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Energy-momentum-tensor trace anomaly in spin-1/2 quantum electrodynamics

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We relate the energy-momentum-tensor trace anomaly in spin-1/2 quantum electrodynamics to the functions $\beta(\alpha)$, $\delta(\alpha)$ defined through the Callan-Symanzik equations, and prove finiteness of $\theta_{\mu\nu}$ when the anomaly is taken into account.

I. INTRODUCTION

Spin-$\frac{1}{2}$ quantum electrodynamics, characterized by the Lagrangian density[1]

$$\mathcal{L}_{\rm inv}(x)=\bar{\psi}(x)(i\gamma\cdot\partial-m_0)\psi(x)-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)-e_0\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)\,, \tag{1.1}$$

is one of the simplest field theory models in which to study anomalies. The axial-vector divergence anomaly in this theory has been extensively analyzed[2]; we wish in this note to discuss some properties of the energy-momentum-tensor trace anomaly.[3] Taking for the energy-momentum tensor $\theta_{\mu\nu}$ the symmetric form

$$\begin{aligned}\theta_{\mu\nu}&=\theta^{e\gamma}_{\mu\nu}+\theta^{\gamma}_{\mu\nu}\,,\\ \theta^{\gamma}_{\mu\nu}&=\tfrac{1}{4}\eta_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}-F_{\lambda\mu}F^{\lambda}{}_{\nu}\,,\\ \theta^{e\gamma}_{\mu\nu}&=\tfrac{1}{4}i[\bar{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu+ie_0A_\nu)\psi+\bar{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu+ie_0A_\mu)\psi\\&\quad-\bar{\psi}(\overleftarrow{\partial}_\nu-ie_0A_\nu)\gamma_\mu\psi-\bar{\psi}(\overleftarrow{\partial}_\mu-ie_0A_\mu)\gamma_\nu\psi]\,,\end{aligned} \tag{1.2}$$

a simple application of the equations of motion gives the so-called "naive" trace formula

$$\theta_\mu{}^\mu=m_0\bar{\psi}\psi\,. \tag{1.3}$$

As has been shown by the authors of Ref. 3, Eq. (1.3) is not correct as it stands, but instead must be modified by the addition of an anomalous term[4] proportional to $Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}$. Our aim in this paper is to derive an explicit formula for the trace anomaly, valid to all orders in perturbation theory, expressed in terms of the functions $\beta(\alpha)$ and $\delta(\alpha)$ of the fine-structure constant defined through the Callan-Symanzik equations.

In Sec. II we give a simple heuristic derivation of our result, which, as we shall see, is most naturally written in terms of a subtracted operator $N[F_{\lambda\sigma}F^{\lambda\sigma}]$. There, we will be thinking in terms of using massive regulator fields. Some related details are given in the appendixes.

Then in Sec. III we will give a more careful derivation using normal-product methods[5] and dimensional regularization.[6] In n space-time dimensions, we have

$$\theta_\mu{}^\mu=-(n-4)\mathcal{L}_{\rm inv}-3(\tfrac{1}{2}i\bar{\psi}\overleftrightarrow{\not{D}}\psi-m_0\bar{\psi}\psi)+m_0\bar{\psi}\psi\,. \tag{1.4}$$

The anomaly is the term $-(n-4)\mathcal{L}_{\rm inv}$, which would vanish if $\mathcal{L}_{\rm inv}$ were finite. We wish to express the anomaly in terms of renormalized operators.

Our derivation will give as a byproduct a proof that $\theta_{\mu\nu}$ as defined by Eq. (1.2) is finite to all orders of perturbation theory even when the trace anomaly is taken into account. The earlier proof by Callan, Coleman, and Jackiw[7] is incomplete, while the one by Freedman, Muzinich, and Weinberg[8] is not directly applicable to our case.

II. HEURISTIC DERIVATION

The heuristic derivation is obtained by writing down an operator formula for the trace equation and then determining the unknown coefficients appearing in this formula by studying its electron-to-electron and vacuum-to-two-photon matrix elements. As our initial operator ansatz let us write the most general linear combination of gauge-invariant scalar C-even operators with the correct dimensionality,

$$\theta_\mu{}^\mu=C_1m_0\bar{\psi}\psi+C_2Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}+C_3\tfrac{1}{2}i[\bar{\psi}\gamma\cdot(\overrightarrow{\partial}+ie_0A)\psi-\bar{\psi}\gamma\cdot(\overleftarrow{\partial}-ie_0A)\psi-2m_0\bar{\psi}\psi]\,. \tag{2.1}$$

The coefficient of C_3 is formally zero by use of the equations of motion; it represents a discontinuous contribution which is present at zero momentum transfer, but which vanishes for nonzero momentum transfers, and hence does not contribute to physical matrix elements. The precise structure of this term will be determined in Sec. III, but we will ignore it in the heuristic discussion which follows. Focussing on the first two terms, it is easy to see that either C_1 or C_2 is infinite, or Eq. (2.1) cannot be correct as it stands. The reason is that both $\theta_\mu{}^\mu$ and $m_0\bar{\psi}\psi$ are finite operators[9] (that is, their matrix elements are made finite by the usual electron and photon wave-function renormalizations), whereas a simple calculation shows that the *lowest-order* diagrams (illustrated in Fig. 1) contributing to the electron-to-electron and vacuum-to-two-photon matrix elements of $Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}$ are logarithmically divergent, and hence cannot be made finite by wave-function renormalizations alone. This problem is analyzed in more detail in Appendix A, where it is shown that if a photon regulator is introduced to make the diagrams of Fig. 1 finite, then energy-momentum-tensor conservation requires the introduction of extra contributions, proportional to the mass squared of the regulator field, in the $\theta_{\mu\nu}$-regulator photon vertex. These terms may be thought of as arising from the energy-momentum tensor of the regulator field. In the limit of infinite photon regulator mass these contributions survive and, in lowest relevant order, give a second logarithmic divergence, which just cancels the logarithmic divergence of the diagrams in Fig. 1. Thus, C_1 and C_2 remain finite, and the correct form of Eq. (2.1) is actually

$$\theta_\mu{}^\mu = C_1 m_0\bar{\psi}\psi + C_2 N_0[F_{\lambda\sigma}F^{\lambda\sigma}] + \text{discontinuous terms}\,, \qquad (2.2)$$

with $N_0[F_{\lambda\sigma}F^{\lambda\sigma}]$ a subtracted form of the operator $Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}$. Once it is apparent that a subtracted operator appears in Eq. (2.2), it is convenient to reexpress this operator in terms of another subtracted operator $N[F_{\lambda\sigma}F^{\lambda\sigma}]$ defined by

$$\langle e(p)|N[F_{\lambda\sigma}F^{\lambda\sigma}]|e(p')\rangle \underset{p'\to p}{=} \langle e(p)|Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|e(p)\rangle_{\text{tree}} = 0\,,$$
$$\langle 0|N[F_{\lambda\sigma}F^{\lambda\sigma}]|\gamma(p,\epsilon_1)\gamma(-p',\epsilon_2)\rangle \underset{p'\to p}{=} \langle 0|Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle_{\text{tree}} \qquad (2.3)$$

through a relation of the form

$$N_0[F_{\lambda\sigma}F^{\lambda\sigma}] = aN[F_{\lambda\sigma}F^{\lambda\sigma}] + bm_0\bar{\psi}\psi + \text{discontinuous term}\,. \qquad (2.4)$$

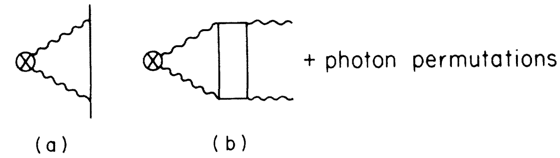

FIG. 1. (a), (b) Logarithmically divergent electron and photon vertex parts, respectively, of the operator $Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}$, the coupling of which is denoted by $\otimes$. Wavy lines indicate photon propagators, and solid lines indicate electron propagators.

