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Fields and observables in the massive Schwinger model*

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The correspondences proposed by Coleman, Jackiw, and Susskind for the massive Schwinger model are examined in terms of an underlying field theory formulated in terms of fermion operators and vector potentials. By considering QED as a limit of a massive vector-meson theory, the stability of the proposed correspondences for the gauge-invariant quantities against a fermion mass addition is established.

I. INTRODUCTION

Coleman, Jackiw, and Susskind[1] have shown that quantum electrodynamics of massive fermions in two-dimensional space-time (massive Schwinger model) should have the same observable content as that of the theory of a scalar field satisfying a modified sine-Gordon equation.

Although it is clear that the observable content of QED should be entirely contained in the algebra of gauge-invariant quantities, it is nevertheless instructive to take a more traditional approach by regarding QED as a theory formulated in terms of fermion fields and electromagnetic potentials. There exist, in principle, many ways of embedding a gauge-invariant algebra in a field algebra. We choose a particular way of doing this, by regarding the massive Schwinger model as the limiting case of a model of massive fermions interacting with vector mesons (massive Thirring-Wess model), which provides us with a better insight into the mechanism of fermion confinement[2] and makes the parallelism between the massive and massless Schwinger model[3] more transparent. Here the explicit breakdown of gauge invariance of the second kind allows one to obtain a local solution for both the fermion and vector-meson fields in a positive-metric Hilbert space. As long as the bare mass of the vector meson is different from zero there is no fermion confinement and the theory exhibits the usual charge-sector structure. The gauge-invariant quantities obtained in the QED limit are found to have the same form as the corresponding quantities of the massless Schwinger model, thus proving the assumed stability of observables against the addition of a fermion mass. In the QED limit confinement appears in our approach as the result of the decoupling of the fermionic degree of freedom responsible for the charge sectors in the Thirring-Wess model.

The material of this paper is arranged as follows. In Sec. II we obtain a formal Coulomb-gauge solution. The nonexistence of a bona fide Coulomb-gauge fermion field will be related (in Sec. V) to the nonexistence of charge sectors in the massive Schwinger model. In Sec. III we investigate in detail the field equations of the massive Thirring-Wess model. Section IV will be devoted to a further discussion of the fermion equation of motion and to the Lorentz invariance of the theory. Our concluding section deals with the recovery of the massive Schwinger model as the limit of the massive Thirring-Wess model. In this section the periodic vs nonperiodic nature of the underlying scalar field equations will play a fundamental role with regard to the existence or nonexistence of charge sectors. Our conventions will be

$$\gamma^0=\sigma_1,\ \gamma^1=i\sigma_2,\ \gamma^5=-\sigma_3,\ \epsilon^{10}=\epsilon_{01}=1\ .$$

II. FORMAL CONSIDERATIONS IN THE COULOMB GAUGE

It is well known that the massless Schwinger model[3,4] can be entirely described in terms of a massive pseudoscalar field $\Sigma(x)$ satisfying the equation of motion

$$(\Box+e^2/\pi)\Sigma(x)=0\ . \tag{2.1}$$

The corresponding electromagnetic current is given by

$$j^\mu(x)=-\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Sigma(x) \tag{2.2}$$

and, in a suitable Lorentz gauge,

$$\psi(x)=(\mu/2\pi)^{1/2}e^{-(i\pi/4)\gamma^5}e^{i\sqrt{\pi}\gamma^5\Sigma(x)}\begin{pmatrix}e^{i\theta_1}\\ e^{i\theta_2}\end{pmatrix}\ , \tag{2.3a}$$

$$A^\mu(x)=-\frac{\sqrt{\pi}}{e}\epsilon^{\mu\nu}\partial_\nu\Sigma(x)\,, \tag{2.3b}$$

$$E(x)=-\frac{e}{\sqrt{\pi}}\Sigma(x)\ . \tag{2.3c}$$

Here ψ, A^μ, and E denote the "fermion," vector, and electric field, respectively, $\mu^2=(e^2/4\pi)e^{2\gamma}$ with γ the Euler constant, and θ_1, θ_2 label different vacuums corresponding to the spontaneous symmetry breakdown of both gauge invariance and γ^5 invariance in the model. In what follows we choose $\theta_1 = \theta_2$ without an essential loss in generality.

Since the observables of this model are gauge invariant, they will depend only on the pseudoscalar field $\Sigma(x)$. Hence, the Hamiltonian density of the theory must necessarily be of the form

$$\mathcal{H}_0(x) = \frac{1}{2} : \left[\pi_\Sigma{}^2 + (\nabla\Sigma)^2 + \frac{e^2}{\pi}\Sigma^2\right] :$$

