and not affect the leading logarithm. Since $w'_+>0$, we can remove w'_+ from the last term of the denominator without changing the sign of $i\epsilon$. Also, the leading behavior of the numerator is given by $\gamma_\mu=\gamma_\nu=\gamma_+$. Therefore, we can factor out of (3.2) all the dependence upon k.

$$A^{(a)}_{++} = -i\frac{g^2}{2}\frac{s^{1/2}}{\mu}\mathrm{Tr}\,\lambda_a\lambda_b\left[k_- - \frac{c}{s^{1/2}} + i\epsilon\right]^{-1} I(\vec{q},0)\,, \tag{3.3}$$

where

$$I(\vec{k}_1,\vec{k}_2)=4\pi G^2\mu(2\pi)^{-4}\int d^2w\int_0^1 d\alpha_1 d\alpha_2\,\delta(1-\alpha_1-\alpha_2)$$
$$\times\left[\frac{4m^2}{\alpha_1\alpha_2}-\frac{\vec{w}\cdot(\vec{w}-\vec{k}_2)+m^2}{\alpha_2{}^2}-\frac{(\vec{w}-\frac{1}{2}\vec{q})\cdot(\vec{w}+\vec{k}_1-\frac{1}{2}\vec{q})+m^2}{\alpha_1{}^2}\right.$$
$$\left.-\frac{\vec{w}\cdot(\vec{w}+\vec{k}_1-\frac{1}{2}\vec{q})+m^2}{\alpha_1\alpha_2}-\frac{(\vec{w}-\vec{k}_2)\cdot(\vec{w}-\frac{1}{2}\vec{q})+m^2}{\alpha_1\alpha_2}\right]$$
$$\times\left[\tfrac{1}{4}\vec{q}^2+\mu^2-\frac{(\vec{w}-\frac{1}{2}\vec{q})^2+m^2}{\alpha_1}-\frac{\vec{w}^2+m^2}{\alpha_2}\right]^{-1}$$
$$\times\left[\tfrac{1}{4}\vec{q}^2+\mu^2-\frac{(\vec{w}+\vec{k}_1-\frac{1}{2}\vec{q})^2+m^2}{\alpha_1}-\frac{(\vec{w}-\vec{k}_2)^2+m^2}{\alpha_2}\right]^{-1}. \tag{3.4}$$

$\alpha_1=w_+'$, and $\vec{k}_1$ is the transverse momentum attached to the forward-moving fermion line (quark) and $\vec{k}_2$ is the transverse momentum attached to the backward-moving fermion line (antiquark), and $\vec{k}_1+\vec{k}_2=\vec{q}$. $I(\vec{k}_1,\vec{k}_2)$ contains a spurious logarithmic divergence which is independent of the momenta $\vec{k}_1$ and $\vec{k}_2$. This divergence cancels when all the diagrams in Fig. 4 are added together.

Using the same procedure, we calculate $A_{\mu\nu}^{(b)}$ and $A_{\mu\nu}^{(c)}$ and get

$$A_{++}^{(b)}=-i\frac{g^2}{2}\frac{s^{1/2}}{\mu}\mathrm{Tr}\,\lambda_b\lambda_a\left[-k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}I(\vec{q},0),$$
$$A_{++}^{(c)}=i\frac{g^2}{2}\frac{s^{1/2}}{\mu}\mathrm{Tr}\,\lambda_a\lambda_b\left\{\left[k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}+\left[-k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}\right\}I(\vec{k}_1+\tfrac{1}{2}\vec{q},\tfrac{1}{2}\vec{q}-\vec{k}_1). \tag{3.5}$$

We add together Eqs. (3.3) and (3.5), and the result is

$$A_{++}=i\frac{g^2}{2}\frac{s^{1/2}}{\mu}\mathrm{Tr}\,\lambda_a\lambda_b\left\{\left[k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}+\left[-k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}\right\}I'(\vec{k}_1+\tfrac{1}{2}\vec{q},\tfrac{1}{2}\vec{q}-\vec{k}_1), \tag{3.6}$$

where $I'(\vec{k}_1,\vec{k}_2)=I(\vec{k}_1,\vec{k}_2)-I(\vec{k}_1+\vec{k}_2,0)$. The function $I'(\vec{k}_1,\vec{k}_2)$ does not contain the spurious logarithmic divergence.

