Equivalence of the sine-Gordon and Thirring models and cumulative mass effects

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We investigate the equivalence of the sine-Gordon and Thirring models on the basis of the short-distance behavior in the massive Thirring model. We find that for $\dim\bar{\psi}\psi \geq 1$ there is a new additive renormalization effect originating from the occurrence of nonleading mass singularities in the spinor vacuum expectation values. In the sine-Gordon language this effect makes its appearance as a "cumulative" mass effect. It leads to a breakdown of the naive variational method.

I. INTRODUCTION

Quasiclassical approximations suggest that certain quantum-field theories lead to new particle states which, relative to the quasilocal algebra of fields,[1] cannot be interpreted as conventional bound states. There is no element in the field algebra of such models which connects the vacuum to the new states. Rather, the new states lie in another superselection sector of the local algebra.[2,3]

Present experience shows that a sufficient condition for such a situation to arise in the occurrence of topological stability of classical solutions called "kink" solutions.[4] Classical nonlinear field theories leading to such behavior are known in two and four dimensions. The present quantum treatment of these models[5,6] appears somewhat ugly (to the eyes of a local quantum-field theorist) because it lacks formal covariance and locality between different sectors. Technically speaking, these methods do not produce a "grand Hamiltonian" which works for all sectors, but rather they collect the total Hamiltonian as a direct sum over sector Hamiltonians for a given number of kink states.

The crucial question is whether this quasiclassical appearance is an intrinsic feature of the model. In this case one might talk about a quantum-field theory of "extended objects" as being a distinctive different class from conventional local quantum-field theories. A second possibility is that there exists a local shift operator[2] linking the different superselecting sectors which serves at the same time as an interpolating field for the new states.

Arguments in favor of the latter possibility have been given for the sine-Gordon equation which, under the introduction of local charge-carrying fields, turns into the massive Thirring model.[7-9] These considerations are somewhat formal since they discuss the equivalence problem in the spirit of singular mass perturbation.

In the massive Thirring model one could perhaps try to solve the equivalence problem by studying the infrared problem of the mass in the spirit of a thermodynamic limit[10] and showing that the limiting composites of the axial-vector current potential have the desired properties. Since this model appears to be a good candidate for the first explicitly soluble nontrivial quantum-field theory,[11] this may even be a feasible program. However, we feel that in view of the general nature of the problem at hand, it is desirable to dissociate the equivalence discussion from the more detailed constructive approach. In the following we will present an approach which is based on short-distance properties.

In Sec. II we review briefly the formalism of the boson representation of two-dimensional massless fermions.[9] In this context we emphasize the role of cluster properties.

Section III contains a discussion of the axial-vector current potential ϕ and its exponential functions in the massive Thirring model. A generalization of Schur's lemma for local noncanonical fields will lead us to the validity of the sine-Gordon equation for ϕ for the case $\dim\psi_1^\dagger\psi_2 < 1$. The case $\dim\psi_1^\dagger\psi_2 \geq 1$ leads to the appearance of nonleading singularities in the Thirring model requiring an additive renormalization of the composite field $\psi_1^\dagger\psi_2$. We show that this effect has repercussions in the sine-Gordon equation. The composite field corresponding to the classical exponential functions (and also the sine function) exhibits a cumulative mass effect for $\dim\bar{\psi}\psi \to 1$, which renders the multiplicative renormalization of Coleman insufficient. For the free massive Dirac field the correct expression precisely agrees with that obtained recently by Lehmann and Stehr[12] from an explicit computation. The regime $\dim\bar{\psi}\psi \geq 1$ is noncanonical for the sine-Gordon fields and hence the conventional renormalization picture breaks down.[13] This opens the interesting possibility that the new "phase" $\dim\bar\psi\psi > 2$, which leads to a crossing of the previously nonleading terms with the leading terms, defines an interesting model for a nonrenormalizable theory.[14]

II. BOSON REPRESENTATION OF THE ZERO-MASS LIMIT

In order to study the short-distance properties in the massive Thirring model, we consider first its zero-mass limit. The details of this discussion have been carried out by Klaiber[15] and may be summarized by the following formula for the two-component Fermi field:

$$\psi_i(x) = e^{i\chi_i^{(+)}(x)}\psi_{0i}(x)e^{i\chi_i^{(-)}}, \quad i=1,2. \tag{2.1}$$

Here $\psi_0(x)$ is the free zero-mass Dirac field and $\chi(x)$ is an operator expressed in the "potentials" (l = left, r = right)

$$j_l(u) = \frac{1}{2\pi}\int_0^\infty e^{-ipu}C_l(p)\frac{dp}{2p} + \text{H.c.}, \tag{2.2a}$$

$$j_r(v) = \frac{1}{2\pi}\int_0^\infty e^{-ipv}C_r(p)\frac{dp}{2p} + \text{H.c.}, \tag{2.2b}$$

with $u=t+x$, $v=t-x$. The C's, which obey Bose commutation relations, can be written as bilinear expressions in the fermion creation and annihilation operators of ψ_0. The (regularized) two-point functions of the free fields j and their commutation relations with ψ_0 are

$$\langle j_l(u)j_l(u')\rangle = -\frac{1}{4\pi}\left[\ln m(u-u'-i\epsilon)+\frac{i\pi}{2}\right], \tag{2.3a}$$

$$\langle j_r(v)j_r(v')\rangle = -\frac{1}{4\pi}\left[\ln m(v-v'-i\epsilon)+\frac{i\pi}{2}\right], \tag{2.3b}$$

$$\langle j_l(u)j_r(v)\rangle = 0, \tag{2.3c}$$

$$\frac{1}{\sqrt{\pi}}[j_l^{(\pm)}(u), \psi_{01}(u')] = \pm\frac{1}{2\pi i}\left[\ln m(u-u'\pm i\epsilon)\mp\frac{i\pi}{2}\right]\psi_{01}(u'), \tag{2.4a}$$

$$\frac{1}{\sqrt{\pi}}[j_r^{(\pm)}(v), \psi_{02}(v')] = \pm\frac{1}{2\pi i}\left[\ln m(v-v'\pm i\epsilon)\mp\frac{i\pi}{2}\right]\psi_{02}(v'), \tag{2.4b}$$

$$[j_l(u), \psi_{20}(v)] = 0 = [j_r(v), \psi_{10}(u)], \tag{2.4c}$$

and corresponding relations with $\psi^\dagger$.