The states of the physical Hilbert space are obtained by applying gauge-invariant operators to the vacuum. The algebra of gauge-invariant observables is generated by the bilocals

$$T(x, y) \simeq \psi(x)\exp\left[ie\int_x^y dz^\mu A_\mu(z)\right]\psi^*(y) , \qquad (2.4)$$

where the symbol $\simeq$ indicates that the operator product still needs to be defined. Recalling the expressions (2.3), a convenient definition of (2.4) is provided by

$$T(x, y) = N(x-y) : \exp\left\{i\sqrt{\pi}\left[\gamma_x^5\Sigma(x) - \int_x^y dz_\mu \epsilon^{\mu\nu}\partial_\nu\Sigma(z) - \gamma_y^5\Sigma(y)\right]\right\} : , \qquad (2.5a)$$

with

$$N(z) = -\frac{1}{2\pi}\begin{pmatrix} \dfrac{-i}{z^0+z^1} & -\mu \\ -\mu & \dfrac{-i}{z^0-z^1} \end{pmatrix}, \qquad (2.5b)$$

where the singularities have been chosen such as to ensure that the bilocals transform as the bilinear product of spin-$\frac{1}{2}$ fields. The current (2.2) as well as the scalar and pseudoscalar densities are then simply obtained as the gauge-invariant limits

$$j^\mu(x) = -\lim_{\epsilon\to 0}\{\mathrm{Tr}[\gamma^0\gamma^\mu T(x+\epsilon, x)] - \langle 0|\mathrm{Tr}[\gamma^0\gamma^\mu T(x+\epsilon, x)]|0\rangle\} , \qquad (2.6)$$

$$-\lim_{\epsilon\to 0}\mathrm{Tr}[\gamma^0 T(x+\epsilon, x)] \equiv N[\bar\psi(x)\psi(x)] = -\frac{\mu}{\pi} : \cos[2\sqrt{\pi}\Sigma(x)] : , \qquad (2.7)$$

$$-\lim_{\epsilon\to 0}\mathrm{Tr}[\gamma^0\gamma^5 T(x+\epsilon, x)] \equiv N[\bar\psi(x)\gamma^5\psi(x)] = -\frac{i\mu}{\pi} : \sin[2\sqrt{\pi}\Sigma(x)] : \qquad (2.8)$$

Following Coleman, Jackiw, and Susskind[1] we describe the massive Schwinger model by the Hamiltonian density

$$\mathcal{H}(x) = \mathcal{H}_0(x) + MN[\bar\psi(x)\psi(x)] , \qquad (2.9)$$

where the mass term is given by Eq. (2.7). The Hamiltonian (2.9) leads to the equation of motion

$$(\Box + e^2/\pi)\Sigma(x) = -\frac{2}{\sqrt{\pi}}\mu M : \sin[2\sqrt{\pi}\Sigma(x)] : . \qquad (2.10)$$

Equation (2.10) implies the nonconservation of the axial-vector current, $j^{5\mu} = (1/\sqrt{\pi})\partial^\mu\Sigma$,

$$\partial_\mu j^{5\mu} = \frac{-e^2}{\pi\sqrt{\pi}}\Sigma - iMN[\bar\psi\gamma^5\psi] ,$$

where $N[\bar\psi\gamma^5\psi]$, given by Eq. (2.8), is the usual γ^5-invariance breaking term arising from the fermion mass in the Hamiltonian, and the term proportional to $e^2\Sigma$ is the two-dimensional analog of the axial-vector anomaly.[5]

Although one expects that the gauge-invariant quantities such as the current, the electric field, and the bilocals are still of the form (2.2), (2.3c), and (2.5), with Σ now a solution to the equation of motion (2.10), the correspondences (2.3a) and (2.3b) for the gauge-dependent fields can no longer be maintained as we now show.

Starting from Maxwell's equation in the Lorentz gauge,

$$\Box A_\mu(x) = -ej_\mu(x) ,$$

we obtain, using the correspondence (2.2) for the electromagnetic current,

$$\Box\epsilon^{\mu\nu}\partial_\mu A_\nu = \frac{e}{\sqrt{\pi}}\Box\Sigma ,$$

which has the solution

$$A^\mu(x) = \frac{e}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Box^{-1}\Sigma(x) . \qquad (2.11)$$

Hence, unless Σ is a free field, A_μ is a nonlocal operator in space and time, and the contemplated correspondence given by Eq. (2.3b) no longer holds. Since unequal-time dynamics is involved, we cannot expect to solve the Dirac equation with A^μ given by Eq. (2.11) because the correspondence (2.3a) is evidently also violated.

However, we can use the gauge freedom of the theory in order to remove the nonlocality in time, which will lead us in a natural way to the Coulomb-gauge formulation of the massive Schwinger model.[6] The Coulomb-gauge computations are, however, of at most a heuristic value, since it is known that due to the growth of the Coulomb potential in two dimensions, no bona fide fermion operators can be constructed in this gauge (see Sec. V for a more detailed discussion). This is, of course, intimately connected with the confinement of the underlying fermions in this model.