Now, $A_{\mu\nu}B^{\mu\nu}=\frac{1}{4}A_{++}B^{--}+O(1)$, so the invariant transition amplitude T is

$$T=i\frac{g^4}{32}(2\pi)^{-4}\pi^2\frac{s}{\mu^2}\mathrm{Tr}\,\lambda_a\lambda_b\,\mathrm{Tr}\,\lambda_a\lambda_b$$
$$\times\int\frac{d^4k}{\pi^2}I'(\vec{k}+\tfrac{1}{2}\vec{q},\tfrac{1}{2}\vec{q}-\vec{k})I'(-\vec{k}-\tfrac{1}{2}\vec{q},\vec{k}-\tfrac{1}{2}\vec{q})\left\{\left[k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}+\left[-k_--\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}\right\}$$
$$\times\left\{\left[k_+-\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}+\left[-k_+-\frac{c}{s^{1/2}}+i\epsilon\right]^{-1}\right\}[(k+\tfrac{1}{2}q)^2+i\epsilon]^{-1}[(k-\tfrac{1}{2}q)^2+i\epsilon]^{-1}. \tag{3.7}$$

We have included a factor of $\frac{1}{2}$ in (3.7) because of double counting. We point out, that except for the functions $I'(\vec{k}+\frac{1}{2}\vec{q},\frac{1}{2}\vec{q}-\vec{k})$ and the SU(3) factor $\mathrm{Tr}\,\lambda_a\lambda_b\,\mathrm{Tr}\,\lambda_a\lambda_b$, the amplitude (3.7) is the same as that for the fermion-fermion, isospin-nonflip scattering amplitude. Therefore, we use the results of Sec. II to get

$$T=-2ig^4(2\pi)^{-4}\pi^3(s/\mu^2)\tilde{K}_1(t), \tag{3.8}$$

where

$$\tilde{K}_1(t)=\int\frac{d^2k}{\pi}I'(\vec{k}+\tfrac{1}{2}\vec{q},\tfrac{1}{2}\vec{q}-\vec{k})I'(-\vec{k}-\tfrac{1}{2}\vec{q},-\tfrac{1}{2}\vec{q}+\vec{k})[(\vec{k}+\tfrac{1}{2}\vec{q})^2]^{-1}[(\vec{k}-\tfrac{1}{2}\vec{q})^2]^{-1}. \tag{3.9}$$

In order g^6, the procedure is exactly the same. The non-Abelian diagrams, i.e., those containing 3-gluon and 4-gluon vertices are easily generalized. The result is

$$T=12ig^6(2\pi)^{-8}\pi^5(s/\mu^2)(-t)\tilde{K}_2(t)\ln s, \tag{3.10}$$

where

$$\tilde{K}_2(t)=\int\frac{d^2k_1d^2k_2}{\pi^2}I'(\vec{k}_1+\tfrac{1}{2}\vec{q},\tfrac{1}{2}\vec{q}-\vec{k}_1)I'(-\vec{k}_2-\tfrac{1}{2}\vec{q},-\tfrac{1}{2}\vec{q}+\vec{k}_2)$$
$$\times[(\vec{k}_1+\tfrac{1}{2}\vec{q})^2]^{-1}[(\vec{k}_1-\tfrac{1}{2}\vec{q})^2]^{-1}[(\vec{k}_2+\tfrac{1}{2}\vec{q})^2]^{-1}[(\vec{k}_2-\tfrac{1}{2}\vec{q})^2]^{-1}. \tag{3.11}$$