This leads to the final operator form for the trace equation

$$\theta_\mu{}^\mu = K_1 m_0\bar{\psi}\psi + K_2 N[F_{\lambda\sigma}F^{\lambda\sigma}] + \text{discontinuous term}\,, \qquad (2.5)$$

with the subtracted operator $N[F_{\lambda\sigma}F^{\lambda\sigma}]$ uniquely specified by the conditions of Eq. (2.3).

We proceed now to determine the coefficients K_1 and K_2 in Eq. (2.5) by taking matrix elements of Eq. (2.5) between appropriate sets of states. Taking first the matrix element between electron states in the limit of zero momentum transfer, and using[10]

$$\langle e(p)|\theta_\mu{}^\mu|e(p')\rangle \underset{p'\to p}{=} \eta^{\mu\nu}\left(\frac{p_\mu p_\nu + p_\nu p_\mu}{2m}\right) = m\,, \qquad (2.6)$$

and Eq. (2.3) we find

$$K_1\langle e(p)|m_0\bar{\psi}\psi|e(p)\rangle = m\,. \qquad (2.7)$$

However, as shown by Sato[11] and as explained in Appendix B, it is easy to see from the Callan-Symanzik equation for the electron propagator that

$$\langle e(p)|m_0\bar{\psi}\psi|e(p)\rangle = \frac{m}{1+\delta(\alpha)}\,, \qquad (2.8)$$

with $\delta(\alpha)$ the function of the fine-structure constant α defined by[12]

$$1+\delta(\alpha) = \frac{m}{m_0}\frac{\partial m_0}{\partial m}\,. \qquad (2.9)$$

Combining Eqs. (2.7) and (2.8), we conclude that[13]

$$K_1 = 1+\delta(\alpha) = 1+\frac{3\alpha}{2\pi}+\cdots\,. \qquad (2.10)$$

Next we take the matrix element of Eq. (2.2) between the vacuum and the two-photon state, again in the limit of zero momentum transfer. Now as Iwasaki[14] has shown, the general form of the vertex $\langle 0|\theta_{\mu\nu}|\gamma(p_1,\epsilon_1)\gamma(p_2,\epsilon_2)\rangle$ is

$$\langle 0|\theta_{\mu\nu}|\gamma(p_1,\epsilon_1)\gamma(p_2,\epsilon_2)\rangle = [\tfrac{1}{2}(F^{1\rho}_{\mu}F^2_{\nu\rho} + F^{1\rho}_{\nu}F^2_{\mu\rho}) - \tfrac{1}{4}\eta_{\mu\nu}F^{1\lambda\sigma}F^2_{\lambda\sigma}]A(q^2) + F^{1\lambda\sigma}F^2_{\lambda\sigma}(p_1-p_2)_\mu(p_1-p_2)_\nu B(q^2) + \tfrac{1}{2}(F^1_{\mu\alpha}F^2_{\nu\beta} + F^1_{\nu\alpha}F^2_{\mu\beta})q^\alpha q^\beta C(q^2)\,, \qquad (2.11)$$

$$q = p_1+p_2,\quad p_1{}^2 = p_2{}^2 = 0,\quad F^i_{\alpha\beta} = (p_i)_\alpha(\epsilon_i)_\beta - (p_i)_\beta(\epsilon_i)_\alpha,\quad i=1,2\,.$$

As Iwasaki notes, Eq. (2.11) implies that the vacuum-to-two-photon matrix element of $\theta_\mu{}^\mu$ is

$$\langle 0|\theta_\mu{}^\mu|\gamma(p_1,\epsilon_1)\gamma(p_2,\epsilon_2)\rangle = (\epsilon_1\cdot\epsilon_2 p_1\cdot p_2 - \epsilon_1\cdot p_2\epsilon_2\cdot p_1) \times q^2[-2B(q^2)+\tfrac{1}{2}C(q^2)] , \quad (2.12)$$

which vanishes at $q^2=0$. Hence, from the vacuum-to-two-photon matrix element of Eq. (2.5) we get, using Eq. (2.6),

$$0=[1+\delta(\alpha)]\langle 0|m_0\bar\psi\psi|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle + K_2\langle 0|Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle_{\text{tree}} . \quad (2.13)$$

Now as shown by Adler *et al.*[15] and again as explained in Appendix B, from the Callan-Symanzik equation for the photon propagator one sees that

$$\langle 0|m_0\bar\psi\psi|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle = -\frac{1}{4}\frac{\beta(\alpha)}{1+\delta(\alpha)} \times\langle 0|Z_3{}^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle_{\text{tree}} , \quad (2.14)$$

with $\beta(\alpha)$ defined by[12,16]

$$\beta(\alpha)=\frac{1}{\alpha}m\frac{\partial\alpha}{\partial m} = \frac{1}{\alpha}[1+\delta(\alpha)]m_0\frac{\partial\alpha}{\partial m_0} = \frac{2\alpha}{3\pi}+\frac{\alpha^2}{2\pi^2}+\cdots . \quad (2.15)$$

Comparing Eq. (2.13) with Eq. (2.14), we learn that

$$K_2=\tfrac{1}{4}\beta(\alpha) , \quad (2.16)$$

and thus our final result for the trace equation is

$$\theta_\mu{}^\mu=[1+\delta(\alpha)]m_0\bar\psi\psi+\tfrac{1}{4}\beta(\alpha)N[F_{\lambda\sigma}F^{\lambda\sigma}] + \text{discontinuous term} . \quad (2.17)$$

The first two terms in the power-series expansion of the coefficient of the $F_{\lambda\sigma}F^{\lambda\sigma}$ term in Eq. (2.17) agree with the fourth-order calculation of Chanowitz and Ellis.[17]

The above derivation is evidently closely analogous to the derivation,[18] by use of the Callan-Symanzik equations, of the nonrenormalization theorem for the axial-vector divergence anomaly

$$\frac{\partial}{\partial x_\mu}j_\mu^5(x)=2im_0j^5(x) + \frac{\alpha_0}{4\pi}F^{\xi\sigma}(x)F^{\tau\rho}(x)\epsilon_{\xi\sigma\tau\rho} . \quad (2.18)$$