Nevertheless, we consider the formal gauge transformation

$$A_\mu \to A_\mu^c = \frac{e}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu \Box^{-1}\Sigma + \frac{1}{e}\partial_\mu \Lambda .$$

Demanding that $A_1^c = 0$ we find, as a possible choice for $\Lambda(x)$,

$$\Lambda(x) = -\frac{e^2}{\sqrt{\pi}}\int_{x^1}^\infty dy^1 \Box^{-1}\dot{\Sigma}(x^0, y^1) .$$

Hence, in the Coulomb gauge,

$$A_1^c = 0 ,$$

$$A_0^c(x) = -\frac{e}{\sqrt{\pi}}\partial_1 \Box^{-1}\Sigma - \frac{e}{\sqrt{\pi}}\int_{x^1}^\infty dy^1 \Box^{-1}\ddot{\Sigma}(x^0, y^1)$$

$$= -\frac{e}{\sqrt{\pi}}\int_{x^1}^\infty dy^1 \Sigma(x^0, y^1) .$$

Using the techniques of the following section one finds that the ansatz

$$\psi^c(x) = \exp[-i(\pi/4)\gamma^5] : \exp\left[i\sqrt{\pi}\gamma^5\Sigma(x) + i\sqrt{\pi}\int_{x^1}^\infty dy^1 \dot{\Sigma}(x^0, y^1)\right] : \tag{2.13}$$

also provides a formal solution of the corresponding Dirac equation. Returning to the formal definition (2.4) for the bilocals, and recalling that the line integral vanishes identically in the Coulomb gauge since $A_1^c = 0$, we have for equal times and a straight integration path

$$T(x, y) \propto \psi^c(x)\psi^{c*}(y) ,$$

which is a result consistent with our input, Eq. (2.5).

III. THE THIRRING-WESS MODEL WITH MASSIVE FERMIONS

A better insight into the behavior of gauge-dependent operators, which at the same time throws additional light on the problem of confinement, can be obtained by regarding quantum electrodynamics, both massless and massive, as the limit of a vector-meson theory (Thirring-Wess model[7,4,8]). This means that we explicitly break gauge invariance of the second kind by a bare mass m_0 for the vector meson, which in fact allows us to obtain a well-defined local solution of the coupled Proca-Dirac equations. We shall then recover from here the corresponding expression for the gauge-invariant observables of QED in the limit $m_0 \to 0$. On the other hand, the gauge-dependent operators will show a pathological behavior which is entirely due to gauge excitations and hence has no observable consequences.

From the known results on the massless Thirring-Wess model[7,4] we expect our physical Hilbert space to contain a bosonic and a fermionic degree of freedom, Σ and ϕ, of which the fermionic degree of freedom is expected to decouple from the gauge-invariant observables in the limit $m_0 \to 0$. Accordingly, we make the following ansatz for the fermion field[9,10]:

$$\psi_\alpha(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} e^{-i(\pi/4)\gamma^5_{\alpha\alpha}} : e^{i\chi_\alpha(x)} : , \tag{3.1a}$$

where $\mu^2 = (e^2/4\pi)e^{2\gamma}$ and

$$\chi_\alpha(x) = \frac{\alpha}{2}\gamma^5_{\alpha\alpha}\Sigma(x) + \frac{\beta}{2}\gamma^5_{\alpha\alpha}\phi(x) + \frac{2\pi}{\beta}\int_{x^1}^\infty dy^1 \dot{\phi}(t, y^1) . \tag{3.1b}$$

Throughout this paper normal ordering will be understood with respect to the mass parameter μ, the implicit assumption being that the massive Thirring-Wess model to be discussed here has, as the short-distance fixed point, the corresponding massless Thirring-Wess model,[11] and in the terminology of Schroer and Truong,[12] there are no cumulative mass effects in the spinor fields. In particular this implies that systematic use will be made in this paper of

$$[\varphi^{(+)}(0, x^1), \varphi^{(-)}(0)] \simeq -\frac{1}{4\pi}\{\ln[\mu(-x^1 - i0)] + \ln[\mu(-x^1 + i0)]\} , \tag{3.2}$$

where φ stands for Σ or ϕ.

The line integral (with prescribed coefficient) is required if $\psi(x)$ is to correspond to an anticommuting fermion field[9]:

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}_{x^0 = y^0} = 0, \quad x^1 \neq y^1 .$$

In particular, the choice $\alpha = 0$, $\beta = 2\sqrt{\pi}$ yields the free canonical fermion field.

Now, the Thirring-Wess model with massless fermions corresponds to the Hamiltonian density

$$\mathcal{H}_0(x) = \tfrac{1}{2} : [\pi_\Sigma{}^2 + \pi_\phi{}^2 + (\nabla\Sigma)^2 + (\nabla\phi)^2 + m^2\Sigma^2] : ,$$

where

$$m^2 = e^2/\pi + m_0{}^2 \ . \tag{3.3}$$

Following the same reasoning as in Sec. II, the massive Thirring-Wess model will be described by

$$\mathcal{H}(x) = \mathcal{H}_0(x) - \frac{M\mu}{\pi} : \cos(\alpha\Sigma + \beta\phi) : \ . \tag{3.4}$$

From the Hamiltonian (3.4) we obtain the equations of motion

$$(\Box + m^2)\Sigma(x) = -\alpha\frac{M\mu}{\pi} : \sin[\alpha\Sigma(x) + \beta\phi(x)] : , \tag{3.5a}$$

$$\Box\phi(x) = -\beta\frac{M\mu}{\pi} : \sin[\alpha\Sigma(x) + \beta\phi(x)] : . \tag{3.5b}$$

