PHYSICAL REVIEW D VOLUME 15, NUMBER 6 15 MARCH 1977

Invariants and classification of Yang-Mills fields

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(Received 9 December 1976)

We present a polynomial basis for the algebraic invariants of Yang-Mills fields. We also study the asymptotic behavior of pure Yang-Mills fields and the eigenvector structure of matrices constructed from the fields. Based on these results we present a classification of Yang-Mills fields. We also analyze the form of the radiation part of the Yang-Mills field.

I. INTRODUCTION

Yang-Mills fields are beginning to have considerable significance in particle physics, and it is important to know as much about them as possible. The closest familiar analog to Yang-Mills fields, or non-Abelian gauge fields, is the Maxwell field, and one would like to understand how to generalize well-known results for the Maxwell case to the Yang-Mills situation. In the Maxwell theory, the two fundamental algebraic invariants, $F_{\mu\nu}F^{\nu\mu}$ and $F_{\mu\nu}F^{*\nu\mu}$, are of great help[1] in analyzing the structure and behavior of the field $F_{\mu\nu}$. For example, for bounded sources, the asymptotic part of the field (the radiation field) is simply characterized by the vanishing of the two invariants. The classification of Maxwell fields is based on the behavior of these invariants. It can also be based on the number of eigenvectors of $F_\mu{}^\nu$. While this classification has not been of crucial significance for electromagnetic theory, the analogous classification in general relativity (Petrov classification) was historically of utmost importance in understanding what constituted radiation in curved space.[2]

For the Yang-Mills field, where the gauge group is SU(2) or O(3), we show that there are 9 independent invariants, rather than 2. These are scalars under both Lorentz and gauge (internal-symmetry) transformations. We present a polynomial basis for these invariants. Then we study the asymptotic behavior of pure Yang-Mills fields and are led to a three-fold classification. We show how to characterize these three types of fields by the behavior of the invariants and by the eigenvector structure of associated matrices.

Eguchi[3] has proposed a classification of Yang-Mills fields based on the maximum number of fields $F^i_{\mu\nu}$ which can simultaneously be made radiative by some gauge transformation. While this is of course a possible classification, it treats the Lorentz group and gauge groups on a very different footing. It does not ask whether there is a Lorentz frame in which the fields satisfy some simple property under the gauge group. In the present approach, we focus on the invariants under the direct product of Lorentz and gauge groups and so both groups are treated symmetrically. Moreover, the classification proposed here is connected to the asymptotic behavior of pure Yang-Mills fields, and to the eigenvector structure of matrices constructed from the fields. For these reasons, we feel that the scheme proposed in this paper is the natural generalization of the traditional classification of Maxwell fields.

One of the crucial questions in non-Abelian gauge theories is the large-distance behavior, the confinement problem. Perhaps the present analysis will provide a useful framework in which to cast discussion of this problem. In the accompanying paper, it is shown that the vanishing or nonvanishing of one of the Yang-Mills invariants is crucial in deciding whether the Yang-Mills fields $F^i_{\mu\nu}$ suffice to uniquely determine the Yang-Mills potentials A^i_μ.

The detailed outline of the paper is as follows: In Sec. II, we review the situation for the Maxwell field, outlining several equivalent ways of classifying Maxwell fields and showing how this classification depends on the invariants. In Sec. III, we study the Yang-Mills invariants and present a polynomial basis for them. In Sec. IV, we use the analogy to the Maxwell case to explore several classification schemes for Yang-Mills fields. These different classifications do not turn out to be identical, but they are compatible. In Sec. V, we study the structure of what seems to be the natural Yang-Mills generalization of the Maxwell radiation field. The appendixes contain details of calculations referred to in the text. The entire analysis is confined to the simplest non-Abelian group, namely SU(2) or O(3).

II. REVIEW OF THE MAXWELL FIELD

Given a Maxwell field at a point x,

$$F_{\mu\nu}(x)=\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & H_z & -H_y\\ E_y & -H_z & 0 & H_x\\ E_z & H_y & -H_x & 0\end{pmatrix}, \tag{2.1}$$

it is well known that there are only two independent invariants[1]

$$F_{\mu\nu}F^{\nu\mu} \equiv 2(E^2 - H^2) \tag{2.2}$$

and

$$F_{\mu\nu}F^{*\nu\mu} = 4\vec{E}\cdot\vec{H}\ , \tag{2.3}$$

where

$$F^{*\nu\mu} = \tfrac{1}{2}\epsilon^{\nu\mu\rho\sigma}F_{\rho\sigma}, \quad \epsilon^{0123} = 1\ . \tag{2.4}$$

Any other Lorentz-invariant polynomial in $F_{\mu\nu}$ is a polynomial in these invariants. As long as both invariants do not vanish simultaneously they completely characterize the field $F_{\mu\nu}$ at a point, up to Lorentz transformation. If both invariants vanish, one has to distinguish between radiation fields and the case when $F_{\mu\nu} \equiv 0$.

Since the invariants essentially characterize the field up to Lorentz transformations, one might classify Maxwell fields by the value of their invariants. That would give us a double infinity of fields. In practice, we only distinguish two kinds: a radiation field, and everything else. The radiation field is characterized by the vanishing of both invariants. There are at least two ways to arrive at this rather gross classification: one by considering the asymptotic behavior of Maxwell fields in space, the other by studying the eigenvector structure of $F_{\mu\nu}$ or a related matrix. Because we wish to study the Yang-Mills system in analogy to the Maxwell field, we explain these briefly.