However, there are two important ways in which the trace anomaly differs from the axial-vector divergence anomaly. First, the trace anomaly *is* renormalized in higher orders of perturbation theory, and in fact would vanish, leaving only the "soft" operator $[1+\delta(\alpha)]m_0\bar\psi\psi$ as the trace, if $\beta(\alpha)$ satisfied the eigenvalue condition[12,19]

$$\beta(\alpha)=0 . \quad (2.19)$$

Second, whereas the axial anomaly involves the *divergent* operator $Z_3{}^{-1}F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}$, with the consequence that matrix elements of j_μ^5 are not renormalized by wave-function renormalization factors alone, the trace anomaly involves the *convergent* (once-subtracted) operator $N[F_{\lambda\sigma}F^{\lambda\sigma}]$, consistent with the finiteness of matrix elements of the energy-momentum tensor. The appearance of a subtracted operator in Eq. (2.17), as well as closely analogous results of Lowenstein and Schroer in ϕ^4 scalar field theory,[19] suggests that it should be natural to derive Eq. (2.17) within the framework of the normal-product formalism.[5] This is the subject to which we now turn.

III. NORMAL-PRODUCT DERIVATION

In all subsequent discussion we assume that the vacuum expectation value of any operator we consider has been implicitly subtracted off.

In this section we will express $\theta_\mu{}^\mu$ as a linear combination of normal-product operators. Underlying this derivation are the following two observations:

(1) The expression for $\theta_\mu{}^\mu$ in terms of normal products is determined entirely by its insertions at zero momentum into Green's functions: The only operators that can occur are gauge invariant and of dimension at most 4; but the only such operator which vanishes at zero momentum is $\partial^\mu(\bar\psi\gamma_\mu\psi)$, and this operator has the wrong charge-conjugation properties.

(2) The Callan-Symanzik equation is the Ward identity which expresses the nonconservation of the dilatation current[20] and the divergence of the dilation current is essentially $\theta_\mu{}^\mu$. So, if we express this Ward identity in terms of an insertion of $\int\theta_\mu{}^\mu d^4x$, then comparison with the Callan-Symanzik equation in its standard form will give $\theta_\mu{}^\mu$ (at zero momentum) in terms of renormalized operators.

We will use dimensional renormalization[21] to define both the normal products and the renormalized Green's functions. This is by no means essential: All that is required is that the subtractions performed implicitly by the normal products agree with those obtained by an explicit redefinition of the fields and parameters of the bare theory.

We will frequently consider insertions at zero momentum of operators in Green's functions. In

Lowenstein's[22] terminology these are differential vertex operations (DVO's).

First we must define the theory by adding a gauge-fixing term

$$\mathcal{L}_{gf} \equiv -\tfrac{1}{2}(\partial\cdot A)^2/\xi_0 \tag{3.1}$$

to the Lagrangian so the theory is given by

$$\mathcal{L} = \mathcal{L}_{inv} + \mathcal{L}_{gf} \ . \tag{3.2}$$

As usual ξ_0 is renormalized by writing

$$\xi_0 = Z_3 \xi_R \ . \tag{3.3}$$

We are now ready to start the proof.

Consider the equation of dimensional analysis for an unrenormalized (but dimensionally regularized) Green's function G_0:

$$0 = \left(\kappa \frac{\partial}{\partial\kappa} - D_G + m_0 \frac{\partial}{\partial m_0} + (2-\tfrac{1}{2}n)e_0 \frac{\partial}{\partial e_0}\right) G_0 \ . \tag{3.4}$$

Here D_G is the mass dimension of G_0.

By the action principle we can express $\partial/\partial e_0$ and $\partial/\partial m_0$ in terms of operator insertions. Thus,

$$m_0 \frac{\partial}{\partial m_0} = -im_0 \bar\psi\psi^{\sim}(0) \ , \tag{3.5}$$

$$e_0 \frac{\partial}{\partial e_0} = -ie_0 \bar\psi A\!\!\!/ \psi^{\sim}(0) \ , \tag{3.6}$$

where the superscript tilde means that the operator has been Fourier transformed into momentum space. Then

$$0 = \left(\kappa \frac{\partial}{\partial\kappa} - D_G - i[m_0\bar\psi\psi + (2-\tfrac{1}{2}n)e_0\bar\psi A\!\!\!/ \psi]^{\sim}(0)\right) G_R \ , \tag{3.7}$$

where we have multiplied the equation by $Z_2^{-1/2}$ for each external fermion line of G_0, and by $Z_3^{-1/2}$ for each external photon. Thus, Eq. (3.7) is an equation for the renormalized Green's function G_R.

To rewrite (3.7) in terms of $\theta_\mu{}^\mu$ we will need the counting identities.[22] These are simple consequences of the equations of motion, and can be written in terms of either bare fields or normal products. In QED these identities are

$$\begin{aligned} N_e &= (\tfrac{1}{2}\bar\psi \overleftrightarrow{D\!\!\!/} \psi + im_0\bar\psi\psi)^{\sim}(0) \\ &= (\tfrac{1}{2}N[\bar\psi \overleftrightarrow{D\!\!\!/}\psi] + imN[\bar\psi\psi])^{\sim}(0) \ , \end{aligned} \tag{3.8}$$

$$\begin{aligned} N_\gamma &= [\tfrac{1}{2}iF_{\mu\nu}{}^2 + i(\partial\cdot A)^2/\xi_0 + ie_0\bar\psi A\!\!\!/ \psi]^{\sim}(0) \\ &= \{\tfrac{1}{2}iN[F_{\mu\nu}{}^2] + iN[(\partial\cdot A)^2]\xi_R \\ &\quad + ie\mu^{2-n/2}N[\bar\psi A\!\!\!/ \psi]\}^{\sim}(0) \ . \end{aligned} \tag{3.9}$$

Here N_e and N_γ denote respectively the number of external electron lines of a Green's function and the number of external photon lines. Also, μ is the unit of mass,[21] which is used by dimensional regularization to make explicit the dimension of e_0, while keeping dimensionless the renormalized charge e; thus we have $e_0 = \mu^{2-n/2} eZ_3(e,n)^{-1/2}$. These identities are for operations applied to Green's functions, i.e., for DVO's.

We can now write

$$\begin{aligned} \tilde\theta_\mu{}^\mu(0) &= (2-\tfrac{1}{2}n)iN_\gamma + \tfrac{1}{2}(1-n)iN_e \\ &\quad + \{(2-\tfrac{1}{2}n)(\partial\cdot A)^2/\xi_0 + m_0\bar\psi\psi \\ &\quad + (2-\tfrac{1}{2}n)e_0\bar\psi A\!\!\!/ \psi\}^{\sim}(0) \ . \end{aligned} \tag{3.10}$$

Notice that the right-hand side of (3.10) contains (a) the operators occurring in Eq. (3.7), (b) N_e and N_γ, which have been expressed in terms of renormalized operators, and (c) $(n-4)(\partial\cdot A)^2/\xi_0$. The only operator in an inconvenient form is the last one.