Our task will thus consist in obtaining, given the ansatz (3.1), a solution of the coupled Dirac-Proca equations

$$(-i\gamma\cdot\partial + \tilde{M})\psi(x) = e\gamma^\mu N[B_\mu(x)\psi(x)] , \tag{3.6a}$$

$$\partial_\nu F^{\mu\nu} + m_0{}^2 B^\mu = -ej^\mu(x) , \tag{3.6b}$$

$$F^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu$$

consistent with the equations (3.5), where the precise meaning of $N[B_\mu\psi]$ in terms of a limiting procedure will be given later on. The gauge-invariant current is given by the limit (2.6), with the corresponding bilocal now defined to be

$$T_{\alpha\beta}(x, y) = N_{\alpha\beta}(x-y) : \exp\left\{i\left[\chi_\alpha(x) + e\int_x^y dz^\mu B_\mu(z) - \chi_\beta(y)\right]\right\} : , \tag{3.7}$$

with $N(z)$ as in Eq. (2.5b). In this model $\tilde{M}$ will be found to be equal to M; *a priori* it could, however, be zero or infinite, depending on the scale dimension of the mass term (see discussion, Sec. IV).

In order to be able to use the usual differentiation formulas for exponentials of bona fide operators, we find it convenient to smear the fields in the exponent with some suitable normalized function $h_x(\eta)$, whose support about the point x^1 we allow to shrink to zero at the end of the calculation:

$$\varphi_h(x^0, x^1) = \int_{-\infty}^{\infty} d\eta\, h_x(\eta)\varphi(x^0, \eta), \quad \int d\eta\, h_x(\eta) = 1$$

$$\varphi(x) = \lim_{h\to\delta} \varphi_h(x) ,$$

where $h \to \delta$ stands for

$$\operatorname*{supp}_x\{h\} \to 0.$$

In this notation we may then rewrite our ansatz (3.1) in the form

$$\psi(x)_\alpha = \lim_{h\to\delta} \psi(x; h)_\alpha \ , \tag{3.8a}$$

where

$$\psi(x; h)_\alpha = Z_\psi{}^{-1/2}(h) e^{i\chi_h(x)_\alpha} \ , \tag{3.8b}$$

and where $Z_\psi{}^{1/2}(h)$ plays the role of the wave-function renormalization constant. Its specific form will not be needed here.

Proceeding in the spirit of Ref. 9, we consider the action of the differential operator $-i\gamma\cdot\partial$ on $\psi(x;h)$. Using the identity

$$\delta e^{A(x)} = \int_0^1 d\lambda\, e^{(1-\lambda)A(x)}\,\delta A(x) e^{\lambda A(x)} , \tag{3.9}$$

we obtain

$$-i\gamma\cdot\partial\psi(x;h) = \gamma^\mu \int d\eta\, h_x(\eta) \int_0^1 d\lambda\, e^{i\lambda\chi_h(x)} \left\{\gamma^5[\tfrac{1}{2}\alpha\partial_\mu\Sigma(x^0,\eta) + \tfrac{1}{2}\beta\partial_\mu\phi(x^0,\eta)] + \frac{2\pi}{\beta}\partial_\mu \int_\eta^\infty dy^1 \dot{\phi}(x^0, y^1)\right\} \times e^{-i\lambda\chi_h(x)}\psi(x;h) \ . \tag{3.10}$$

Using the equation of motion (3.5b) and noting that $\gamma^\mu\gamma^5 = -\gamma_\nu\epsilon^{\nu\mu}$, we may cast Eq. (3.10) into the form

$$-i\gamma\cdot\partial\psi(x;h) = \tfrac{1}{2}e\gamma_\mu[B_h^\mu(x)\psi(x;h) + \psi_h(x;h)B_h^\mu(x)] + \mathfrak{M}(x;h)\psi(x;h) , \tag{3.11}$$

where

$$B_h^\mu(x) = \frac{1}{e}[\tfrac{1}{2}\alpha\epsilon^{\mu\nu}\partial_\nu\Sigma_h(x) + \tfrac{1}{2}\beta(1 - 4\pi/\beta^2)\epsilon^{\mu\nu}\partial_\nu\phi_h(x)] \tag{3.12}$$

and

$$\mathfrak{M}_{\alpha\beta}(x;h) = -2\mu M\gamma^0_{\alpha\beta}\int d\eta\, h_x(\eta)\int_\eta^\infty dy^1\int_0^1 d\lambda\, e^{i\lambda\chi_h(x)_\beta}\colon \sin[\alpha\Sigma(x^0,y^1)+\beta\phi(x^0,y^1)]\colon e^{-i\lambda\chi_h(x)_\beta}\,. \tag{3.13}$$