A. Asymptotics of the Maxwell field

Suppose one has sources bounded in space. Then one can show rather generally that the solutions of Maxwell's equations far away from the sources

$$\begin{aligned} F^{\mu\nu}{}_{,\nu} &= 0\ , \\ F^{*\mu\nu}{}_{,\nu} &= 0, \end{aligned} \tag{2.5}$$

which satisfy the outgoing-wave boundary condition, can be written for large r as

$$F_{\mu\nu}(r,t,\theta,\phi) \sim \frac{F^1_{\mu\nu}(r-t,\theta,\phi)}{r} + \frac{F^2_{\mu\nu}(r-t,\theta,\phi)}{r^2} + O\left(\frac{1}{r^3}\right)\ , \tag{2.6}$$

and one finds that the invariants constructed from the asymptotic field $F^1_{\mu\nu}/r$ vanish, while those from the field $F^1_{\mu\nu}/r + F^2_{\mu\nu}/r^2$ can take on any value. Thus the classification of asymptotic fields by invariants yields the two types—radiation and everything else.

B. Eigenvectors of the Maxwell field

Consider the eigenvector equation

$$F_\mu{}^\nu \eta_\nu = \lambda\eta_\mu\ . \tag{2.7}$$

The associated secular equation is

$$\lambda^4 - \lambda^2(E^2 - H^2) - (\vec{E}\cdot\vec{H})^2 = 0. \tag{2.8}$$

One finds that unless both invariants vanish, (2.7) gives rise to 4 linearly independent eigenvectors η_ν (not all real). However, if both invariants vanish and $F_\mu{}^\nu$ does not vanish, then there are only 2 linearly independent eigenvectors. Thus a classification of F by the number of linearly independent eigenvectors again yields the two types of fields—radiation and everything else.

There is also an attractive alternate approach to the eigenvector problem, called the spinor formalism.[4] Define the matrices

$$\sigma^0_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.9}$$

$$\sigma^1_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.10}$$

$$\sigma^2_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.11}$$

$$\sigma^3_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{2.12}$$

Then define

$$\psi_{AB,\,A'B'} = \sigma^\mu_{AA'}\sigma^\nu_{BB'}F_{\mu\nu}\ . \tag{2.13}$$

Under Lorentz transformations, the unprimed indices transform under SL(2, C), while the primed indices transform under the complex conjugate. From the antisymmetry and reality of $F_{\mu\nu}$ one shows that

$$\psi_{AB,A'B'} = \epsilon_{AB}\phi_{A'B'} + \epsilon_{A'B'}\overline{\phi}_{AB}\ , \tag{2.14}$$

where ϕ is a complex symmetric tensor, ϵ is the invariant antisymmetric tensor

$$\epsilon_{00} = \epsilon_{11} = 0, \quad \epsilon_{01} = -\epsilon_{10} = 1\ , \tag{2.15}$$

and $\overline{\phi}$ denotes complex conjugate of ϕ. $F_{\mu\nu}$ and ϕ_{AB} each carry the same information. It is easily shown that

$$\begin{aligned} \det\phi \equiv \phi_{00}\phi_{11} - \phi_{01}{}^2 &= -\tfrac{1}{4}(\vec{E} + i\vec{H})\cdot(\vec{E} + i\vec{H}) \\ &= -\tfrac{1}{4}(\vec{E}^2 - \vec{H}^2 + 2i\,\vec{E}\cdot\vec{H})\ . \end{aligned} \tag{2.16}$$

Thus $\det\phi$ contains both invariants and vanishes only for a radiation field. This provides us with an elegant classification of asymptotic fields. For if we write

$$\phi_{AB} = \frac{\phi^1_{AB}}{r} + \frac{\phi^2_{AB}}{r^2} + \cdots\ , \tag{2.17}$$

then ϕ^1 is a radiation field, and so

$$\det\phi^1 = 0\ . \tag{2.18}$$

But ϕ^1 is a 2×2 matrix, and so its determinant vanishes only if it has rank 1 (assuming $\phi^1 \neq 0$).

By symmetry,

$$\phi^1_{AB} = \tau_A \tau_B \tag{2.19}$$

for some spinor τ. One can then easily show that in general

$$\phi^2_{AB} = \tau_A \eta_B + \tau_B \eta_A \tag{2.20}$$

for some spinor η. The fact that the spinor τ appearing in (2.20) is the same as the spinor appearing in (2.19) is an example of the peeling theorem.[5] Since any 2×2 symmetric matrix can be written as in (2.20), ϕ^2 is not algebraically special. Now consider the eigenvalue problem

$$\phi_{AB}\eta^B = \lambda \eta_A \,, \tag{2.21}$$

where

$$\eta^B = \epsilon^{BC}\eta_C, \ \ \epsilon^{01} = -\epsilon^{10} = 1, \ \ \epsilon^{00} = \epsilon^{11} = 0 \,. \tag{2.22}$$

Again λ is Lorentz invariant. The secular equation is

$$\lambda^2 + \det\phi = 0 \,. \tag{2.23}$$

$\det\phi$ vanishes only for the radiation case, and it then follows from (2.19) that ϕ has only one linearly independent eigenvector τ. Otherwise it has two. Thus whether we study the eigenvectors of $F_{\mu\nu}$ or of ϕ_{AB}, we are led to the dichotomous classification—radiation and everything else.

III. INVARIANTS OF THE YANG-MILLS FIELDS

Suppose we are given the Yang-Mills fields $F^i_{\mu\nu}(x)$ at a point x. For each i, $F^i_{\mu\nu}$ is an antisymmetric tensor with respect to the Lorentz group L. For fixed $\mu\nu$, $F^i_{\mu\nu}$ transforms according to the adjoint representation of the Yang-Mills (gauge) group. In this paper, we assume that the gauge group is SU(2) or O(3), so that F^i transforms like the $J=1$ representation of the rotation group.