However,[23] an application of the gauge Ward identities to each $\partial\cdot A$ in turn proves that $(\partial\cdot A)^2/\xi_0$ has only a single-loop divergence, and that

$$\frac{1}{2\xi_0}(\partial\cdot A)^2 = \frac{1}{2\xi_R}N[(\partial\cdot A)^2] - \frac{ie^2\xi_R}{16\pi^2(n-4)}(\bar\psi \overleftrightarrow{D\!\!\!/}\psi + 2im_0\bar\psi\psi) \ . \tag{3.11}$$

Hence,

$$0 = \left[\kappa\frac{\partial}{\partial\kappa} - D_G - i\tilde\theta_\mu{}^\mu(0) + \left(\frac{3}{2} - \frac{e^2\xi_R}{8\pi^2}\right)N_e\right] G_R + O(n-4) \ . \tag{3.12}$$

We have not yet proved $\theta_\mu{}^\mu$ to be finite, so we cannot set $n=4$ here.

Next, we recall the Callan-Symanzik equation[24,25] for G_R:

$$0 = \left(\kappa\frac{\partial}{\partial\kappa} - D_G - \beta\frac{\partial}{\partial e} + (1+\gamma_m)m\frac{\partial}{\partial m} + \gamma_3\xi_R\frac{\partial}{\partial\xi_R} - \tfrac{1}{2}\gamma_2 N_e - \tfrac{1}{2}\gamma_3 N_\gamma\right) G_R \ . \tag{3.13}$$

Comparison of the last two equations shows that $\tilde\theta_\mu{}^\mu(0)$ is finite at $n=4$, and that

$$\begin{aligned} \tilde\theta_\mu{}^\mu(0) &= -i\beta\frac{\partial}{\partial e} + i(1+\gamma_m)m\frac{\partial}{\partial m} + i\gamma_3\xi_R\frac{\partial}{\partial\xi_R} \\ &\quad - i\left(\tfrac{1}{2}\gamma_2 + \tfrac{3}{2} - \frac{e^2\xi_R}{16\pi^2}\right)N_e - \frac{i}{2}\gamma_3 N_\gamma \\ &= \Big[-\beta N[\bar\psi A\!\!\!/ \psi] + (1+\gamma_m)m_0\bar\psi\psi - \tfrac{1}{2}\gamma_3 N[(\partial\cdot A)^2]/\xi_R \\ &\quad - i\left(\tfrac{1}{2}\gamma_2 + \tfrac{3}{2} - \frac{e^2\xi_R}{16\pi^2}\right)N_e - \tfrac{1}{2}i\gamma_3 N_\gamma\Big]^{\sim}(0) \ . \end{aligned} \tag{3.14}$$

Here the renormalized action principle has been used to express derivatives with respect to e etc. in terms of normal products. Also, we have used the result[9] that $m_0\bar{\psi}\psi = mN[\bar{\psi}\psi]$.

Finally, we use (a) the identities (3.8) and (3.9) to express N_e and N_γ in terms of normal products, (b) the result $\beta = e\gamma_3/2$, and (c) the observation made earlier that the zero-momentum expression for $\theta_\mu{}^\mu$ determines the expression at all momenta. We get[26]

$$\theta_\mu{}^\mu = \tfrac{1}{4}\gamma_3 N[F_{\mu\nu}{}^2] + (1+\gamma_m)m_0\bar{\psi}\psi - [\gamma_2 + 3 - e^2\xi_R/(8\pi^2)] \times (\tfrac{1}{2} iN[\bar{\psi}\overleftrightarrow{\not{D}}\psi] - mN[\bar{\psi}\psi]) \ . \tag{3.15}$$

Use of the fermion equations of motion gives[27]

$$\theta_\mu{}^\mu = \tfrac{1}{4}\gamma_3 N[F_{\mu\nu}{}^2] + (1+\gamma_m)m_0\bar{\psi}\psi \ , \tag{3.16}$$

the same operator formula as was found in Eq. (2.17) above.

Note added in proof. After this work was completed, we learned that essentially identical results have been obtained by N. K. Nielsen (unpublished).

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APPENDIX A

We analyze here the consequences of including a photon regulator to make finite the divergent diagrams of Figs. 1(a) and 1(b). It proves convenient to use a regulator scheme similar to that used[28] in studying the axial-vector divergence anomaly, and specified as follows:

(i) The smallest closed fermion loops illustrated in Fig. 2(a) are given their usual gauge-invariant, renormalized values.

(ii) The larger fermion loops, such as illustrated in Fig. 2(b), are calculated to be photon gauge-invariant and hence finite.

(iii) All photon propagators are regularized: Photon propagators emerging singly from vertices, as in Fig. 2(c), are regularized by the replacement

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 - M^2} = \frac{-M^2}{p^2(p^2-M^2)} \ , \tag{A1}$$

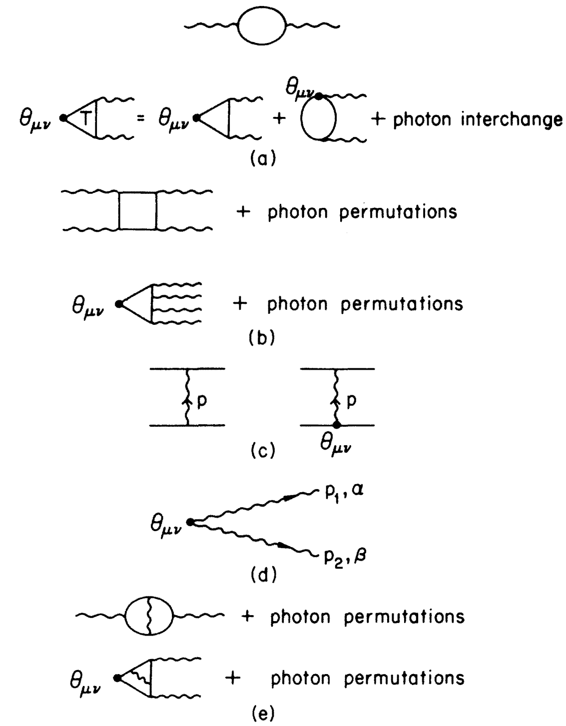

FIG. 2. (a) Smallest closed fermion loops which are given their gauge-invariant fully renormalized values. (b) Larger fermion loops which are evaluated to be gauge invariant. (c) Photons emerging from single-photon vertices, which are regulated according to Eq. (A1). (d) Photon pair emerging from $\theta_{\mu\nu}$, which is regulated according to Eq. (A2). (e) Fermion-loop diagrams with photon radiative corrections.