Commuting through the left-hand exponential and performing the λ integration one obtains

$$\mathfrak{M}(x;h)_{\alpha\beta}\psi_\beta(x;h) = -2\mu M\gamma^0_{\alpha\beta}\int d\eta\, h_x(\eta)\int_\eta^\infty dy^1\, \frac{[\colon\cos(\alpha\Sigma(y)+\beta\phi(y))\colon, \psi_\beta(x;h)]}{i[\chi_h(x)_\beta, \alpha\Sigma(y)+\beta\phi(y)]}\Bigg|_{x^0=y^0}\,. \tag{3.14}$$

The equal-time commutator in the denominator is evaluated to be

$$i[\chi_h(x)_\beta, \alpha\Sigma(y)+\beta\phi(y)]_{\mathrm{ET}} = 2\pi\int_{-\infty}^{y} d\eta'\, h_x(\eta')\,.$$

Hence, interchanging the order of integrations, one obtains

$$\mathfrak{M}(x;h)\psi(x;h) = -\frac{\mu M}{\pi}\gamma^0\int_{-\infty}^\infty dy^1[\colon\cos(\alpha\Sigma(y)+\beta\phi(y))\colon, \psi(x;h)]_{y^0=x^0}\,. \tag{3.15}$$

The commutator (3.15) is evidently nothing but the contribution of the mass term in the Hamiltonian (3.4) to the equation of motion for ψ. Hence we expect it to give a contribution of the form $\tilde{M}\psi(x)$ in the limit $h\to\delta$, with $\tilde{M}$ some constant which is expected to be infinite, finite, or zero, depending on where the scale dimension of the mass operator

$$d = (\alpha^2+\beta^2)/4\pi \tag{3.16}$$

is greater than, equal to, or less than one, the value $d=1$ being the canonical dimension. It is simple to check that the only contribution to the integral in Eq. (3.15) arises from the neighborhood of $y^1=x^1$, reflecting the locality of the mass operator with respect to the fermion field. Assuming for the moment $d\leq 1$, one finds that the limit $h\to\delta$ may be taken under the integral sign. Making use of the short-distance behavior (3.2) we then arrive at the expression

$$\begin{aligned}\mathfrak{M}(x)_{\alpha\beta}\psi_\beta(x) = &-\frac{1}{2}\frac{\mu M}{\pi}\left(\frac{\mu}{2\pi}\right)^{1/2}\gamma^0_{\alpha\beta}\int_{-\infty}^\infty dy^1\colon e^{i\alpha\Sigma(x)+i\beta\phi(x)+i\chi(x)_\beta}\colon e^{-i(\pi/4)\gamma^5_{\beta\beta}}\\ &\times\left\{\frac{[\mu(x^1-y^1-i0)]^{(1+d\gamma^5_{\beta\beta})/2}}{[\mu(x-y+i0)]^{(1-d\gamma^5_{\beta\beta})/2}} - \frac{[\mu(x-y+i0)]^{(1+d\gamma^5_{\beta\beta})/2}}{[\mu(x-y-i0)]^{(1-d\gamma^5_{\beta\beta})/2}}\right\}\\ &+\frac{1}{2}\frac{\mu M}{\pi}\left(\frac{\mu}{2\pi}\right)^{1/2}\gamma^0_{\alpha\beta}\int_{-\infty}^\infty dy^1\colon e^{-i\alpha\Sigma(x)-i\beta\phi(x)+i\chi(x)_\beta}\colon e^{-i(\pi/4)\gamma^5_{\beta\beta}}\\ &\times\left\{\frac{[\mu(x-y-i0)]^{(1-d\gamma^5_{\beta\beta})/2}}{[\mu(x-y+i0)]^{(1+d\gamma^5_{\beta\beta})/2}} - \frac{[\mu(x-y+i0)]^{(1-d\gamma^5_{\beta\beta})/2}}{[\mu(x-y-i0)]^{(1+d\gamma^5_{\beta\beta})/2}}\right\},\end{aligned} \tag{3.17}$$

where the summation over β is understood and d is given by Eq. (3.16). Before proceeding to evaluate the integral (3.17), we now turn to the Proca equation (3.6b).

Starting from the definition (2.6) for the current a straightforward calculation yields

$$j^\mu(x) = -\frac{1}{2\pi}[\alpha\epsilon^{\mu\nu}\partial_\nu\Sigma(x)+\beta\epsilon^{\mu\nu}\partial_\nu\phi(x)]\,. \tag{3.18}$$

Note that the current has already been normalized such as to correspond to the commutation relations

$$[j^0(x),\psi(y)]_{\mathrm{ET}} = -\delta(x^1-y^1)\psi(y)\,.$$

This commutation relation, with ψ a well-defined operator, reflects the fact that, contrary to the case of the Schwinger model, there exist nontrivial charge sectors, i.e., there is no confinement.