The connection of the potentials with the zero-mass currents (whose boson representation does not require infrared regularization and therefore does not show any dependence on the regularization parameter m) is

$$j_+(u) = \tfrac{1}{2}(j_0+j_1) = \frac{1}{\sqrt{\pi}}\,\partial_u j_l(u), \tag{2.5a}$$

$$j_-(v) = \tfrac{1}{2}(j_0-j_1) = \frac{1}{\sqrt{\pi}}\,\partial_v j_r(v). \tag{2.5b}$$

The potentials j are not uniquely determined by the current. By interpreting the integrals for small momenta in a suitable distribution-theoretical way, we have chosen one particular definition of the potentials.

The χ_i are given by

$$\chi_1 = C_1 j_l(u) + C_2 j_r(v), \tag{2.6a}$$

$$\chi_2 = C_2 j_l(u) + C_1 j_r(v), \tag{2.6b}$$

where the C's are real parameters. Introducing a $\gamma_5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ matrix, this can also be written as

$$\chi = \alpha j(x) - \beta\gamma_5\tilde{j}(x), \quad j = j_l + j_r, \quad \tilde{j} = j_l - j_r, \tag{2.7}$$

in terms of new real parameters α and β.

The result of the ordinary Wick contraction [commuting $\chi^{(\pm)}$ through the spinors by using (2.3) and (2.4)] gives precisely Klaiber's formula[15] for the Wightman functions. The negative-metric aspect of the regularized potential ϕ introduced by infrared regularization drops out on the level of the exponential functions. The structure of the free ψ_0-expectation value, which requires the same number of ψ_i and $\psi_i^\dagger$ in each index, ensures that the exponential part of the Wightman function is itself positive-definite. Klaiber proved this[15] by working with infrared-cutoff operators in a Fock space, and he compensates the resulting violation of Lorentz invariance by adding multiples of the charge operator with suitably chosen functions. In addition to spinors, currents and finite line integrals over currents are also operators which exist in ordinary Fock space.

Boson representation[9] now means that the ψ_0 field itself may be written in terms of j's, so that we obtain all together

$$\psi_1(u,v) = \left(\frac{m}{2\pi}\right)^{1/2} :\exp\{i[\gamma_1 j_l(u)+\gamma_2 j_r(v)]\}:, \tag{2.8a}$$

$$\psi_2(u,v) = \left(\frac{m}{2\pi}\right)^{1/2} :\exp\{-i[\gamma_2 j_l(u)+\gamma_1 j_r(v)]\}:. \tag{2.8b}$$

The double dots mean boson Wick ordering.

By considering the two-point function

$$\langle\psi_1(u,v)\psi_1^\dagger(u',v')\rangle = \frac{m}{2\pi}\exp\left\{-d_\psi \ln[-(u-i\epsilon)(v-i\epsilon)m^2] - s_\psi \ln\frac{u-i\epsilon}{v-i\epsilon}\right\}, \tag{2.9}$$

with

$$d_\psi + s_\psi = \frac{\gamma_1{}^2}{4\pi} = \frac{(\alpha-\beta)^2}{4\pi}, \qquad (2.10a)$$

$$d_\psi - s_\psi = \frac{\gamma_2{}^2}{4\pi} = 1 + \frac{(\alpha+\beta)^2}{4\pi} - \frac{1}{\sqrt{\pi}}(\alpha+\beta), \qquad (2.10b)$$

we obtain the positive dimension d_ψ and the arbitrary continuous spin s_ψ as a function of our parameters. In order to obtain an "intrinsic" parametrization we may (by choosing a suitable sign convention) express γ_1 and γ_2 in terms of d_ψ and s_ψ. The conventionally discussed case is $s_\psi = \frac{1}{2}$. The dimension d_ψ may also be related to the coupling constant g appearing in the Thirring equation of motion; however, this relation depends on what definition one uses for the interacting currents J_μ as a space-time limit in the spinor fields[15] (i.e., whether one uses the definitions of Johnson, Sommerfield, Klaiber, or anyone else). Note that ψ_2 has the same dimension, but the opposite spin. Only the sign of the coupling constant in the Thirring field equation, i.e., the distinction between attraction and repulsion, is intrinsic and does not depend on the renormalization and short-distance limiting procedures, respectively. It is convenient to adopt the following normalization for the interacting current:

$$J_\mu(x) = \text{multiple of } j_\mu \text{ with } [J_0, \psi] = -\delta(x-y)\psi;$$
$$J_{\mu 5} = \epsilon_{\mu\nu} J^\nu \text{ and } J_{\mu 5} = (1/\sqrt{\pi})\partial_\mu \phi,$$

where the pseudo-current potential ϕ is only canonically normalized for $\psi = \psi_0$.