An invariant is a function of the $F^i_{\mu\nu}$, which is a scalar under $\mathrm{L}\times\mathrm{O}(3)$. No explicit reference to Yang-Mills potentials A^i_μ is permitted, nor are derivatives. This is in analogy to the Maxwell case, where for example

$$(\partial_\mu F^{\rho\sigma})(\partial^\mu F_{\rho\sigma})$$

is certainly both Lorentz invariant and gauge invariant, but is not considered when classifying "the invariants" of the Maxwell field. I will show that there are 9 independent invariants.

First, a heuristic argument. There are 18 independent components $F^i_{\mu\nu}$. The Lorentz group L has 6 parameters and the O(3) group has 3. So one could perhaps choose a Lorentz frame and O(3) frame in which $9=6+3$ of the components vanished. There would then be $9=18-9$ remaining components. Any invariant could be evaluated in this special frame, and therefore be a function of these 9 components.

This argument shows that there are at least 9 independent invariants. There might be more if one could show that, for any $F^i_{\mu\nu}$, there was a nontrivial subgroup of the group $\mathrm{L}\times\mathrm{O}(3)$ which left F invariant. The general theorem is[6]

No. of invariants = No. of components
− dimension of group
+ dimension of little group.

This is what happens in the Maxwell field. There $F_{\mu\nu}$ has 6 components, L has 6 dimensions, and the little group has dimension 2. For example, if $F_{\mu\nu}$ is not radiative, there is a frame in which $E_x \neq 0$, $H_x \neq 0$, and all other components vanish. But this F is invariant under Lorentz transformations along x and rotations around x, which form a group of dimension 2.

Returning to the Yang-Mills case, instead of showing that the little group is trivial, I will give a more constructive proof.[7] Given the arbitrary components $F^i_{\mu\nu}$, one can to go a Lorentz frame where $\vec{E}^1$ and $\vec{H}^1$ point along the x axis. (I am assuming that there are no relations among the $F^i_{\mu\nu}$, and so F^1 is not a radiation field.) Then by a combination of rotations around x and Lorentz transformations along x, one can make $H^2_y = E^2_y = 0$. Then $F^i_{\mu\nu}$ has only 12 independent components in this frame. Now consider the O(3) tensors

$$K_{ij} = \tfrac{1}{2}F^i_{\mu\nu}F^{j\nu\mu} = \vec{E}^i\cdot\vec{E}^j - \vec{H}^i\cdot\vec{H}^j \,, \tag{3.1}$$

$$J_{ij} = \tfrac{1}{2}F^i_{\mu\nu}F^{*j\nu\mu} = \vec{E}\cdot\vec{H}^j + \vec{E}^j\cdot\vec{H}^i . \tag{3.2}$$

Clearly both J and K are Lorentz scalars and, under an O(3) transformation O, they transform as

$$J \to OJO^T \,, \tag{3.3}$$

$$K \to OKO^T \,. \tag{3.4}$$

But any real symmetric matrix can be diagonalized by a real orthogonal matrix. Therefore, one can choose O so that $J_{12}=J_{13}=J_{23}=0$. This imposes 3 more (independent) conditions on the 12 components of $F^i_{\mu\nu}$ defined above, and so there are no more than 9 invariants. Since there are at least 9 invariants, there are exactly 9.

The 9 invariants could be taken to be the independent O(3) scalars one can form from J and K, namely

$$\mathrm{Tr}(J),\ \mathrm{Tr}(J^2),\ \det(J),\ \mathrm{Tr}(K),\ \mathrm{Tr}(K^2),$$
$$\det(K),\ \mathrm{Tr}(JK),\ \mathrm{Tr}(J^2K),\ \mathrm{Tr}(JK^2)\,.$$

Here J and K are considered as 3×3 matrices. One easily shows that these are independent by showing that when J is diagonal, it is possible to find a configuration of the fields $F^i_{\mu\nu}$ such that any one of the components $\{J_{ii}, K_{ij}\}$ can be chosen nonzero while the remaining 8 vanish.

However, these invariants do not form a polynomial basis. For example, consider the invariants

$$t = \tfrac{1}{6}\epsilon_{ijk}(F^{i\nu}_{\mu} F^{j\rho}_{\nu} F^{k\mu}_{\rho}), \tag{3.5}$$

$$t' = -\tfrac{1}{6}\epsilon_{ijk}(F^{*i\nu}_{\mu} F^{*j\rho}_{\nu} F^{*k\mu}_{\rho}) . \tag{3.6}$$

These are invariant under $L \times O(3)$. But since they are odd under $F^i \to -F^i$, they are not polynomials in the previous invariants which are all even under $F^i \to -F^i$. However, one can check that

$$2t \cdot t' = \det(J) - \mathrm{Tr}(K^2 J) + \mathrm{Tr}(K)\mathrm{Tr}(JK) + \tfrac{1}{2}\mathrm{Tr}(J)\mathrm{Tr}(K^2) - \tfrac{1}{2}\mathrm{Tr}(J)[\mathrm{Tr}(K)]^2, \tag{3.7}$$

$$t'^2 - t^2 = \det(K) - \mathrm{Tr}(J^2 K) + \mathrm{Tr}(J)\mathrm{Tr}(JK) + \tfrac{1}{2}\mathrm{Tr}(K)\mathrm{Tr}(J^2) - \tfrac{1}{2}\mathrm{Tr}(K)[\mathrm{Tr}(J)]^2 . \tag{3.8}$$