with M the regulator mass. Pairs of photon propagators emerging from the energy-momentum tensor $\theta_{\mu\nu}$, as in Fig. 2(d), are regularized by the replacement

$$\frac{1}{p_1{}^2} V_{\mu\nu\alpha\beta}(p_1,p_2)\frac{1}{p_2{}^2} \to \frac{1}{p_1{}^2} V_{\mu\nu\alpha\beta}(p_1,p_2)\frac{1}{p_2{}^2} - \frac{1}{p_1{}^2 - M^2} V^M_{\mu\nu\alpha\beta}(p_1,p_2)\frac{1}{p_2{}^2 - M^2} \ , \tag{A2}$$

with the regulator vertex $V^M_{\mu\nu\alpha\beta}$ chosen so that (apart from photon gauge terms, which do not contribute to on-shell matrix elements) the algebraic structure of the gravitational Ward identities implied by conservation of $\theta_{\mu\nu}$ is preserved. Specifically, the Feynman rules for vertices of $\theta_{\mu\nu}$ give

$$V_{\mu\nu\alpha\beta}(p_1,p_2)=-\tfrac{1}{2}\eta_{\mu\nu}(p_1\cdot p_2\eta_{\alpha\beta}-p_{1\beta}p_{2\alpha})+\tfrac{1}{2}(p_1\cdot p_2\eta_{\mu\alpha}\eta_{\nu\beta}+p_{1\mu}p_{2\nu}\eta_{\alpha\beta}-p_{1\mu}p_{2\alpha}\eta_{\nu\beta}-p_{2\nu}p_{1\beta}\eta_{\mu\alpha}$$
$$+p_1\cdot p_2\eta_{\nu\alpha}\eta_{\mu\beta}+p_{1\nu}p_{2\mu}\eta_{\alpha\beta}-p_{1\nu}p_{2\alpha}\eta_{\mu\beta}-p_{2\mu}p_{1\beta}\eta_{\nu\alpha})\,, \tag{A3}$$

which when contracted with $(p_1+p_2)^\mu$ gives

$$(p_1+p_2)^\mu V_{\mu\nu\alpha\beta}(p_1,p_2)=\text{gauge terms}+\tfrac{1}{2}p_1^{\,2}(p_{2\nu}\eta_{\alpha\beta}-p_{2\alpha}\eta_{\nu\beta})+\tfrac{1}{2}p_2^{\,2}(p_{1\nu}\eta_{\alpha\beta}-p_{1\beta}\eta_{\nu\alpha})\,. \tag{A4}$$

We wish to construct $V^M_{\mu\nu\alpha\beta}(p_1,p_2)$ so that

$$(p_1+p_2)^\mu V^M_{\mu\nu\alpha\beta}(p_1,p_2)=\text{gauge terms}+\tfrac{1}{2}(p_1^{\,2}-M^2)(p_{2\nu}\eta_{\alpha\beta}-p_{2\alpha}\eta_{\nu\beta})+\tfrac{1}{2}(p_2^{\,2}-M^2)(p_{1\nu}\eta_{\alpha\beta}-p_{1\beta}\eta_{\nu\alpha})\,, \tag{A5}$$

which gives for the divergence of Eq. (A2)

$$(p_1+p_2)^\mu\left(\frac{1}{p_1^{\,2}}V_{\mu\nu\alpha\beta}(p_1,p_2)\frac{1}{p_2^{\,2}}-\frac{1}{p_1^{\,2}-M^2}V^M_{\mu\nu\alpha\beta}(p_1,p_2)\frac{1}{p_2^{\,2}-M^2}\right)$$
$$=\text{gauge terms}+\tfrac{1}{2}(p_{2\nu}\eta_{\alpha\beta}-p_{2\alpha}\eta_{\nu\beta})\left(\frac{1}{p_2^{\,2}}-\frac{1}{p_2^{\,2}-M^2}\right)+\tfrac{1}{2}(p_{1\nu}\eta_{\alpha\beta}-p_{1\beta}\eta_{\nu\alpha})\left(\frac{1}{p_1^{\,2}}-\frac{1}{p_1^{\,2}-M^2}\right), \tag{A6}$$

which has the same structure as the divergence of $(1/p_1^{\,2})V_{\mu\nu\alpha\beta}(1/p_2^{\,2})$, apart from the replacement of the photon propagators by regularized propagators. One easily finds that the lowest-order polynomial in momenta satisfying Eq. (A5) is

$$V^M_{\mu\nu\alpha\beta}(p_1,p_2)=V_{\mu\nu\alpha\beta}(p_1,p_2)-\tfrac{1}{2}M^2(\eta_{\nu\mu}\eta_{\alpha\beta}-\eta_{\mu\alpha}\eta_{\nu\beta}-\eta_{\mu\beta}\eta_{\nu\alpha})\,. \tag{A7}$$

Thus, the requirement that the regularization scheme respect gravitational Ward identities introduces an explicit M^2 dependence into the $\theta_{\mu\nu}$-photon vertex. This is, of course, just the contribution to $\theta_{\mu\nu}$ expected from the mass term in the regulator field Lagrangian.

(iv) The regularization prescription adopted above makes radiative correction diagrams such as illustrated in Fig. 2(e) finite for finite M, but divergent as $M\to\infty$, with the divergences canceled by appropriate counterterms appearing in the renormalization constant $Z_3(M)$. We note, however, that since explicitly renormalized values for the single-loop diagrams of Fig. 2(a) are always used, Z_3 contains no counterterms referring to these diagrams. In effect, we have adopted a type of intermediate renormalization procedure, in which Z_3 contains counterterms only for those vacuum polarization graphs which involve internal virtual photons.

Having specified the regularization procedure, we can now turn to a study of the lowest-order divergent $\theta_\mu{}^\mu$ insertions of Fig. 1. It suffices to consider these insertions at zero four-momentum transfer, since the *difference* between zero and nonzero four-momentum transfer will converge. Focussing on the trace-to-two-photon vertex on the left-hand side of the dashed line in Fig. 3(a), we find in one-fermion-loop order that there are two classes of $\theta_{\mu\nu}$ couplings which contribute, as illustrated in Fig. 2(b) and Fig. 2(c). [We note in passing that an explicit check of $\theta_{\mu\nu}$ conservation for the diagrams of Figs. 2(b) and 2(c) shows that the structure of the Ward identities is guaranteed by the regularization scheme sketched above, with no need for any additional vertex modifications beyond that given by Eq. (A7).] Taking the trace on $\mu\nu$ of Fig. 3(b), using the trace anomaly formula of Eq. (2.17) to leading order, and dropping gauge terms gives

$$\eta^{\mu\nu}[3(b)]=\frac{-2\alpha}{3\pi}\eta_{\alpha\beta}\frac{M^4}{p^2(p^2-M^2)^2}\,. \tag{A8}$$

In the absence of regulators, the trace of Fig. 3(c) would vanish, but when regulators are included it is nonvanishing, on account of the term propor-

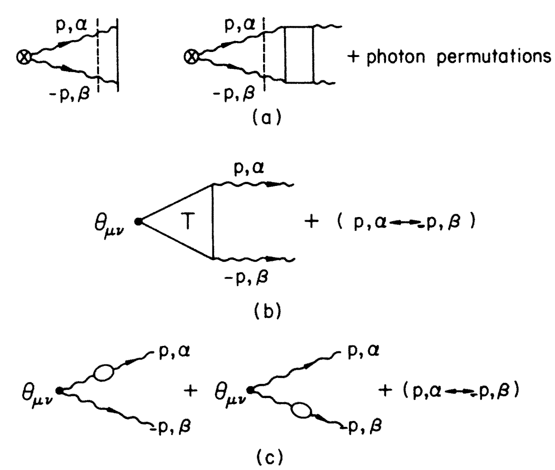

FIG. 3. (a) The divergent diagrams of Fig. 1, at zero four-momentum transfer. We focus on the trace-to-two-photon vertex on the left-hand side of the dashed line. (b), (c) Classes of one-fermion-loop diagrams which contribute to the left-hand side of the dashed line in (a).