Formula (2.8) still needs an explanatory remark. The fields $j_{l,r}$ are fields which exist in an indefinite-metric space, whereas the application of their exponential function to the vacuum must generate a positive-definite Hilbert space. This is so because exponentials of free zero-mass scalar fields have a built-in charge superselection structure.[16] This charge structure may be understood in several equivalent ways. One way is to work with massive free Bose fields and define the zero-mass expectation value by an $m \to 0$ limit, allowing for a multiplicative m-dependent renormalization of the exponential field.[7] Another way is to say that there should be a *unique* vacuum from which the exponentials generate a positive Hilbert space.[16] In such a situation the linked cluster decomposition property holds for vacuum expectation values. This property requires the vanishing of all correlation functions in which the number of ψ_i is not the same as that of $\psi_i^\dagger$. Actually, formula (2.8) is even implicitly contained in Klaiber's paper[15] if one uses his $d_\psi = 0 = s_\psi$ field σ_i and eliminates the free field ψ_{0i} in (2.1) in favor of σ_i:

$$\psi_i^K = [\text{right-hand side of (2.8)}] \times \sigma_i. \qquad (2.11)$$

The σ_i are "spurions" and do not do anything since the selection rules they carry are already true properties of (2.8).[17]

Note that ψ_1 and ψ_2 have relative commutation instead of anticommutation properties. The change via a Klein transformation yields anticommuting ψ_1 and ψ_2. A compact way of writing the resulting formula is a generalization of Mandelstam's[8] expression to arbitrary Lorentz spin s_ψ:

$$\psi = \left(\frac{m}{2\pi}\right)^{1/2} :\exp\left[-i\frac{\gamma_1+\gamma_2}{2}\gamma^5\varphi - i\frac{\gamma_1-\gamma_2}{2}\int_x^\infty \dot{\varphi}(x')\,dx'\right]: \qquad (2.12)$$

with

$$\varphi = j_l + j_r.$$

The boundary term at ∞ from the potentials is just the charge term needed for implementing the Klein transformation.

The computation of the axial charge by a point-separation limiting procedure gives

$$J_{\mu 5} = N[\bar\psi\gamma_\mu\gamma^5\psi] = \frac{\gamma_1+\gamma_2}{2\pi}\partial_\mu\varphi,$$
$$\phi = -\frac{\gamma_1+\gamma_2}{2\sqrt{\pi}}\varphi. \qquad (2.13)$$

For Sec. III we need to know that the dimension of $\bar\psi\psi$,

$$N[\bar\psi\psi] = \frac{m}{2\pi} :e^{-2i\sqrt{\pi}\phi}:, \qquad (2.14a)$$

is given by the Schwinger term in the axial-vector current:

$$[J_{05}(x), J_{15}(y)]_{t_x=t_y} = \frac{\dim\bar\psi\psi}{\pi}\partial_{x^1}\delta(x^1-y^1). \qquad (2.14b)$$

The computation of composite fields in the boson formalism parallels that of the fermion formalism, for example ($u = u_1 - u_2$, $v = v_1 - v_2$, $s = 0$),

$$\psi_1^\dagger(x_1)\psi_1(x_2) = \frac{m}{2\pi}\left[\frac{m^{-2}}{-(u-i\epsilon)(v-i\epsilon)}\right]^{d_\psi}\left(\frac{v-i\epsilon}{u-i\epsilon}\right)^{s_\psi} :\exp[i\gamma_1(j_l(u_2)-j_l(u_1)) + i\gamma_2(j_r(v_2)-j_r(v_1))]:$$
$$= \frac{m}{2\pi}\left[\frac{m^{-2}}{-(u-i\epsilon)(v-i\epsilon)}\right]^{d_\psi}\left(\frac{v-i\epsilon}{u-i\epsilon}\right)^{s_\psi}[1 + i\gamma_1(u_2-u_1)\partial_u j_l + i\gamma_2(v_2-v_1)\partial_v j_r + \cdots], \qquad (2.15)$$

i.e., the leading nontrivial term contains the current operator. Another interesting case is

$$\psi_1^\dagger(x)\psi_2(y)=\frac{m}{2\pi}\left[\frac{m^{-2}}{-(u-i\epsilon)(v-i\epsilon)}\right]^{\gamma_1\gamma_2/4\pi}\{:\exp[i(\gamma_1-\gamma_2)j_l(u_1)+i(\gamma_2-\gamma_1)j_r(v_1)]:+O(u_1-u_2,v_1-v_2)\}. \tag{2.16}$$

The dimension of the higher terms always increases by one unit. Note that in this case the original operator as well as the leading short-distance term (and the singular function in front) are scalars which are independent of the spin s_ψ.

For the conventional case $s_\psi=\frac{1}{2}$, it is easy to see that the roots of (2.10) for γ_i lead to

$$\begin{aligned}\dim\psi_2^\dagger\psi_1&=\dim\psi_1^\dagger\psi_2\\&=\dim\,[\text{leading operator in (2.16)}]\\&=\frac{(\gamma_2-\gamma_1)^2}{4\pi}\\&=2d_\psi\mp(4d_\psi{}^2-1)^{1/2},\end{aligned} \tag{2.17}$$

where the minus sign holds for the attractive sign of the quadrilinear coupling constant in the Thirring field equation.[15] The only remaining bilinear operator is $\psi_1\psi_2$, which is also scalar and has the dimension

$$\dim\psi_1\psi_2=2d_\psi\pm(4d_\psi{}^2-1)^{1/2}. \tag{2.18}$$

Here the plus sign holds for the attractive case. The boson representation of the Thirring model has the advantage that the range of the dimension of composite fields Θ of the theory becomes evident:

$$\{\dim\Theta\}=\left[\frac{1}{4\pi}(n\gamma_1+m\gamma_2)^2+l\right], \tag{2.19}$$

$$n,m=\pm1,\pm2,\ldots,\quad l=0,1,2,\ldots.$$

The l refers to derivatives of the exponential fields and factors $\partial_u j_l$, $\partial_v j_r$.

For $s_\psi=\frac{1}{2}$ we have

$$\gamma_1=(4\pi+\gamma_2{}^2)^{1/2}, \tag{2.20}$$

$$|\gamma_2|=\sqrt{4\pi}\,(d_\psi-\tfrac{1}{2})^{1/2}. \tag{2.21}$$

Clearly the lowest-dimensional fields are $\psi_1^\dagger\psi_2$ in the attractive region and $\psi_1\psi_2$ for the repulsion. The borderline case between the two regions is the free ψ field $d_\psi=\frac{1}{2}$.