The QED-like diagrams are somewhat more complicated. Now, the photon-photon scattering amplitude in order g^6 is zero by Furry's theorem.[12] Because we are dealing with a non-Abelian gauge theory we get a nonzero result, namely,

$$T=-12ig^6(2\pi)^{-8}\pi^5(s/\mu^2)\tilde{J}(t)\ln s\,, \tag{3.12}$$

where

$$\tilde{J}(t)=\int\frac{d^2k_1d^2k_2}{\pi^2}[2I'(\vec{k}_1,\vec{q}-\vec{k}_1)I'(-\vec{k}_2,-\vec{q}+\vec{k}_2)-I'(\vec{k}_1+\vec{k}_2,\vec{q}-\vec{k}_1-\vec{k}_2)I'(-\vec{k}_1-\vec{k}_2,-\vec{q}+\vec{k}_1+\vec{k}_2)]$$
$$\times[\vec{k}_1^{\,2}]^{-1}[\vec{k}_2^{\,2}]^{-1}[(\vec{q}_1-\vec{k}_1-\vec{k}_2)^2]^{-1}. \tag{3.13}$$

Each term in $\tilde{J}(t)$ is separately infrared divergent. However, taken together, the two terms conspire to make $\tilde{J}(t)$ infrared finite.

We combine equations (3.8), (3.10), and (3.12) to obtain the leading $\phi\phi$ scattering amplitude to order g^6. The result is

$$T_{\phi\phi}=-2ig^4(2\pi)^{-4}\pi^3\frac{s}{\mu^2}\left\{\tilde{K}_1(t)-\frac{3g^2}{8\pi^2}\ln s\,[(-t)\tilde{K}_2(t)-\tilde{J}(t)]\right\}. \tag{3.14}$$

IV. CONCLUSIONS AND SUMMARY

The most important feature of the result (3.14) is that it is finite in the limit $t\to 0$. This implies, using the optical theorem, that the total cross section for $\phi\phi$ scattering is finite. To calculate the total cross section, we need to know the behavior of the functions $\tilde{K}_1(t)$, $\tilde{J}(t)$, and $(-t)\tilde{K}_2(t)$ in the limit $t\to 0$.

Now, $\tilde{K}_2(t)\sim\ln^2(\mu^2/-t)$ in the limit $-t\ll\mu^2$, so that $(-t)\tilde{K}_2(t)\to 0$ as $t\to 0$. It is interesting that the piece of the $\phi\phi$ scattering amplitude which comes from adding all the non-Abelian-type diagrams does not contribute to the total cross section. Whether this feature will persist in higher orders remains to be seen. The functions $\tilde{K}_1(t)$ and $\tilde{J}(t)$, which come from adding together all the QED-like diagrams, are nonzero in the limit $t\to 0$,

$$\tilde{K}_1(t=0)=8G^4(2\pi)^{-8}\pi^4N_1(b)\,, \tag{4.1}$$

$$\tilde{J}(t=0)=-8G^4(2\pi)^{-8}\pi^4N_2(b)\,, \tag{4.2}$$

where $b=(\mu^2-4m^2)^{1/2}/\mu$. The positive functions $N_1(b)$ and $N_2(b)$ are extremely complicated integrals which have been calculated numerically for $b=\frac{1}{2}$: $N_1(\frac{1}{2})\approx 8.9$, $N_2(\frac{1}{2})\approx 23.7$. Both functions diverge when $b=0$, i.e., when the fermion lines are on their mass shells.

The total cross section for $\phi\phi$ scattering is

$$\sigma_{\phi\phi}=64g^4G^4(2\pi)^{-12}\pi^7\left[N_1(b)-\frac{3g^2}{8\pi^2}\ln s\,N_2(b)\right]. \tag{4.3}$$

The non-Abelian character of the theory manifests itself through the noncommutivity of the SU(3) matrices λ_a, producing the ln$s$ factor in sixth order. Therefore, the total cross section for the high-energy scattering of two color singlets is proportional to a constant with a correction that decreases like ln$s$. Of course, at this point, it is impossible to tell how the cross section behaves in higher orders. In particular, we cannot tell whether the cross section exponentiates; this would yield a total cross section which decreases like a small power of s. In contrast, the data for high-energy pp scattering shows a constant total cross section with a correction that seems to rise like ln$s$.[13]

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