tional to M^2 in Eq. (A7), and one finds

$$\eta^{\mu\nu}[3(c)] = \frac{-4M^4\eta_{\alpha\beta}\tilde{\Pi}^{(2)}(p^2/m^2,\alpha)}{(p^2-M^2)^3}\ . \qquad (A9)$$

Setting $-p^2=x$, and using the fact that

$$\tilde{\Pi}^{(2)}(p^2/m^2,\alpha) \underset{x\to\infty}{\simeq} -\frac{\alpha}{3\pi}\ln x - c\ , \qquad (A10)$$

with c a constant, the sum of Eqs. (A8) and (A9) becomes

$$\eta^{\mu\nu}[3(b)]+\eta^{\mu\nu}[3(c)] = \frac{2\alpha}{3\pi}\eta_{\alpha\beta}\frac{M^4}{x(x+M^2)^2} - \frac{4\eta_{\alpha\beta}M^4(\alpha/3\pi\ln x + c)}{(x+M^2)^3} = 2\eta_{\alpha\beta}M^4\frac{d}{dx}\left(\frac{\alpha/3\pi\ln x+c}{(x+M^2)^2}\right). \qquad (A11)$$

Now the leading single logarithmic divergence of either of the diagrams in Fig. 3(a) comes from an integral of the form

$$\int^{\infty} x dx\{\eta^{\mu\nu}[3(b)]+\eta^{\mu\nu}[3(c)]\}\psi(x)\ , \qquad (A12)$$

where $\psi(x)\sim c_1/x+\cdots$ represents the right-hand side of the dashed line. But substituting Eq. (A11) and the leading term of $\psi(x)$ into Eq. (A12), we get a result proportional to

$$M^4\int^{\infty} dx\frac{d}{dx}\left(\frac{\alpha/3\pi\ln x+c}{(x+M^2)^2}\right) = M^4\left(\frac{\alpha/3\pi\ln x+c}{(x+M^2)^2}\right)_{x\ \text{finite}}^{x=\infty}, \qquad (A13)$$

which approaches a finite limit as the regulator mass M approaches infinity. In other words, the logarithmically divergent contributions to the trace coming from Figs. 3(b) and Figs. 3(c) precisely cancel: in effect, the extra M^2 term in the $\theta_{\mu\nu}$-regulator photon vertex of Eq. (A7) generates, in the limit as $M\to\infty$, a subtraction counterterm for the divergent operator $Z_3^{-1}F_{\lambda\sigma}F^{\lambda\sigma}$. The mechanism operating here is evidently a photon analog of the fermion regulator behavior[3] which can be thought of as producing the trace anomaly in the first place.

APPENDIX B

We give here the derivation of Eqs. (2.8) and (2.14), and also illustrate Iwasaki's theorem on the vanishing of $\langle 0|\theta_\mu{}^\mu|\gamma\gamma\rangle$ in a special case. To derive Eq. (2.8) we follow the method of Sato.[11] Introducing the scalar vertex part $\tilde{\Gamma}_s(p_1,p_2)$, we have

$$\langle e(p)|m_0\bar{\psi}\psi|e(p)\rangle = \tilde{\Gamma}_s(p,p)|_{\not{p}=m}\ . \qquad (B1)$$

Writing

$$\tilde{\Gamma}_s(p,p) = -iZ_2m_0\frac{\partial}{\partial m_0}[S_F'(p)]^{-1} = -Z_2m_0\frac{\partial}{\partial m_0}Z_2^{-1}[\not{p}-m-\tilde{\Sigma}(p)]\ , \qquad (B2)$$

with S_F' the unrenormalized electron propagator and $\tilde{\Sigma}$ the renormalized electron proper self-energy, and substituting into Eq. (B1), we get

$$\tilde{\Gamma}_s(p,p)|_{\not{p}=m} = m_0\frac{\partial m}{\partial m_0} + \left(m_0\frac{\partial}{\partial m_0}\tilde{\Sigma}(p)\right)\Big|_{\not{p}=m}\ . \qquad (B3)$$

Now by the chain rule we have

$$\left(m_0\frac{\partial}{\partial m_0}\tilde{\Sigma}(p)\right)\Big|_{\not{p}=m} = \left(m_0\frac{\partial m}{\partial m_0}\frac{\partial\tilde{\Sigma}}{\partial m} + m_0\frac{\partial\alpha}{\partial m_0}\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right)_{\not{p}=m}, \qquad (B4)$$

while from the fact that $\tilde{\Sigma}$ is homogeneous of degree 1 in $\not{p}$ and m we get

$$\tilde{\Sigma} = \left(m\frac{\partial}{\partial m} + \not{p}\frac{\partial}{\partial\not{p}}\right)\tilde{\Sigma}\ . \qquad (B5)$$

Combining Eqs. (B4) and (B5), we see that the renormalization conditions on $\tilde{\Sigma}$,

$$\tilde{\Sigma}|_{\not{p}=m} = \frac{\partial\tilde{\Sigma}}{\partial\not{p}}\Big|_{\not{p}=m} = 0 \qquad (B6)$$

imply that

$$\left(m_0\frac{\partial}{\partial m_0}\tilde{\Sigma}(p)\right)\Big|_{\not{p}=m} = 0\ . \qquad (B7)$$

[In Eq. (B6) we have assumed the Yennie gauge, in which $\tilde{S}_F'(\not{p})$ has a true pole at $\not{p}=m$; this restriction is immaterial, since the final result of Eq. (B8) is manifestly gauge invariant.] Thus, the second term in Eq. (B3) vanishes, giving the desired result

$$\langle e(p)|m_0\bar{\psi}\psi|e(p)\rangle = m_0\frac{\partial m}{\partial m_0} = \frac{m}{1+\delta(\alpha)}. \qquad (B8)$$

To derive Eq. (2.14) we follow a similar procedure. Introducing the zero-momentum-transfer scalar to two-photon vertex $\tilde{\Gamma}_{\gamma\gamma s}(p^2/m^2,\alpha)$, we have

$$\langle 0|m_0\bar{\psi}\psi|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle = \tfrac{1}{4}\alpha\tilde{\Gamma}_{\gamma\gamma s}(0,\alpha) \times\langle 0|Z_3^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle_{\text{tree}}\ . \qquad (B9)$$