As will become clearer in the next section, the zero-mass limit correctly describes only the leading short-distance singularities of the actual massive model. Therefore arguments based on the structure of the massless model (as, for example, Coleman's perturbation argument[7]) are expected to break down as soon as the considered quantities develop nonleading singularities. This is the basic reason why the particular form of the operator products in Coleman's equivalence of the massive Thirring model with the sine-Gordon model holds in the regime $\dim\bar\psi\psi<1$ (better than superrenormalizable), but breaks down for $\dim\bar\psi\psi\geq1$ (superrenormalizable). For that case one has to use new normal products with an additive renormalization. For the case of a free field we will explicitly identify these operators as being identical to those constructed by Lehmann[17] and Stehr in terms of conventional (but intrinsically not so appropriate) triple-ordered normal products.

The changes in sine-Gordon language for $\dim\bar\psi\psi\geq1$ are not purely academic; they are related to the breakdown of Coleman's canonical formalism and undermine the basis for variational computations based on coherent states.[18] Contrary to the limitation[7] $\dim\bar\psi\psi<2$ obtained on the basis of such estimates for lower bounds of the energy, we believe that the dimensional crossing of nonleading singularities with the leading ones just indicates the onset of a "nonrenormalizable" phase of the model where zero-mass and short-distance behavior become even further separated than in the region $1\leq\dim\bar\psi\psi<2$.

Before we go over to the detailed discussion of the massive model we make an amusing observation[19] on the massless boson representation. For sufficiently strong attractive coupling constant (i.e., $\dim\bar\psi\psi$ sufficiently small), one finds in addition to $\bar\psi\psi$ many other composite fields $\Theta^{(n)}$ with small dimension:

$$\dim\Theta^{(n)}=n^2\times\dim\bar\psi\psi. \tag{2.22}$$

These operators, which appear in the boson representation (2.19) for $m=-n$, $l=0$, are constructed in the spinor formalism as the lowest-dimensional terms in the short-distance expansion of powers of $\psi_1\psi_2^\dagger$:

$$N[\psi_1(x_1)\psi_2^\dagger(x_1)]\cdots N[\psi_1(x_n)\psi_2^\dagger(x_n)]\xrightarrow[x_i\to x]{}f(x_1\cdots x_n)\Theta^{(n)}(x)+\cdots. \tag{2.23}$$

In terms of the potential ϕ of the axial-vector current, these operators have the form

$$:e^{i2\sqrt{\pi}n\phi}:,$$

i.e., in the language of boson representation of the basic field they represent higher harmonics corresponding to polynomials in

$$\psi_1\psi_2^\dagger = \left(\frac{m}{2\pi}\right)^{1/2} : e^{i2\sqrt{\pi}\,\phi} : . \tag{2.24}$$

Added to the Thirring Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{Th}} + \sum_n m_n \mathcal{O}^{(n)}(x) + \text{H.c.}, \tag{2.25}$$

they lead to superrenormalizable interactions which generalize the mass perturbation $m_1 = m$. Considered as perturbations on the free Dirac field, they are nonrenormalizable.

III. SINE-GORDON EQUATION FROM SHORT-DISTANCE PROPERTIES IN THE CASE $\dim\psi_1\psi_2^\dagger < 1$

Consider the quantum version of the classical pseudopotential ϕ of a zero-curl two-dimensional axial-vector current:

$$\partial^\mu J_\mu = \partial^\mu \epsilon_{\mu\nu} J_5^\nu(x) = 0, \quad J_{\nu 5} = \frac{1}{\sqrt{\pi}}\,\partial_\nu \phi . \tag{3.1}$$

As in Sec. II, the normalization of the axial-vector current, and therefore of ϕ, is such that

$$Q = \int J_0(x)\,dx = \int J_{1,5}\,dx \tag{3.2}$$

is the charge operator with integer eigenvalues. As in the classical theory, there exists a Lorentz pseudoscalar potential ϕ with the following properties[20]:

(a) It is a local field.

(b) It is a nonlocal relative to the charged fields:

$$\frac{1}{\sqrt{\pi}}\,[\phi(x), \psi(y)] = \theta(x^{(1)} - y^{(1)})[Q, \psi(y)] \tag{3.3}$$

for $(x-y)^2 < 0$.

The field ϕ is unique; formally, it is $\phi(x) = \sqrt{\pi}\int_{-\infty}^{x} J_{\nu 5}\,ds^\nu$.

Up to now we used only the conservation of the vector current. In order to obtain some more detailed properties of ϕ we have to use specific properties of the massive Thirring model. From the study of the Callan-Symanzik equation of this model, it is known that its zero-mass limit (suitably parametrized, depending on the renormalization procedure for the massive model) is the massless Thirring model.[21] In addition, the leading short-distance singularities of the (composite) fields in the massive model are determined (up to unknown numerical factors) by the massless limit. The two-point function has the form

$$\begin{aligned} \langle \phi(x)\phi(y)\rangle &= \int \rho(s^2)\, i\Delta_{ms}^{(+)}(\xi)\,ds^2 , \\ i\Delta_{ms}^{(+)}(\xi) &= \frac{\pi}{4}\,H_0^1(ms(-\xi^2 + i\epsilon\,\xi_0)^{1/2}), \end{aligned} \tag{3.4}$$

and

$$\rho(s^2) = (s^2)^{a-3}c \quad \text{for } s\to\infty ,$$

where $a = \dim\psi_1^\dagger\psi_2$.