Substitution of (3.12) and (3.18) into the Proca equation (3.6b) leads to

$$(\Box + m_0^2)[\tfrac{1}{2}\alpha\Sigma(x)+\tfrac{1}{2}\beta(1-4\pi/\beta^2)\phi(x)] = -\frac{e^2}{2\pi}[\alpha\Sigma(x)+\beta\phi(x)]\,.$$

The only way to make this equation compatible with the equations of motion (3.5) is to have

$$\alpha^2/4 = e^2/m^2,\quad \beta^2/4\pi = m_0^2/m^2\,. \tag{3.19}$$

Note that α and β are completely independent of the fermion mass parameter M. According to Eq. (3.16), the values (3.19) correspond to a canonical dimension of the mass operator. Hence the mass term, Eq. (3.17), is readily evaluated to be

$$\mathfrak{M}(x)\psi(x) = -M\psi(x)\ .$$

Collecting all results, we have for the complete solution [we choose the positive square root in (3.19)]

$$\psi(x)_\alpha = e^{-i(\pi/4)\gamma^5_{\alpha\alpha}} : \exp\left[i\frac{e}{m}\gamma^5_{\alpha\alpha}\Sigma(x) + i\sqrt{\pi}\,\gamma^5_{\alpha\alpha}\frac{m_0}{m}\phi(x) + \sqrt{\pi}\frac{m}{m_0}\int_{x^1}^\infty dy^1\dot{\phi}(x^0,y^1)\right] : \left(\frac{\mu}{2\pi}\right)^{1/2}, \tag{3.20}$$

$$B^\mu(x) = -\frac{1}{m}\left[\epsilon^{\mu\nu}\partial_\nu\Sigma(x) - \frac{e}{\sqrt{\pi}\,m_0}\epsilon^{\mu\nu}\partial_\nu\phi(x)\right] \tag{3.21}$$

satisfying the equations

$$(-i\gamma\cdot\partial + M)\psi(x) = \tfrac{1}{2}e\gamma^\mu N[B_\mu(x)\psi(x)],$$

$$\partial_\nu F^{\mu\nu} + m_0 B^\mu = -ej^\mu,$$

where, in accordance with Eq. (3.11), $N[B_\mu\psi]$ is to be identified with the limit

$$N[B_\mu(x)\psi(x)] \equiv \lim_{h\to\delta}\,[B^\mu_h(x)\psi(x;h) + \psi_h(x;h)B^\mu_h(x)], \tag{3.22}$$

and where

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}\left[\frac{e}{m\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Sigma(x) + \left(\frac{m_0}{m}\right)\epsilon^{\mu\nu}\partial_\nu\phi(x)\right],$$

Σ and ϕ now satisfying the equations of motion (3.5) with α and β given by (3.19).

IV. FURTHER PROPERTIES OF THE FERMION FIELD

a. The mass term. For the particular values $\alpha = 0$, $\beta < \sqrt{8\pi}$, the ansatz (3.1) provides a solution of the massive Thirring model.[13,9] Since in the derivation of the equation of motion the values of the parameters α and β were unspecified, we may directly read off from Eq. (3.11) the corresponding equation of motion for the Thirring field. It is interesting to notice that for $\alpha = 0$ and $\beta < \sqrt{4\pi}$ the mass term (3.17) vanishes, since the dimension d of the mass operator [Eq. (3.16)] is less than one. Thus we are led to an equation of motion having the same structure as that of the massless Thirring model. This of course does not mean that for $\beta < \sqrt{4\pi}$ we are describing massless fermions. It rather means that in a truly renormalizable field theory the equations of motion for the fields do not specify the theory uniquely. In fact, from semiclassical arguments[14] one expects that precisely for very small values of β one obtains very heavy fermions. On the other hand, for $\beta > \sqrt{4\pi}$ the mass term (3.17) obviously diverges. Following the standard belief[13] that the massive Thirring model can be defined also in the range $\sqrt{4\pi} \leq \beta < \sqrt{8\pi}$, there must be a compensating divergence in the term (3.22) in Eq. (3.11) with $\alpha = 0$.

b. Lorentz covariance. In this subsection we shall examine the Lorentz transformation properties of the fermion field (3.1). Although we expect the operator (3.1) to transform like a field of Lorentz spin $\frac{1}{2}$, this transformation property is not evident from its definition. Indeed, the classical analog of the field (3.1) would transform in the massless-fermion case as a scalar, due to the conservation of the current $\partial_\mu\phi$; and in the massive case, the line integral over the zero component of this current would completely spoil its covariance properties. On the other hand, a correct quantum treatment will show that the locality of the theory will ensure the path independence of the line integral and that the short-distance behavior of the operators is responsible for the spin-$\frac{1}{2}$ character of the field. These points can be heuristically understood as follows: Consider the operator

$$:\exp\left[i\frac{2\pi}{\beta}\int_{x^1}^\infty dy^1\dot{\phi}(x^0,y^1)\right]: \;= Z'\exp\left[i\frac{2\pi}{\beta}\int_{x^1}^\infty dy^1\dot{\phi}(x^0,y^1)\right], \tag{4.1}$$

where a smearing of the kind (3.8) is understood. A Lorentz transformation acting on the field (4.1) at the origin of space-time will simply rotate the integration path. For an infinitesimal Lorentz transformation with velocity v, the variation of the field is given by

$$\delta : \exp\left[i\frac{2\pi}{\beta}\int_0^\infty dz^1\dot{\phi}(0,z^1)\right]:$$