Consequently one can take the following as independent invariants:

$$\mathrm{Tr}(J),\ \mathrm{Tr}(J^2),\ \det(J),\ \mathrm{Tr}(K),\ \mathrm{Tr}(K^2),\ \det(K),\ \mathrm{Tr}(JK),\ t,\ t'. \tag{3.9}$$

We now show that these are in fact a polynomial basis for the invariants. The proof is by exhaustion. The only invariant tensors under L are $\epsilon_{\mu\nu\rho\sigma}$ and $g_{\mu\nu}$. The invariant tensors under O(3) are ϵ_{ijk} and δ_{ij}. Thus the only quadratic invariants must be

$$F^i_{\mu\nu} F^j_{\rho\sigma} \times \begin{cases} \delta_{ij}\, g^{\mu\rho} g^{\nu\sigma} , \\ \delta_{ij}\, \epsilon^{\mu\nu\rho\sigma} , \end{cases} \tag{3.10}$$

which are multiples of $\mathrm{Tr}(K)$ and $\mathrm{Tr}(J)$, respectively. The only cubic invariants must be of the form

$$F^i_{\mu\nu} F^j_{\rho\sigma} F^k_{\alpha\beta} \epsilon_{ijk} \times \begin{cases} g^{\nu\rho} g^{\sigma\alpha} g^{\beta\mu}, \\ \epsilon^{\nu\rho\sigma\alpha}\, g^{\beta\mu} . \end{cases} \tag{3.11}$$

These are multiples of t and t'.

A more tedious argument, relegated to Appendix A, shows that there are no fourth-order invariants besides those we have already enumerated.

There are no new invariants of degree 5. For if there were, one of the remaining invariants of degree 6 could be eliminated in terms of this one. But that would require the existence of an invariant of degree 1, which does not exist. Consequently the invariants listed in (3.9) form the polynomial basis.

IV. CLASSIFICATION OF YANG-MILLS FIELDS

Having found the invariants, we could decide to classify the Yang-Mills fields $F^i_{\mu\nu}$ by the values of the invariants. This would give rise to a ninefold infinity of inequivalent fields. To find a more manageable classification, we pursue the analogy with the Maxwell field. As shown in Sec. II, there were at least two different ways to arrive at the traditional classification: the asymptotic approach, and the eigenvector approach. As it so happened, these two classifications agreed. This need not have been the case, and will not be the case here. But we will show the compatibility of the two approaches. Asymptotic consideration will lead to a classification of Yang-Mills fields into 3 types. We will show that the eigenvector structure is different for each of these three types. But there is no reason to believe that these 3 types exhaust the possible eigenvector structures.

A. Asymptotic of Yang-Mills fields

In studying the asymptotic solutions to Maxwell's equations, we showed that the $1/r$ part of the field was algebraically special, that is, both invariants vanished. There were no algebraic constraints on the $1/r + 1/r^2$ parts of the field. This led to a twofold classification of Maxwell fields into radiation field (both invariants vanish) and everything else. In this section, we perform the analogous analysis for Yang-Mills fields. We find that both the fields to order $1/r$ and to order $1/r^2$ are algebraically special, while there is no constraint on the fields to order $1/r^3$. This leads to a three-fold classification of Yang-Mills fields:

type I: all invariants vanish;
type II: the invariants satisfy

$$\begin{aligned} &\det(J) = \det(K) = t = t' = 0, \\ &\mathrm{Tr}(JK) = \mathrm{Tr}(J)\mathrm{Tr}(K), \\ &\mathrm{Tr}(K^2) - [\mathrm{Tr}(K)]^2 = \mathrm{Tr}(J^2) - [\mathrm{Tr}(J)]^2 , \end{aligned} \tag{4.1}$$

while $\mathrm{Tr}(J)$, $\mathrm{Tr}(K)$, $\mathrm{Tr}(J^2)$ are arbitrary;
type III: everything else.

By analogy to the Maxwell case, we are inclined to call fields of type I radiation fields.

The calculations follow. Let us define the Yang-Mills potential A^i_μ such that the Yang-Mills fields $F^i_{\mu\nu}$ are given by

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon_{ijk} A^j_\mu A^k_\nu . \tag{4.2}$$

The free-field equations (valid asymptotically) are

$$\partial^\mu F^i_{\nu\mu} + g\epsilon_{ijk} A^{j\mu} F^k_{\nu\mu} = 0. \tag{4.3}$$

We make the ansatz analogous to the Maxwell case for outgoing waves

$$A^i_\mu = \frac{B^i_\mu(u,\theta,\alpha)}{r} + \frac{C^i_\mu(u,\theta,\alpha)}{r^2} + \frac{D^i_\mu(u,\theta,\alpha)}{r^3} + O\left(\frac{1}{r^4}\right) \tag{4.4}$$

$(u = r - t)$.

For simplicity, we choose the gauge so that $A_0^i = 0$. This is always possible.[8] Notice that since A_μ^i appears in the field equation, we are forced to make the ansatz for A_μ^i and not directly for $F_{\mu\nu}^i$. We now substitute (4.4) into (4.2), and both into (4.3). We exclude solutions independent of u since they do not distinguish between incoming and outgoing waves. Expressing the vectors in a spherical basis, we find after some straightforward but tedious algebra:

1. Order 1/r.

$$B_r^i(u, \theta, \alpha) = 0 , \quad B_\theta^i, B_\phi^i \text{ arbitrary.} \tag{4.5}$$

This implies

$$E_\theta^i = H_\phi^i , \quad E_\phi^i = -H_\theta^i , \quad E_r^i = H_r^i = 0 . \tag{4.6}$$

Computing the invariants listed in (3.9), we find that they all vanish.