The asymptotic form of ρ is a straightforward consequence of the definition of ϕ, which on the vacuum takes the simple form

$$\begin{aligned} \phi(x)|0\rangle &= \int_{-\infty}^{\infty} D(x-x')\partial^\mu J_{\mu 5}(x')\,d^2x'|0\rangle \\ &= 2im\int D(x-x')N[\bar\psi\gamma_5\psi](x')\,d^2x'|0\rangle , \end{aligned} \tag{3.5}$$

or alternatively is directly obtainable from

$$\Box\phi|0\rangle = 2imN[\bar\psi\gamma_5\psi]|0\rangle . \tag{3.6}$$

Here we use the "normal-product" notation for the composite field $\bar\psi\gamma_5\psi$.

Up to a constant in front, the ϕ behaves precisely as a canonical zero-mass scalar field for short distances as long as

$$\dim\psi_1^\dagger\psi_2 = \dim\bar\psi\gamma_5\psi < 2 . \tag{3.7}$$

Only in this region does the zero-mass Thirring model give the correct picture of the current: The short-distance behavior of the current is (up to normalizations) that of a derivative of a free scalar field. Hence the two-point function subtraction, i.e., the ordinary (double dot) Wick-ordering of ϕ, is sufficient to define local polynomials $:\phi^n:$, i.e.,

$$\begin{aligned} :\phi^2(x): &= \lim_{y\to x}[\phi(x)\phi(y) - \langle\phi(x)\phi(y)\rangle], \\ :\phi^4(x): &= \lim_{x_j\to x} \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\ &\quad - \prod_{\substack{i<j\subset I\\ n,m\subset J}} \phi(x_i)\phi(x_j)\langle\phi(x_n)\phi(x_m)\rangle, \text{ etc.} \end{aligned} \tag{3.8}$$

Even in this restricted region of coupling the short-distance behavior of 4-composite shows nonleading singular mass terms. For example,

$$\begin{aligned} \langle N[\bar\psi\gamma_5\psi](x)N[\bar\psi\gamma_5\psi](y)\rangle &= \int_0^\infty \hat\rho(s^2)\, i\Delta_{ms}^{(+)}(\xi)\,ds^2 , \\ \hat\rho(s^2) \sim (s^2)^2\rho(s^2) &= (s^2)^{a-1}c\cdots, \quad s^2\to\infty , \end{aligned} \tag{3.9}$$

and therefore[22]

$$\begin{aligned} \langle N[\bar\psi\gamma_5\psi](x)N[\bar\psi\gamma_5\psi](y)\rangle &\sim (\xi^2)^{-a}c_1 \\ &\quad + (\xi^2)^{2-a}c_2(m^2)^{2-a}\ln^m(\xi^2 m^2) \\ &\quad + \cdots . \end{aligned}$$

The appearance of mass "daughter dimensions" can of course also be seen from the Callan-Symanzik equations. For $\dim\bar\psi\psi < 1$ (attractive coupling regime), there are no such persistent mass effects in the two-point function of the bilinear spinor expressions. In this regime we have:

Main statement. For $\dim\bar\psi\psi<1$ ($g<0$, i.e., attractive Thirring coupling) there exists a g-dependent (or what is the same thing, d_ψ-dependent) constant $C(g)$ such that

$$N[\psi_1^\dagger\psi_2](x)=C(g)m(:e^{-2\sqrt{i}\,\pi\,\phi(x)}:-1). \qquad (3.10)$$

Clearly the sine-Gordon equation is a consequence of

$$\frac{1}{\sqrt{\pi}}\Box\phi=\partial^\mu J_{\mu5}=2imN[\bar\psi\gamma_5\psi] \qquad (3.11)$$

and this main statement.

A note of caution: The existence of the exponential expression as a bona fide operator is not an entirely trivial matter. However, since we promised in the Introduction not to concern ourselves with methods of constructive quantum field theory, we will be content simply with the remark that for $\dim\bar\psi\psi<1$ exponentials can be handled by constructive field theorists.[23]

For the proof of the main statement we note that the commutation relation (3.3) of ϕ with ψ leads to

$$:e^{-i2\sqrt{\pi}\lambda\phi(x)}:\psi(y)$$
$$=\exp[2\sqrt{\pi}\,i\lambda\,\theta(x'-y')]\psi(y):e^{-i2\sqrt{\pi}\,\lambda\phi(x)}:, \qquad (3.12)$$

where x' and y' are spatial components of x and y. For $\lambda=1$ this equation says that the Wick-ordered exponential is local relative to the ψ. This local exponential field has the same short-distance dimension as $\bar\psi\psi$. This follows from the fact that

$$\langle:e^{-i2\sqrt{\pi}\,\phi(x)}::e^{i2\sqrt{\pi}\,\phi(y)}:\rangle\simeq C\exp[-a\ln(-m^2\xi_-^{\,2})],$$
$$a=\int\rho(s^2)\,ds^2 \qquad (3.13)$$
$$=\text{Schwinger term in }[J_{05}(x),J_1(y)].$$

The Schwinger term is independent of the mass, and according to the zero-mass formula it is related to the dimension of $\bar\psi\gamma_5\psi$:

$$a=\dim\psi_1^\dagger\psi_2\,. \qquad (3.14)$$

The constant C consists of an exponential of a sum over higher-point truncated functions,

$$\langle\phi^n(x)\phi^m(y)\rangle_T\,,\quad n+m>2 \qquad (3.15)$$

which for $y\to x$ have the same singular behavior as the corresponding zero-mass potentials, i.e., they remain finite, but do not vanish as for zero mass.

If we now define a local field (relatively local with respect to ψ) by

$$\mathcal{O}(x)=N[\psi_1^\dagger\psi_2](x)$$
$$-C(g)m(:e^{-2i\sqrt{\pi}\,\phi(x)}:-1), \qquad (3.16)$$

then we may easily choose the $C(g)$ in such a way that the short-distance singularities of the two-point function $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle$ cancel. Consider the following lemma.[24]

Lemma: A local field $\mathcal{O}(x)$, whose two-point function does not have a short-distance singularity, is a multiple of the identity.