$$= -v2M\mu\int_0^\infty dy^1 y^1\int_0^1 d\lambda : \sin[\alpha\Sigma(0,y^1)+\beta\phi(0,y^1)+2\pi\lambda)] : : \exp\left[i\frac{2\pi}{\beta}\int_0^\infty dz^1\dot{\phi}(0,z^1)\right]: ,$$

where we have used the differentiation formula (3.9) and the equation of motion (3.5b). We observe that in the whole integration range over y^1, with the possible exception of the end point, the sine will average to zero. Contrary to what happens in the mass term Eq. (3.13), the appearance of an extra moment in y^1 ensures also that no contribution will come from the end point for a dimension of the sine operator less than two. This means that the field (4.1) transforms like a scalar under Lorentz transformations. Note that the λ shift is a quantum effect. In the classical analog of (4.1) the absence of a λ integration would completely spoil its covariance properties. The field

$$:\exp\{i\gamma^5[\tfrac{1}{2}\alpha\Sigma(x)+\tfrac{1}{2}\beta\phi(x)]\}: \equiv :e^{i\gamma^5\Phi(x)}: \tag{4.2}$$

is obviously also a scalar.

The fermion field (3.1) can be expressed as the limit of a short-distance expansion involving the product of the fields in (4.1) and (4.2):

$$:e^{i\chi_\alpha(x)}: = \lim_{\epsilon\to 0} f_\alpha(\epsilon) : e^{i\gamma^5_{\alpha\alpha}\Phi(x+\epsilon)}: : \exp\left[i\frac{2\pi}{\beta}\int_{x^1}^\infty dy^1\dot{\phi}(x^0,y^1)\right]: ,$$

where $f(\epsilon)$ is given by

$$f_\alpha(\epsilon) = \exp\{\tfrac{1}{4}\gamma^5_{\alpha\alpha}\ln[(\epsilon_0+\epsilon_1)/(\epsilon_0-\epsilon_1)]\} .$$

It is the direction dependence of this factor which is responsible for the spin-$\frac{1}{2}$ character of the field. We now give a more detailed treatment of the above considerations.

To this end we consider the action of a Lorentz transformation on the smeared field $\psi(x;h)$ [Eq. (3.8)]:

$$U[\Lambda]\psi(x;h)U^{-1}[\Lambda] = Z_\psi^{-1}(h)e^{i\chi_h(x,v)}$$

$$= Z_\psi^{-1}(h)\exp\left\{i\int d\eta\, h_x(\eta)\left[\gamma^5\Phi(\Lambda x^0,\Lambda\eta)+\frac{2\pi}{\beta}\int_\eta^\infty dy^1(\Lambda^{-1})^0_\nu\partial^\nu\phi(\Lambda x^0,\Lambda y^1)\right]\right\} , \tag{4.3}$$

where v is the velocity parametrizing the Lorentz transformation. For an infinitesimal transformation we have, using (3.9),

$$\delta\psi(x;h) = i\int_0^1 d\lambda\, e^{i\lambda\chi_h(x)}v\left[\frac{d}{dv}\chi_h(x,v)\right]_{v=0} e^{-i\lambda\chi_h(x)}\psi(x;h)$$

$$= -iv\int d\eta\, h_x(\eta)\int_0^1 d\lambda\, e^{i\lambda\chi_h(x)}\left[(x^0\nabla+\eta\partial_0)\Phi(x^0,\eta)+\frac{2\pi}{\beta}\int_\eta^\infty dy^1[y^1\ddot{\phi}(y)+x^0\nabla\dot{\phi}(y)+\nabla\phi(y)]_{y^0=x^0}\right]$$

$$\times e^{-i\lambda\chi_h(x)}\psi(x;h) . \tag{4.4}$$

Using the equation of motion (3.5b) we may rewrite Eq. (4.4), using methods similar to those in Sec. III, as

$$\delta\psi(x;h) = -\frac{iv}{2}\int d\eta\, h_x(\eta)\left\{(x^0\nabla+\eta\partial_0)\Phi(x^0,\eta)-\frac{2\pi}{\beta}(\eta\nabla+x^0\partial_0)\Phi(x^0,\eta),\psi(x;h)\right\}$$

$$+iv\frac{\mu M}{\pi}\int_{-\infty}^\infty dy^1 y^1[:\cos(2\Phi(x^0,y^1)):,\psi(x;h)] . \tag{4.5}$$

It is rewarding that the right-hand side of Eq. (4.5) is nothing but the commutator $-iv[M_{01},\psi(x;h)]$, where M_{01} is the generator of Lorentz boosts, constructed in standard fashion from the ϕ and Σ fields.