2. Order 1/r².

$$\frac{\partial^2}{\partial u^2} C_r^i + \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial u}B_\theta^i + \frac{\partial}{\partial\phi}\frac{\partial}{\partial u}B_\phi^i\right) - g\epsilon_{ijk}\left(B_\theta^k\frac{\partial}{\partial u}B_\theta^j + B_\phi^k\frac{\partial}{\partial u}B_\phi^j\right) = 0, \tag{4.7}$$

C_θ^i, C_ϕ^i arbitrary.

Then to order $1/r^2$, we find

$$E_\theta^i = H_\phi^i, \quad E_\phi^i = -H_\theta^i, \quad E_r^i = \frac{1}{r^2}\frac{\partial}{\partial u} C_r^i , \quad H_r^i = \frac{1}{r^2}\left[\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta B_\phi^i - \frac{\partial}{\partial\phi}B_\theta^i\right) + g\epsilon_{ijk}B_\theta^j B_\phi^k\right]. \tag{4.8}$$

Since only $(\partial^2/\partial u^2)\, C_r^i$ is determined by the differential equation, we find that in general $E_r^i \neq H_r^i \neq 0$.

Computing the invariants, we find that the relations are given by (4.1).

If we compute $\vec{E}^i, \vec{H}^i$ to order $1/r^3$, and compute the invariants to order $1/r^4$, we find that they can take on any possible value.

This classification is easily understood using the spinor formalism. We write

$$\phi_{AB}^i = \frac{\phi_{AB}^{(1)i}}{r} + \frac{\phi_{AB}^{(2)i}}{r^2} + \frac{\phi_{AB}^{(3)i}}{r^3} + \cdots . \tag{4.9}$$

Since all the invariants vanish for the $1/r$ part of the field, we shall show in Sec. V that

$$\phi_{AB}^{(1)i} = \xi^i \tau_A \tau_B , \tag{4.10}$$

where ξ^i is a complex O(3) vector, whereas τ is a spinor independent of i. Then in agreement with the peeling theorem[9]

$$\phi_{AB}^{(2)i} = \tau_A \eta_B^i + \tau_B \eta_A^i \tag{4.11}$$

for some arbitrary spinors η^i. But this is not algebraically the most general case, since τ is still independent of i. This form leads to the relations (4.1) between the invariants.

B. Eigenvector analysis

The second way to classify Maxwell fields was by studying the number of eigenvectors of $F_\mu{}^\nu$ or of the associated matrix in the spinor formalism $\phi_A{}^B$. In applying the eigenvector approach to the Yang-Mills field, we are immediately faced with a problem. Which matrix depending on $F_{\mu\nu}^i$ shall we study? The problem is to convert the single index i into a pair of matrix indices in such a way that the resulting matrix has eigenvalues which are Yang-Mills invariants. The simplest way to do this is to define

$$M_{\alpha\mu,\,\beta\nu} = \sum_i F_\mu^{i\nu} D(T^i)_{\alpha\beta} , \tag{4.12}$$

where $D(T^i)$ is a representation of the Lie algebra of the gauge group, in this case O(3) or SU(2). An O(3) transformation O of F^i simply becomes a similarity transform of M by the matrix representative of O in the representation considered. Then the eigenvalues of such a transformed M will coincide with the eigenvalues of the untransformed M. For simplicity, we have considered only the spin-$\frac{1}{2}$ and spin-1 representations of O(3). In each of these cases, we have studied both the usual Lorentz tensor formalism and the spinor formalism obtained by writing

$$\phi_{\alpha A,\,\beta B} = \sum_i \phi_A^{iB} D(T^i)_{\alpha\beta} . \tag{4.13}$$

In increasing order of dimension, the matrices are

$$M^{(4)}_{\alpha A,\,\beta B} = \sum_k (\phi^k)_A{}^B (\sigma^k)_{\alpha\beta} , \tag{4.14}$$

$$M^{(6)}_{iA,\,jB} = \sum_k (\phi^k)_A{}^B \epsilon_{ijk} , \tag{4.15}$$

$$M^{(8)}_{\alpha\mu,\,\beta\nu} = \sum_k (F^k)_\mu{}^\nu (\sigma^k)_{\alpha\beta} , \tag{4.16}$$

$$M^{(12)}_{i\mu,\,j\nu} = \sum_k (F^k)_\mu{}^\nu \epsilon_{ijk} . \tag{4.17}$$

The index of M indicates its dimension as a square matrix. It is easy to verify that eigenvalues of M

TABLE I. The secular equations corresponding to the matrices in Eq. (4.18).