With this lemma we have established the main statement (the multiple of the identity is zero: $\langle\mathcal{O}\rangle=0$). This lemma can be thought of as a generalization of Schur's lemma of canonical quantum-field theory to local noncanonical quantum-field theory.

The picture of purely multiplicative renormalization of composite fields with small dimension, for example,

$$N[\psi_1^\dagger\psi_2]=\lim_{x\to0}|x|^{2b-a}\psi_1^\dagger(x)\psi_2(0), \qquad (3.17)$$
$$b=\dim\psi,\quad a=\dim\psi_1^\dagger\psi_2$$

breaks down owing to the onset of nonleading mass singularities for $a\geq1$, and we obtain

$$N[\psi_1^\dagger\psi_2]=\lim_{x\to0}|x|^{2b-a}[\psi_1^\dagger(x)\psi_2(0)-\langle\ \rangle], \qquad (3.18)$$

i.e., there are nonvanishing chiral-symmetry-breaking expectation values to be subtracted. For example, in the canonical free-field case ($g=0$) the multiplicative factor is unity and we have

$$N[\psi_1^\dagger\psi_2]=\lim_{x\to0}[\psi_1^\dagger(x)\psi_2(x)-\langle\psi_1^\dagger(x)\psi_2(0)\rangle], \qquad (3.19)$$

where the last term is the logarithmic divergent part of the free massive propagator. Equation (3.18) breaks down only if the mass insertion term of the Callan-Symanzik equation cannot be neglected, i.e., for $a\geq2$. What is the repercussion of this change on the sine-Gordon level? From (3.4) we can say that even though the canonical character of ϕ is maintained, the canonical conjugate π has a short-distance expansion with a divergent term:

$$\langle\pi(x,0)\pi(y,0)\rangle=\text{divergent},\quad \pi=d\phi/dt\,.$$

This divergence indicates potential short-distance trouble, although the proper canonical formalism breaks down only for $a\geq\frac{3}{2}$. There is an interesting manifestation of the nonleading singularities in the form of cumulative mass effects in the exponential functions of ϕ for $a\geq1$. They will be the main topic of the next section.

IV. NONLEADING SINGULARITIES AND CUMULATIVE MASS EFFECTS IN EXPONENTIAL FUNCTIONS

We now come to the central part of our discussion, namely the consequences of the nonleading short-distance singularities in the definition of (3.18) for the exponentials of the axial-vector currents. We expect, by analogy to the spinors, that the Wick-ordered exponential for $\lambda \to 1$ blows up as a c number:

$$\vdots e^{-2i\sqrt{\pi}\lambda\phi}\vdots \sim F(1-\lambda)I, \tag{4.1}$$

with

$$F(1-\lambda)\xrightarrow[\lambda=1]{} \infty.$$

Let us reinforce this picture with an explicit computation for the "free" axial potential (i.e., belonging to the free massive Dirac equation):

$$\phi(x)=\frac{i}{4\sqrt{\pi}}\int\int d\theta_p d\theta_q\left[\frac{-1}{\sinh[\frac{1}{2}(\theta_p-\theta_q)]}(e^{i(p-q)x}a_p^\dagger a_q+e^{-i(p-q)x}b_q^\dagger b_p)\right.$$
$$\left.+\frac{1}{\cosh[\frac{1}{2}(\theta_p-\theta_q)]}(e^{i(p+q)x}a_p^\dagger b_q^\dagger+e^{-i(p+q)x}a_q b_p)\right]. \tag{4.2}$$

For this model it is standard practice[25] to introduce in addition to the Wick ordering as, for example, used and explained by Coleman,[7] the "triple ordering:"

$$\vdots\phi^n(x)\vdots=\lim_{x_1,\ldots,x_n\to x}\vdots\phi(x_1)\cdots\phi(x_n)\vdots, \tag{4.3a}$$

with

$$\vdots\phi(x_1)\cdots\phi(x_n)\vdots=\phi(x_1)\cdots\phi(x_n)-\sum_{r=1}^{n}\sum_{p}\langle\phi(x_{i_1})\cdots\phi(x_{i_r})\rangle\vdots\phi(x_{j_1})\cdots\phi(x_{j_{n-r}})\vdots, \tag{4.3b}$$

where $\sum_p$ is the summation over all partitions of $1,\ldots,n$ into disjoint subsets with

$$i_1<i_2<\cdots<i_r,\quad j_1<j_2<\cdots<j_{n-r}.$$

The simplicity obtained from the introduction of triple-dot ordering appears in computing truncated functions. Consider for example the correlation function

$$\langle\phi(x)\vdots\phi^n(0)\vdots\rangle. \tag{4.4}$$

On the one hand, we may use the fact that triple ordering implies partial ordering of the fermion operators. In (4.4) we therefore pick only the term in $:\phi^n:$ proportional to $a^\dagger b^\dagger$. A straightforward combinatorial consideration yields an expression in terms of a generating function:

$$\langle 0|a_q b_q\vdots\phi^n(0)\vdots|0\rangle=\frac{1}{(-2i\sqrt{\pi})^n}\frac{\partial^n}{\partial\lambda^n}\frac{1}{2\pi}\frac{\sin\pi\lambda\, e^{\lambda(\theta_p-\theta_q)}}{\cosh\frac{1}{2}(\theta_p-\theta_q)}\bigg|_{\lambda=0}, \tag{4.5a}$$

or[26]

$$\langle\phi(x)\vdots e^{-2i\sqrt{\pi}\lambda\phi(0)}\vdots\rangle=-2i\frac{\sin\pi\lambda}{\sqrt{\pi}}\int_0^\infty\frac{\cosh 2\lambda\beta}{\cosh^2\beta}i\Delta_\mu^{(+)}(x)d\beta, \tag{4.5b}$$

with $\mu=2m\cosh\beta$. On the other hand, the formula (4.3) together with the recursive definition of truncated functions immediately yields