However, $\tilde{\Gamma}_{\gamma\gamma s}(p^2/m^2,\alpha)$ is related to the photon renormalized proper self-energy $\tilde{\Pi}(p^2/m^2,\alpha)$ by the Callan-Symanzik equation[12]

$$\frac{1}{1+\delta(\alpha)}\left(m\,\frac{\partial}{\partial m}+\alpha\beta(\alpha)\,\frac{\partial}{\partial\alpha}\right)\frac{1}{\alpha}\left[1+\tilde{\Pi}(p^2/m^2,\alpha)\right] = \tilde{\Gamma}_{\gamma\gamma s}(p^2/m^2,\alpha)\ . \quad (\text{B10})$$

On setting $p^2=0$ in Eq. (B10) and using the renormalization condition

$$\tilde{\Pi}(0,\alpha)=0\ , \quad (\text{B11})$$

we get

$$\alpha\tilde{\Gamma}_{\gamma\gamma s}(0,\alpha)=-\frac{\beta(\alpha)}{1+\delta(\alpha)}\ , \quad (\text{B12})$$

which when substituted into Eq. (B9) gives the desired relation

$$\langle 0|m_0\bar{\psi}\psi|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle = -\frac{1}{4}\,\frac{\beta(\alpha)}{1+\delta(\alpha)} \times\langle 0|Z_3^{-1}F_{\lambda\sigma}F^{\lambda\sigma}|\gamma(p,\epsilon_1)\gamma(-p,\epsilon_2)\rangle_{\text{tree}}\ . \quad (\text{B13})$$

It is also instructive to rearrange Eq. (B10) into a slightly different form by writing

$$\Pi^{\mu\nu}(p,-p)=(p^\mu p^\nu-p^2\eta^{\mu\nu})\tilde{\Pi}(p^2/m^2,\alpha)\ ,$$
$$\Delta^{\mu\nu}(p,-p)=(p^\mu p^\nu-p^2\eta^{\mu\nu})\alpha\tilde{\Gamma}_{\gamma\gamma s}(p^2/m^2,\alpha)\ , \quad (\text{B14})$$

giving

$$\Delta^{\mu\nu}(p,-p)=\frac{1}{1+\delta(\alpha)}\left[\left(2-p\cdot\frac{\partial}{\partial p}\right)+\beta(\alpha)\left(\alpha\,\frac{\partial}{\partial\alpha}-1\right)\right] \times\pi^{\mu\nu}(p,-p)-\frac{\beta(\alpha)}{1+\delta(\alpha)}(p^\mu p^\nu-p^2\eta^{\mu\nu})\ . \quad (\text{B15a})$$

An equivalent form, suggested by Eq. (2.17), is

$$[1+\delta(\alpha)]\Delta^{\mu\nu}(p,-p)+\beta(\alpha)(p^\mu p^\nu-p^2\eta^{\mu\nu}) =\left[2-p\cdot\frac{\partial}{\partial p}+\beta(\alpha)\left(\alpha\,\frac{\partial}{\partial\alpha}-1\right)\right]\Pi^{\mu\nu}(p,-p)\ . \quad (\text{B15b})$$

Equation (B15) is an exact expression, at zero momentum transfer, for the vacuum-to-two-photon matrix element of the "naive" or canonical trace $m_0\bar{\psi}\psi$.[29] Substituting the second-order perturbation formula

$$\tilde{\Pi}^{(2)}(p^2/m^2,\alpha) =\frac{-2\alpha}{\pi}\int_0^1 dx\,x(1-x)\ln\left(1-\frac{p^2x(1-x)}{m^2}\right) \quad (\text{B16})$$

into Eqs. (B10) or (B16), we recover the usual second-order perturbation theory formula

$$\alpha\tilde{\Gamma}^{(2)}_{\gamma\gamma s}(p^2/m^2,\alpha) =-\frac{4\alpha}{\pi}\int_0^1 dx\,x(1-x)\,\frac{m^2}{m^2-p^2x(1-x)}\ . \quad (\text{B17})$$

As a simple, explicit check on Iwasaki's theorem we have calculated the second-order vacuum-to-two-photon matrix element of $\theta_{\mu\nu}$ at zero momentum transfer. (This can either be done directly by diagrammatic techniques, or more simply by using the Ward identities[30] following from conservation of $\theta_{\mu\nu}$.) Denoting this matrix element by $T^{(2)}_{\mu\nu\alpha\beta}(p)$, with p, $-p$ the (virtual) photon four-momenta and with α and β the photon polarization indices, we find that

$$T^{(2)}_{\mu\nu\alpha\beta}(p)=t_{\mu\nu\alpha\beta}(p)\tilde{\Pi}^{(2)}(p^2/m^2,\alpha) +(p_\alpha p_\beta-p^2\eta_{\alpha\beta})p_\mu p_\nu 2\,\frac{\partial}{\partial p^2}\tilde{\Pi}^{(2)}(p^2/m^2,\alpha)\ ,$$
$$t_{\mu\nu\alpha\beta}=p^2(\eta_{\mu\nu}\eta_{\alpha\beta}-\eta_{\mu\alpha}\eta_{\nu\beta}-\eta_{\nu\alpha}\eta_{\mu\beta}) -2\eta_{\alpha\beta}p_\mu p_\nu-\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\mu\alpha}p_\nu p_\beta +\eta_{\nu\alpha}p_\mu p_\beta+\eta_{\nu\beta}p_\mu p_\alpha+\eta_{\mu\beta}p_\nu p_\alpha\ . \quad (\text{B18})$$

Taking the trace we obtain

$$\eta^{\mu\nu}T^{(2)}_{\mu\nu\alpha\beta}(p)=(p_\alpha p_\beta-p^2\eta_{\alpha\beta})2p^2 \times\frac{\partial}{\partial p^2}\tilde{\Pi}^{(2)}(p^2/m^2,\alpha)\ , \quad (\text{B19})$$

which evidently vanishes for on-shell photons ($p^2=0$) as asserted by Iwasaki's general argument. To exhibit the splitting of Eq. (B19) into "naive" and anomalous trace terms, we substitute Eq. (B16) and rearrange by comparison with Eq. (B17), giving

$$\eta^{\mu\nu}T^{(2)}_{\mu\nu\alpha\beta}(p)=-(p_\alpha p_\beta-p^2\eta_{\alpha\beta}) \times\left(\alpha\tilde{\Gamma}^{(2)}_{\gamma\gamma s}(p^2/m^2,\alpha)+\frac{2\alpha}{3\pi}\right) \quad (\text{B20})$$

as expected from Eqs. (2.17) and (B9) in second order.

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[1]We follow throughout the metric and γ-matrix conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[2]For reviews, see S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, Mass., 1970), p. 3; S. B. Treiman,

R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton Univ. Press, Princeton, N. J., 1972), p. 97.

[3]R. J. Crewther, Phys. Rev. Lett. 28, 1421 (1972); M. S. Chanowitz and J. Ellis, Phys. Lett. 40B, 397 (1972); M. S. Chanowitz and J. Ellis, Phys. Rev. D 7, 2490 (1973).