Dimension	Secular equation
4	$\lambda^4 + a_2\lambda^2 + a_1\lambda + a_0 = 0$
	$a_2 = -\frac{1}{2}[\mathrm{Tr}(K) + i\,\mathrm{Tr}(J)]$
	$a_1 = -i(t + it')$
	$a_0 = \frac{1}{8}[\mathrm{Tr}(K^2) - \mathrm{Tr}(J^2) + 2i\,\mathrm{Tr}(JK)] - \frac{1}{16}[\mathrm{Tr}(K) + i\,\mathrm{Tr}(J)]^2$
6	$\lambda^6 - 2a\lambda^4 - 2b\lambda^3 + a^2\lambda^2 + 2ab\lambda + b^2 = 0$
	$a = -\frac{1}{4}[\mathrm{Tr}(K) + i\,\mathrm{Tr}(J)]$
	$b = -\frac{1}{4}(t + it')$
8	$\lambda^8 - 2c\lambda^6 - 4d\lambda^5 + \lambda^4(c^2 - 2e) + 4cd\lambda^3 + \lambda^2(4d^2 + 2ce) + 4de\lambda + e^2 = 0$
	$c = \mathrm{Tr}(K)$
	$d = -it$
	$e = \frac{1}{2}\{\mathrm{Tr}(J^2) - \frac{1}{2}[\mathrm{Tr}(J)]^2\}$
12	$\sum_{n=0}^{12} C_n\lambda^n = 0$
	$C_{12} = 1,\ C_{11} = 0,\ C_{10} = 2\mathrm{Tr}(K),\ C_9 = 2t$
	$C_8 = -\frac{1}{2}\mathrm{Tr}(J^2) - \frac{1}{2}K_2 + [\mathrm{Tr}(K)]^2$
	$C_7 = 2\mathrm{Tr}(K)t - \mathrm{Tr}(J)t'$
	$C_6 = \det(K) - 2t'^2 - \frac{1}{2}\mathrm{Tr}(K)[\mathrm{Tr}(J^2) + K_2]$
	$C_5 = -t'\,\mathrm{Tr}(JK) - t\{K_2 + \frac{1}{2}[\mathrm{Tr}(J)]^2\}$
	$C_4 = \mathrm{Tr}(K)[\det(K) - t'^2] - t't\,\mathrm{Tr}(J)$
	$\quad + \frac{1}{4}\{K_2 + \frac{1}{2}[\mathrm{Tr}(J)]^2\}\{\mathrm{Tr}(J^2) - \frac{1}{2}[\mathrm{Tr}(J)]^2\}$
	$\quad - \frac{1}{4}[\mathrm{Tr}(JK) - \mathrm{Tr}(J)\mathrm{Tr}(K)]^2$
	$C_3 = t'(\det(J) + \frac{1}{2}\mathrm{Tr}(J)\{\mathrm{Tr}(J^2) - \frac{1}{2}[\mathrm{Tr}(J)]^2\})$
	$\quad + 2t\{\det(K) - t'^2 + \frac{1}{4}\mathrm{Tr}(J)[\mathrm{Tr}(JK) - \mathrm{Tr}(J)\mathrm{Tr}(K)]\}$
	$C_2 = \frac{1}{2}\det(J)[\mathrm{Tr}(JK) - \mathrm{Tr}(J)\mathrm{Tr}(K)]$
	$\quad + \frac{1}{2}[t'^2 - \det(K)]\{\mathrm{Tr}(J^2) - \frac{1}{2}[\mathrm{Tr}(J)]^2\} - \frac{1}{4}[\mathrm{Tr}(J)t]^2$
	$C_1 = -\frac{1}{2}\det(J)\mathrm{Tr}(J)t$
	$C_0 = \frac{1}{4}[\det(J)]^2$
	$K_2 \equiv \mathrm{Tr}(K^2) - [\mathrm{Tr}(K)]^2$

are invariant under L×O(3), since these are implemented as similarity transformations on M.

For each dimension, we are looking for solutions of the matrix equation

$$M^{(n)}\eta = \lambda\eta; \qquad (4.18)$$

e.g., for dimension 4,

$$\sum_{\beta B} M^{(4)}_{\alpha A,\,\beta B}\,\eta_{\beta B} = \lambda\eta_{\alpha A}\ .$$

For each of these matrices, we tabulate the secular equation in Table I. The way these are obtained is explained in Appendix B. We will find that these secular equations for dimensions 4, 6, and 8 do not involve all the Yang-Mills invariants. This is not particularly surprising, since the secular equation is invariant under similarity transformations by the whole group GL(n, C), of which L×O(3) is but a subgroup. (n is the dimension of the matrix M.) Clearly the coefficients of the secular equations, which are the invariants under GL(n, C), are also L×O(3) invariant, but the converse is not true.

We see that the secular equation in the 4-dimensional formalism depends on 6 of the 9 Yang-Mills invariants. For the 6- and 8-dimensional formalisms, the number of invariants drops to 4 and 3, respectively. Only in the 12-dimensional formalism do all the invariants enter.[10] Consequently, if we use the eigenvectors to characterize the fields, then only in the 12-dimensional formalism can we hope to get a complete classification of the different types of Yang-Mills fields. Fortuitously, in the Maxwell case, both invariants entered into the 2- or the 4-dimensional formalism.

As yet, we do not have a complete classification of the number of possible eigenvectors in the 12-dimensional case. This matter is still under investigation. However, we can demonstrate that the 2 algebraically special types of fields found in IV A do have a different eigenvector structure. It is easy to take the solutions for the fields in IV A, insert them into the matrix $M^{(12)}$, and show the following:

For type-I fields, the secular equation is

$$\lambda^{12}=0, \tag{4.19}$$

and the equation $M\eta=0$ has 6 linearly independent eigenvectors.

For type-II fields, the secular equation is

$$\lambda^4(\lambda^4+\lambda^2\mathrm{Tr}(K)-\tfrac{1}{2}\{\mathrm{Tr}(J^2)-[\mathrm{Tr}(J)]^2/2\})^2=0. \tag{4.20}$$

The number of linearly independent eigenvectors[11] is 8, broken down as follows:

$\lambda=0$, 2 linearly independent eigenvectors

$$\lambda=\pm\left(\frac{\{[\mathrm{Tr}(K)]^2+2\mathrm{Tr}(J^2)-[\mathrm{Tr}(J)]^2\}^{1/2}-\mathrm{Tr}(K)}{2}\right)^{1/2},\ \text{2 linearly independent eigenvectors each}$$

$$\lambda=\pm i\left(\frac{\{[\mathrm{Tr}(K)]^2+2\mathrm{Tr}(J^2)-[\mathrm{Tr}(J)]^2\}^{1/2}+\mathrm{Tr}(K)}{2}\right)^{1/2},\ \text{1 linearly independent eigenvector each.}$$

For a general type-III field, one expects 12 distinct eigenvalues and therefore 12 eigenvectors. Thus in the 12-dimensional formalism, types I, II, and III correspond to 6, 8, and 12 linearly independent eigenvectors, respectively.