$$\langle\phi(x):\phi^n(0):\rangle=\langle\phi(x)\phi^n(0)\rangle_T. \tag{4.6}$$

Therefore the short-distance limit leads to

$$\langle\phi(x)\phi^n(0)\rangle_T\underset{x\to 0}{\simeq}\begin{cases}\frac{1}{2\pi}\ln\sqrt{-x^2}, & n=1\\ C_n, & n>1\end{cases} \tag{4.7}$$

where we have used[27]

$$\int_0^\infty\frac{\cosh 2\lambda\beta}{(\cosh\beta)^{2\kappa}}d\beta=2^{2\kappa-2}\frac{\Gamma(\kappa+\lambda)\Gamma(\kappa-\lambda)}{\Gamma(2\kappa)}, \tag{4.8}$$

and the constants C_n can be easily read off by differentiating with respect to κ (thus generating a $\ln\cosh\beta$) and then setting κ equal to unity. We find that our general picture following from asymptotic scale invariance for dim $\bar{\psi}\psi<2$ is confirmed: The truncated n-point functions for $n>2$ approach finite limits. These limits are universal numbers which for the general massive Thirring model would only depend on the dimensionless coupling g. These numbers contain information outside the scale-invariant theory, and we doubt that they can be derived on the basis of the explicit knowledge of the scale-invariant theory alone (including the knowledge of all composite fields). They constitute important information in the understanding of "cumulative mass effects." Consider the Wick product and its connection with the triple product slightly below the value $\lambda = 1$, which formally corresponds to the expression $N[\psi_1^\dagger \psi_2]$:

$$:e^{-2i\sqrt{\pi}\lambda\phi}: = \vdots e^{-2i\sqrt{\pi}\lambda\phi} \vdots \exp\left[\sum_{n=4}^{\infty} \frac{(-2\sqrt{\pi}\, i\lambda)^n}{n!} \langle \phi^n \rangle_T\right]. \tag{4.9}$$

The fastest way to see what happens for $1-\lambda \to 0_+$ is to realize that the series (up to a finite additive constant) can be gotten by

$$\int_0^{1-\lambda} C(\mu)d\mu, \tag{4.10}$$

with

$$C(\mu) = \frac{-i}{2\pi} \frac{\sin \pi\mu}{\sqrt{\pi}} \frac{d}{d\xi}\left[2^{2\xi-2} \frac{\Gamma(\xi+\mu)\Gamma(\xi-\mu)}{\Gamma(2\xi)}\right]_{\xi=1}.$$

The integrand clearly has a divergence at $\mu = 0$ and we obtain

$$\int_0^{1-\lambda} C(\mu)d\mu \sim -\ln(1-\lambda), \tag{4.11}$$

leading to

$$\exp\left[\sum_{n=4}^{\infty} \frac{(-2\sqrt{\pi}\, i\lambda)^n}{n!} \langle \phi^n \rangle_T\right] \sim \frac{1}{(1-\lambda)}, \tag{4.12}$$

i.e., the convergence of the series breaks down.

Looking back now at (4.5b) we see that

$$\lim_{\lambda\to 1} \vdots e^{-2i\sqrt{\pi}\lambda\phi} \vdots = 1, \tag{4.13}$$

at least inside a two-point function. So our picture about the c-number divergence of the $\lambda \to 1$ limit is completely consistent with explicit computations. A first trial in such a situation would be to define a new finite normal product by

$$N[e^{-2i\sqrt{\pi}\phi}] = \lim_{\lambda\to 1} \frac{1}{(\lambda-1)} (:e^{-2i\sqrt{\pi}\lambda\phi}: - \langle :e^{-2i\sqrt{\pi}\lambda\phi}: \rangle 1)$$
$$= \vdots e^{-2i\sqrt{\pi}\phi}\phi \vdots. \tag{4.14}$$

Such a definition would be in agreement with keeping the correct dimension, but is in disagreement with the locality relation to the spinor fields. The way to complete (4.14) to a local operator becomes obvious if we study the differentiated commutation relation (3.12) with triple-dot ordering at $\lambda = 1$:

$$-2i\sqrt{\pi} \vdots \phi(0)^{-2i\sqrt{\pi}\phi(0)} \vdots \psi(x) + 2i\sqrt{\pi}\psi(x) \vdots \phi e^{-2i\sqrt{\pi}\phi(0)} \vdots$$
$$= \psi(x) \vdots e^{-2i\sqrt{\pi}\phi(0)} \vdots. \tag{4.15}$$

By subtracting the differentiated commutation relation for $\lambda = 0$

$$[\vdots \phi(0)(e^{-2\sqrt{\pi}i\phi(0)} - 1) \vdots, \psi(x)] = 0.$$

Clearly the locality of the difference, i.e., of

$$\vdots \phi(0)(e^{-2\sqrt{\pi}i\phi} - 1) \vdots, \tag{4.16}$$

is necessary and sufficient for the triviality (4.13) of the exponential.

The causal completion of (4.14), i.e., that operator which takes over the role of the exponential, can be written as

$$N[e^{-2i\sqrt{\pi}\phi}] = \lim_{\epsilon\to 0} \frac{1}{2i\sqrt{\pi}\epsilon} \{:e^{-2i\sqrt{\pi}(1-\epsilon)\phi}: - \langle :e^{-2\sqrt{\pi}i(1-\epsilon)\phi}: \rangle :e^{-2i\sqrt{\pi}\epsilon\phi}:\} \sim \vdots \phi(e^{-i2\sqrt{\pi}\phi} - 1) \vdots. \tag{4.17}$$