[4]We use e_0 and m_0 to denote the unrenormalized charge and mass, e and m to denote the renormalized charge and mass, and ψ, A_μ, and $F_{\mu\nu}$ to denote unrenormalized fields. In Sec. III, the notation for the functions defined through the Callan-Symanzik equation follows J. C. Collins and A. J. Macfarlane, Phys. Rev. D 10, 1201 (1974); the relation to the notation of Sec. II and the appendixes is $\gamma_3=\beta$, $\gamma_m=\delta$.

[5]The normal-product algorithm was first set up using Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization by W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, Mass., 1970), Vol. I, p. 397. In Sec. III we shall use normal products defined by dimensional renormalization: J. C. Collins, Nucl. Phys. B92, 477 (1975), and P. Breitenlohner and D. Maison, Max-Planck Institut Report No. MPI-PAE PTh25 and NYU Report No. NYU/TR2/76 (unpublished).

[6]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); C. G. Bollini and J. J. Giambiagi, Phys. Lett. 40B, 566 (1972); G. M. Cicuta and E. Montaldi, Lett. Nuovo Cimento 4, 329 (1972).

[7]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970). The proof of finiteness of $\theta_{\mu\nu}$ by these authors uses the trace identity to prove $\theta_\mu{}^\mu$ finite, but they ignore the trace anomaly. Our completion of their proof will be to show that the trace anomaly is finite, and hence that its existence does not affect the proof that $\theta_{\mu\nu}$ is finite.

[8]D. Z. Freedman, I. J. Muzinich, and E. J. Weinberg, Ann. Phys. (N.Y.) 87, 95 (1974); D. Z. Freedman and E. J. Weinberg, *ibid.* 87, 354 (1974). The proof of finiteness of $\theta_{\mu\nu}$ given in these papers does not include the possibility of fermion fields, and it depends on the Ward identities for the conservation of $\theta_{\mu\nu}$. Since these Ward identities are more complicated in the presence of fermions, the proof appears not to have a trivial generalization. In fact, use of the trace identity seems necessary.

[9]Finiteness of $m_0\bar{\psi}\psi$ is proved, for instance, by S. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972). Within the framework of dimensional regularization it is also a special case of a general theorem by J. C. Collins, Nucl. Phys. B80, 341 (1974). Finiteness of $\theta_{\mu\nu}$ is a conjecture which we prove in Sec. III. The results in Sec. III depend in no way on Sec. II.

[10]We use throughout an invariant-state normalization convention. One of us (S.L.A.) wishes to thank M. A. B. Bég for a critical comment on the original version of this paper, which suggested the importance of studying the electron-to-electron, as well as the vacuum-to two-photon, matrix element of the trace equation in the heuristic derivation of Sec. II.

[11]H. Sato, University of Minnesota report (unpublished).

[12]S. L. Adler, Phys. Rev. D 5, 3021 (1972), Eq. (41).

[13]For the δ expansion, see Adler and Bardeen, Ref. 9, Eqs. (15b) and (47), where $\alpha(x)$ is used to denote the quantity called $\delta(x)$ in this paper.

[14]Y. Iwasaki, Phys. Rev. D 15, 1172 (1977).

[15]S. L. Adler, J. Lieberman, Y. J. Ng, and H-S. Tsao, Phys. Rev. D 14, 359 (1976), Appendix A, Eqs. (A30) and (A31). The matrix element corresponding to the effective Lagrangian $\mathfrak{L}_{\rm kin}$ of Eq. (A31) is $i\mathfrak{L}_{\rm kin}$, and the $n=1$ term of this is the vacuum-to-two-photon matrix element of $-im_0\bar{\psi}\psi$.

[16]For the β expansion, see for example, J. D. Bjorken and S. D. Drell, Ref. 1, Eq. (19.160).

[17]M. S. Chanowitz and J. Ellis, Phys. Rev. D 7, 2490 (1973).

[18]A. Zee, Phys. Rev. Lett. 29, 1198 (1972); J. Lowenstein and B. Schroer, Phys. Rev. D 7, 1929 (1973).

[19]Related observations in the ϕ^4 scalar field theory have been given by J. H. Lowenstein, Phys. Rev. D 4, 2281 (1971), and B. Schroer, Lett. Nuovo Cimento 2, 867 (1971). It would be interesting to see whether requiring the softness of the energy-momentum-tensor trace in non-Abelian gauge theories gives nontrivial coupling-constant eigenvalue conditions. The extension of the methods of this paper to quantum chromodynamics has been studied, and the results will be described elsewhere.

[20]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).

[21]Collins, and Breitenlohner and Maison, Ref. 5.

[22]J. H. Lowenstein, Commun. Math. Phys. 24, 1 (1971).

[23]J. C. Collins, Ph.D. thesis, Cambridge University, 1975 (unpublished). The argument is a dimensionally regularized version of that given by L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 29, 89 (1955) [Sov. Phys.-JETP 2, 69 (1956)]; K. Johnson and B. Zumino, Phys. Rev. Lett. 3, 351 (1959); B. Zumino, J. Math. Phys. 1, 1 (1960). A recent paper using dimensional regularization is B. Lautrup, Nucl. Phys. B105, 23 (1976). See also J. H. Lowenstein and B. Schroer, Phys. Rev. D 6, 1553 (1972).

[24]This equation, in the form we use it, is due to S. Weinberg, Phys. Rev. D 8, 3497 (1973). Note that β as used here differs from the β in Sec. II. The coefficients β, γ_2, γ_3, and γ_m are precisely those defined by Collins and Macfarlane (Ref. 4).

[25]We deliberately call Eq. (3.13) the Callan-Symanzik equation rather than the renormalization-group equation. It is true that *with dimensional renormalization* the two equations are essentially identical, for, by Refs. 4, 5, and 9, $m\partial/\partial m = m_0\partial/\partial m_0 = m_0\bar{\psi}\psi^\sim(0)$. Our derivation in this section of the trace anomaly is so arranged that it has a simple generalization to any renormalization prescription, provided that the normal products used are those natural to the prescription. (E.g., in the case of renormalization on-shell, we must define $N[F_{\mu\nu}{}^2]$ by Eq. (2.3).) But, in general (see Ref. 22), the two equations differ, and it is the Callan-Symanzik equation we must use as (3.13). This is because it is the Callan-Symanzik equation that contains the mass term $m_0\bar{\psi}\psi$, and we want to write $\theta_\mu{}^\mu$ as $m_0\bar{\psi}\psi$ plus an anomaly.

[26]Note that by Ref. 23, $\gamma_2-e^2\xi_R/(8\pi^2)$ is precisely γ_2 evaluated in the Landau gauge $\xi_R=0$.

[27]We could not use the equations of motion earlier because we were considering Green's functions containing $\theta_\mu{}^\mu$ at zero momentum.

[28]S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969).

[29]Equation (B15) disagrees, by an over-all factor of $[1+\delta(\alpha)]^{-1}$ and the presence of the $\beta(\alpha)(\alpha\partial/\partial\alpha-1)$ term, with the fourth-order result given in the Appendix of Chanowitz and Ellis, Ref. 17, who have incorrectly generalized to fourth order the lowest-order canonical trace identity.

[30]Freedman, Muzinich, and Weinberg, Ref. 8, Eq. (3.7).