Presumably, there are configurations of the invariants which lead to numbers of linearly independent eigenvectors different from 6, 8, and 12. We have not yet studied these. Anandan[12] has shown that in the 4-dimensional formalism there are field configurations leading to 1, 2, 3, or 4 eigenvectors. Thus our classification into 3 types of fields is not sufficiently fine. More on this will be published later.

V. GENERAL FORM OF RADIATION FIELDS

We have defined a radiation field to be one in which all the 9 Yang-Mills invariants defined in (3.9) vanish. We shall now show that the general form of such fields is

$$\vec{H}^i=\vec{n}\times\vec{E}^i, \tag{5.1}$$

$$\vec{E}^i\cdot\vec{n}=0, \tag{5.2}$$

where n is an arbitrary unit vector, *independent of* i.

Proof. From the vanishing of all invariants, it follows that the matrices J_{ij} and K_{ij} vanish, i.e.,

$$\vec{E}^i\cdot\vec{H}^j+\vec{E}^j\cdot\vec{H}^i=0, \tag{5.3}$$

$$\vec{E}^i\cdot\vec{E}^j-\vec{H}^i\cdot\vec{H}^j=0. \tag{5.4}$$

If we take $i=j$ above, this shows us that for each i, $(\vec{E}^i,\vec{H}^i)$ form a radiation field in the Maxwell sense. Thus, there are unit vectors $\vec{n}^i$ such that

$$\vec{H}^i=\vec{n}^i\times\vec{E}^i,\ \vec{E}^i\cdot\vec{n}^i=0. \tag{5.5}$$

We can assume that none of the E^i vanish, for if one does one can take $\vec{n}^i$ arbitrarily. It remains to show that $\vec{n}^1=\vec{n}^2=\vec{n}^3$.

Substituting (5.5) into (5.3) and (5.4) gives

$$(\vec{n}^j-\vec{n}^i)\cdot(\vec{E}^j\times\vec{E}^i)=0, \tag{5.3$'$}$$

$$(\vec{E}^i\cdot\vec{E}^j)(1-\vec{n}^i\cdot\vec{n}^j)=-(\vec{n}^i\cdot\vec{E}^j)(\vec{n}^j\cdot\vec{E}^i). \tag{5.4$'$}$$

Take $i=1$, $j=2$. Since (5.3′), (5.4′) are rotationally invariant, and the lengths of $\vec{E}^1$, $\vec{E}^2$ scale in both equations, we can choose

$$\vec{E}^1=\vec{k},$$

$$\vec{E}^2=\vec{k}\cos\phi+\vec{i}\sin\phi,$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along x, y, z, respectively. Then, since $\vec{n}_1\cdot\vec{E}_1=0$, $\vec{n}_2\cdot\vec{E}_2=0$,

$$\begin{aligned}\vec{n}_1&=\vec{i}\cos\theta+\vec{j}\sin\theta,\\ \vec{n}_2&=\vec{j}\sin\psi+(\vec{i}\cos\phi-\vec{k}\sin\phi)\cos\psi.\end{aligned} \tag{5.6}$$

Inserting these into (5.3′) and (5.4′) gives

$$\sin\phi(\sin\theta-\sin\psi)=0, \tag{5.7}$$

$$\cos\phi(1-\sin\theta\sin\psi)=\cos\theta\cos\psi. \tag{5.8}$$

From (5.7), either

$$\phi=0\ \text{or}\ \phi=\pi,\ \text{or}\ \sin\theta=\sin\psi \tag{5.9}$$

Inserting any of these into (5.8), we find $\vec{n}^1=\vec{n}^2$. Similarly $\vec{n}^3=\vec{n}^1$.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the help of my colleagues at the University of Pittsburgh, particularly Iwo Birula, Ted Newman, Jeeva Anandan, and Paul Tod.

I wish to thank the Aspen Center for Physics for their hospitality during the time some of this work was carried out.

APPENDIX A: THE FOURTH-ORDER INVARIANTS

We shall show that invariants of degree 4 are linear combinations of

$$\begin{aligned}&\mathrm{Tr}(J^2),\ \mathrm{Tr}(K^2),\ \mathrm{Tr}(JK),\ [\mathrm{Tr}(J)]^2,\\ &[\mathrm{Tr}(K)]^2,\ \mathrm{Tr}(J)\mathrm{Tr}(K)\,.\end{aligned} \tag{A1}$$

Any invariant of degree 4 must be of the form

$$F^i_{\mu\nu}F^j_{\rho\sigma}F^k_{\alpha\beta}F^l_{\gamma\delta}\delta_{ij}\delta_{kl}\,T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}, \tag{A2}$$

where T is a tensor built from $\epsilon^{\mu\nu\rho\sigma}$ and $g^{\mu\nu}$. Since a product of two ϵ's contracted over any index is a linear combination of g's, it suffices to consider T built out of two ϵ's with no repeated index, or one ϵ, or no ϵ's. Moreover, it is very useful to note that

$$F^{i\,*\rho}_{\mu}F^{i\sigma}_{\rho}\propto \mathrm{Tr}(J)g^\sigma_\mu\,, \tag{A3}$$

so that

$$F^i_{\mu\nu}F^i_{\rho\sigma}\epsilon^{\mu\nu\rho\kappa}\propto \mathrm{Tr}(J)g^\kappa_\sigma\,. \tag{A4}$$

Thus if, for example, $T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}$ has a factor $\epsilon^{\mu\nu\rho\kappa}$, that particular invariant is clearly a product of 2 quadratic invariants.