This idea of constructing a local operator by a λ-limiting procedure which replaces the formal Wick-ordered expressions which ceases to make sense can be formulated for all $\lambda = 1, 2, \ldots$. This is of relevance to the solution of the Federbush model,[28] and we will return to this in another publication. For the sine-Gordon model such an extension is not relevant since the leading singularities are overtaken by the nonleading ones already for dim $\psi_1^\dagger\psi_2 = 2$. The discussion leading to the normal product is from a physical point of view still somewhat unsatisfactory because we are approaching a local situation ($\lambda = 1$) as a limit of nonlocal fields. In the spinor formulation (3.18) defines a local limiting procedure for the exponential of ϕ. A natural bosonic candidate is the bilocal product (subtracting the vacuum expectation value)

$$B(x,0) = :e^{-i\sqrt{\pi}\phi(x)}::e^{-i\sqrt{\pi}\phi(0)}: - \langle \cdots \rangle 1. \tag{4.17'}$$

Clearly each factor is free of the cumulative effect, and according to (3.3) anticommutes with the ψ's and $\psi^\dagger$'s for spacelike distances. So the whole bilocal aggregate commutes if the ψ coordinate is spacelike relative to x and zero. We obtain

$$B(x,0) = C\, e^{(1/4)\ln(-x^2)} \exp\left[\sum_4^{\infty} \frac{(-i\sqrt{\pi})^n}{n!} \langle\langle \phi(x) + \phi(0))^n \rangle_T\right](\vdots e^{-i\sqrt{\pi}(\phi(x)+\phi(0))} \vdots - 1). \tag{4.18}$$

The first factor is the factor one naively expects from the analogy to the zero-mass version; the second factor is very difficult to compute [we think it is $O(1)$, but this is irrelevant to the following considerations]. The most interesting object is the triple-ordered bilocal which we can obtain from the B by suitable division. The operator has remarkable properties. Let us write this operator as

$$\vdots\, e^{-\sqrt{\pi}(\phi(x)-\phi(0)+2\phi(0))} \vdots - 1 = -i\sqrt{\pi}\,\vdots (\phi(x)-\phi(0))\, e^{-2i\sqrt{\pi}\,\phi(0)} \vdots + \frac{(-i\sqrt{\pi})^2}{2!} \vdots (\phi(x)-\phi(0))^2 e^{-2i\sqrt{\pi}\,\phi(0)} \vdots + \cdots . \qquad (4.19)$$

From (4.5b) and its first derivative at $\lambda = 1$ the following operator relation is suggested:

$$\vdots (\phi(x)-\phi(0))\, e^{-2i\sqrt{\pi}\,\phi(0)} \vdots = \phi(x) - \vdots\, \phi(0) e^{-2i\sqrt{\pi}\,\phi} \vdots . \qquad (4.20a)$$

Like (4.13), this relation follows rigorously from the fermion-ordering algorithm of Lehmann and Stehr.[12] It looks somewhat strange since it lacks manifest periodicity, but this is just because triple orderings for exponentials are not intrinsic natural normal products. It is better to call the limit $x \to 0$ of this operator following (4.17)

$$N[\, e^{-i2\sqrt{\pi}\,\phi(0)}\,] = 2i\sqrt{\pi}\,\vdots\, \phi(0)(e^{-2i\sqrt{\pi}\,\Phi} - 1) \vdots . \qquad (4.20b)$$

With this notation the vanishing of the higher terms is plausible since the rule

$$\lim_{x\to 0} N[(\phi(x)-\phi(0))\, e^{-2i\sqrt{\pi}\,\phi(0)}\,] \to 0 \qquad (4.21)$$

appears to be "natural." The relation (4.21) and the corresponding higher relations can be directly checked in the needed form

$$\vdots (\phi(x)-\phi(0))^n e^{-2i\sqrt{\pi}\,\phi(0)} \vdots \underset{x\to 0}{\longrightarrow} 0 \text{ for } n > 1 . \qquad (4.22)$$

If one is satisfied with just seeing consistency in special configurations one can generalize and adapt (4.5b) to the problem at hand. The general operator proof again has to be based on the fermion-ordering algorithm.[12]

However, the generalization of this reordering formalism to bilocals is very involved and will be discussed in detail in a future publication. An alternative method to obtain the exponential by a space-time limiting procedure has been given by Swieca.[29] It consists in

$$N[\, e^{-i2\sqrt{\pi}\,\phi(0)}\,] = \lim_{f\to\delta} :e^{-2i\sqrt{\pi}\,\phi(f)}: - \langle 0|\, :e^{-i2\sqrt{\pi}\,\phi(f)}:\, |0\rangle \qquad (4.23)$$

for a suitable local δ-function sequence. The cumulative mass effects will depend in this formulation is a very complicated nonlinear way on f and we have not been able to demonstrate that this limit exists and converges to our N. The derivation of the sine-Gordon equation using the N ordering for the free Dirac equation follows now by explicit computation. From (4.20) and (4.5b) we obtain ($\mu = 2m\cosh\beta$)

$$2m^2 \langle 0|\, \phi(x) N[\sin 2\sqrt{\pi}\,\phi](0)\, |0\rangle = 2m^2\sqrt{\pi}\, 2 \int_0^\infty \Delta_\mu^{(+)}(x)\, d\beta = \sqrt{\pi}\, \langle \phi(x) \Box \phi(0) \rangle , \qquad (4.24)$$

i.e., the sine-Gordon equation in a special configuration. The validity for all configurations, i.e., as an operator relation, is a result of the formalism of Lehmann and Stehr.[12] In the general case dim $\bar\psi\psi < 2$, one again argues along the lines of the properties[30] of $\mathcal{O}(x)$ (3.16). However, now only the leading singularities cancel. The operator $\mathcal{O}$ is hence a local operator in the massive Thirring model with a possible nonleading short-distance singularity (containing explicitly the mass). In the local algebra generated by even functions of ψ and $\psi^\dagger$ such an operator does not exist. Hence $\mathcal{O}$ can only be a multiple of the identity and because of its normalization properties it should vanish. However, this argument is not rigorous since the local completeness of polynomials of a basic field (whose asymptotic dimension does not vanish) has not been established in "axiomatic field theory."

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