Comparing the result (4.5) with the one obtained for the field evaluated at the Lorentz-transformed point, we have

$$U[\Lambda]\psi(x;h)U[\Lambda]^{-1} = \psi(\Lambda x;h) - ivS\psi(x;h) + O(v^2), \tag{4.6}$$

where $S\psi$ will be responsible for the Lorentz spin of the field, and is given by

$$S\psi(x;h)=\tfrac{1}{2}\int dy^1 h_x(y^1)(y^1-x^1)\times\left\{\gamma^5\partial_0\Phi(x^0,y^1)-\frac{2\pi}{\beta}\nabla\phi(x^0,y^1),\psi(x;h)\right\}-\frac{\mu M}{\pi}\int_{-\infty}^{\infty}dy^1(y^1-x^1)\times[:\cos(2\Phi(x^0,y^1)):,\psi(x;h)]\,.$$

In the limit $h\to\delta$ the last term vanishes for a scale dimension of the cosine less than two, as a result of the locality of the theory. The vanishing of this term is also responsible for the path independence discussed previously. The remaining term involving the anticommutator is evaluated to be, in the limit $h\to\delta$,

$$\lim_{h\to\delta} S\psi(x;h)=\tfrac{1}{2}i\gamma^5\psi(x)\,, \tag{4.7}$$

where use has been made of

$$\lim_{h\to\delta} h_x(y)\int d\eta\, h_x(\eta)\left(\frac{1}{y^1-\eta+i0}+\frac{1}{y^1-\eta-i0}\right)=-\delta'(y^1-x^1)\,.$$

Substitution of (4.7) into Eq. (4.6) immediately shows that the field $\psi(x)$ transforms with a Lorentz spin $\frac{1}{2}$. This proves the Lorentz invariance of both the massive Thirring and Thirring-Wess models.

V. THE RECOVERY OF THE SCHWINGER MODEL

In this section we want to investigate the limit of m_0 going to zero of the Thirring-Wess model. We expect that in a sense, to be specified below, we should recover in this limit the Schwinger model.

From Eqs. (3.20) and (3.21) we immediately see that the limit $m_0\to 0$ does not exist for the fields ψ and B^μ themselves. This is by no means surprising since if the limit were to exist we would obtain a local charge-carrying fermion field, which is incompatible with Maxwell's equations being satisfied as operator equations. Correspondingly, the vector field exhibits a divergent behavior in the longitudinal part, which is just needed to render the gauge-invariant bilocals finite in the limit. In fact, using Eqs. (3.20), (3.21) and the definition (3.7) for the bilocals we are left in the limit with our original expression (2.5) in terms of the Σ field alone. The same applies to the electric field and to the current. This means that the ϕ field has completely decoupled from the observables.

The decoupling of ϕ implies that the charge sectors of the Thirring-Wess model have disappeared in the QED limit. This is nothing but confinement. Indeed, the reason for having charged states in the Thirring-Wess model is the periodic structure of the equation of motion (3.5b) for the ϕ field, which allows one to construct well-defined charge-carrying operators in terms of line integrals over $\dot\phi$ extending to infinity. Because of the nonperiodic nature of the equation of motion (3.5a) for the Σ field, no such charge-raising operators will exist in the QED limit.

To clarify the above statements consider a dipole state of the form

$$|d\rangle=\exp\left[i\frac{2\pi}{\alpha}\int_{x^1}^{y^1}dz^1\dot\varphi(0,z^1)\right]|0\rangle\ , \tag{5.1}$$

where φ is a generic field, and a smearing around x^1 and y^1 is understood in order to have a well-defined state. The state $|d\rangle$ represents a negative and positive charge, localized around x^1 and y^1, with respect to the charge-density operator

$$j^0(x)=\frac{\alpha}{2\pi}\partial_1\varphi(x)\,.$$

Writing the Hamiltonian as

$$H=\tfrac{1}{2}\int:\{\dot\varphi^2(z)+[\nabla\varphi(z)]^2+f(\varphi(z))\}:dz^1\ ,$$

where f is an arbitrary function, we compute the expectation value of the Hamiltonian in the dipole state, with respect to the vacuum, to be

$$\langle d|H|d\rangle-\langle 0|H|0\rangle=|x^1-y^1|\langle 0|[:f(\varphi-2\pi/\alpha):-:f(\varphi):]|0\rangle+\cdots\ , \tag{5.2}$$

where the neglected contributions refer to the energy associated with the localization around the points x^1 and y^1; these are the only contributions in the case where f is periodic with period $2\pi/\alpha$, so that the energy remains finite when the two charges become infinitely separated. On the other hand, if f is not a periodic function the energy of the dipole state will grow linearly with increasing charge separation. Hence, in the case of a periodic function f, the charged states of the theory are obtained by removing one of the charges of the dipole state to infinity. Hence, in the case of the Thirring-Wess model, the ϕ field can be used to construct charge sectors. In the QED limit, however, any gauge-invariant state will involve only the Σ field, whose equation of motion (3.5a) does not satisfy the periodicity condition; hence the charge sectors will not survive in this limit. What was called the Coulomb-gauge solution in Sec. II essentially corresponds to the state (5.1) with

$\varphi(x) = \Sigma(x)$ in the limit $y^1 \to \infty$. From (5.2) we immediately realize that this is an infinite energy state and, hence, it has only a formal character. The absence of charge sectors in both the massive and massless Schwinger model is nothing but confinement.[2]

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