We now enumerate the possibilities for $T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}$.

a. Two factors of ϵ. We can ignore any ϵ containing 3 indices of $(\alpha\beta\gamma\delta)$ or 3 of $(\mu\nu\rho\sigma)$. The remaining possibilities are

$$T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}=\begin{cases}\epsilon^{\mu\rho\alpha\gamma}\,\epsilon^{\nu\sigma\beta\delta}\,,\\ \epsilon^{\mu\nu\alpha\beta}\,\epsilon^{\rho\sigma\gamma\delta},\\ \epsilon^{\mu\nu\alpha\gamma}\,\epsilon^{\rho\sigma\beta\delta}.\end{cases} \tag{A5}$$

b. One factor ϵ. The different possibilities are

$$T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}=\begin{cases}\epsilon^{\mu\rho\alpha\gamma}g^{\nu\beta}g^{\sigma\delta},\\ \epsilon^{\mu\nu\alpha\beta}\,g^{\rho\gamma}g^{\sigma\delta},\\ \epsilon^{\mu\nu\alpha\gamma}g^{\rho\beta}g^{\sigma\delta}\,.\end{cases} \tag{A6}$$

c. No factors of ϵ. T must be constructed completely from g's, which must link the pairs $(\mu\nu)$, $(\rho\sigma)$, $(\alpha\beta)$, $(\gamma\delta)$. That is,

$$T=g^{\mu\rho}g^{\nu\sigma}g^{\alpha\gamma}g^{\beta\delta},$$

which links $(\mu\nu)$ with $(\rho\sigma)$ but not with $(\alpha\beta)$ or $(\gamma\delta)$, can be ignored since it clearly leads to a product of 2 quadratic invariants. Therefore

$$T^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}=\begin{cases}g^{\mu\rho}g^{\sigma\alpha}g^{\beta\gamma}g^{\nu\delta}\,,\\ g^{\mu\alpha}g^{\nu\gamma}g^{\rho\beta}g^{\sigma\delta}.\end{cases} \tag{A7}$$

It is easy but tedious to verify that any of the choices in (A5)–(A7) leads to linear combinations of the terms in (A1).

APPENDIX B: COMPUTATION OF THE SECULAR DETERMINANTS

The computation of the secular determinants was carried out by computer. This was particularly necessary in the 12-dimensional formalism since the expressions were quite unwieldy. The approach was a hybrid of Fortran and symbolic calculation. For illustration, we concentrate on the 12-dimensional formalism.

The matrix $M^{(12)}$ is a 12×12 matrix, which by virtue of its symmetries has only 72 nonzero entries. Each of the entries consists of a single component $\pm F^{i\nu}_{\mu}$, each one appearing exactly 4 times. Since the secular equation is an invariant, it can be evaluated in any frame. I chose the frame in which E^1_y, E^1_z, H^1_y, H^1_z, E^2_y, H^2_y vanished. This left $M^{(12)}$ with only 48 nonzero entries in terms of the 12 remaining variables. I wrote a simple nonoptimized Fortran program which, for each power of λ, ran through the rows and columns and returned the list of terms in the naive definition of the determinant $|M-\lambda|$ in which none of factors was zero, together with the sign of the permutation for that particular term. This list was the input for the symbolic program[13] ASHMEDAI. There the expressions involving the 48 components were reduced to expressions involving 12 components. It was then a relatively simple trial-and-error process to express this invariant in terms of the fundamental set of invariants evaluated in the same frame.

As an indication of running times, the largest calculation was the coefficient of λ^2. The calculations were done on the University of Pittsburgh's DEC-10. The Fortran program returned 4800 nonzero terms in under a minute. The ASHMEDAI program reduced these to 1256 distinct terms in 23 minutes. The bulk of the time was spent reading in the 4800 terms and putting them away on disk to overcome storage problems. The calculation itself took less than 4 minutes. Finally, the reduction of this expression of 1256 terms to invariants was done interactively by looking at those terms with large powers of a particularly variable, and subtracting invariants which canceled those large powers.

*Work supported in part by the National Science Foundation under Grant No. MPS75-14073.
†Work supported in part by the Alfred P. Sloan Foundation.

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[6]I wish to thank S. Coleman for pointing this out to me.

[7]This argument was suggested by J. Anandan.

[8]See, e.g., S. Coleman, in Proceedings of the 13th International School of Subnuclear Physics, "Ettore Majorana," Erice, Italy, 1975 (unpublished).

[9]I wish to thank Paul Tod for pointing this out to me.

[10]Not only do all the invariants appear in the secular equation for the 12-dimensional formalism, but the coefficients of that equation allow us to reconstruct each invariant. Since there are 9 invariants and 11 nontrivial coefficients, there must be 2 relations between these 11 coefficients.

[11]For type II fields, there are two inequivalent sets of matrices J, K and J', K' which lead to the same value of the invariants, but which are *not* related by similarity transformation by an orthogonal matrix. There are 8 eigenvectors for the configuration of the fields defined in (4.8). Whether the fields which give rise to the other set (J', K') also yield 8 eigenvectors is not known at present.

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