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Functional bridge between gauge theory and strings in two dimensions*

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We develop a functional-integral bridge that explicitly exhibits the string content of certain two-dimensional gauge theories.

I. INTRODUCTION

In recent years, the thrust of particle theory has turned increasingly toward the problem of extracting hadrons from local quantum field theory. A simple chain of reasoning based on (1) renormalizability, (2) nonrelativistic-quark-model success, (3) scaling, (4) the rate for $\pi^0 \to 2\gamma$, and (5) the experimental absence of quarks, has led us inexorably to a single candidate: that theory of quarks and non-Abelian gluons ennobled in the epithet "quantum chromodynamics."

The hope is that the *fields* carrying color will agglutinate (only) into the color-singlet *particles* known as hadrons, and there is reason to believe that, in the equivalent hadron language, the theory will closely resemble present dual-string models. Indeed, in two-dimensional models of confinement, representative matrix elements of a gauge theory[1] have been shown[2] to coincide with those of a string theory.[3]

Motivated by this success, we were led to ask if there exists a *direct* bridge (at least in such models) from the field language [$\phi(xt)$, $\psi(xt)$, $A_\mu^\alpha(xt)$, etc.] to the particle (string) language [e.g., $x^\mu(\tau)$ for end points, $x^\mu(\tau,\sigma)$ for string]. At first sight, the two types of degrees of freedom seem worlds apart, accurately reflecting one of the conceptual dilemmas surrounding confinement.

But in 1950, Feynman[4] provided the beginnings of such a bridge, when he noticed that, at least for scalar fields, the Green's functions of the field theory can be reexpressed as path integrals over classical trajectories $x^\mu(\tau)$ (and the gauge field). In this neglected language, creation and annihilation are properly described in terms of trajectories moving forward and backward in τ (proper time).

Of course, $x^\mu(\tau)$ is just the coordinate of an end point of a string. It is perhaps not surprising then that (after a final approximate functional integration) the gauge field becomes the string itself. Thus we arrive directly and explicitly at the Bardeen-Bars-Hanson-Peccei[3] (BBHP) string.

In actual fact, the explicit development of the bridge is not trivial for a number of technical and conceptual reasons. In the first place, Feynman gave his representation only for Abelian scalar electrodynamics, and we would like to extend the result to fermions and non-Abelian gluons. We have ideas[5] of how to do this, which involve anticommuting c-number "classical" trajectories, but in this paper, in two dimensions, we will settle on a more modest goal: (Abelian) scalar electrodynamics and the massive Schwinger model. Abelian fermions are so simple in two dimensions that we can work without anticommuting c-numbers. Beyond finding the bridge itself, results are already interesting; although the (light-cone) Schwinger model leads to the BBHP string, (light-cone) scalar electrodynamics involves deviations. (*Classically*, however, they *both* are the BBHP string.)

The conceptual difficulties lie primarily in identifying, in the functional language, a gauge-invariant approximation scheme which will reexpress the theory in terms of strings and string interactions. Since we are working with Abelian gauge theories, we do not have the N^{-1} expansion as a guide; a set of graphs planar in one gauge will not generally be so in another. It turns out that the string corresponds to a *gauge-invariant restriction* that one quark moves always forward in τ, while the other moves always backward (antiquark), and there are no internal quark loops [Fig. 1(a)]. For example, inserting the gauge-invariant requirement that $\dot{x}^+_{(1)}(\tau) > 0$, $\dot{x}^+_{(2)}(\tau) < 0$ (light-cone times) yields the set of graphs which, *in the light-cone gauge*, is planar [Fig. 1(b)]. In this paper, we concentrate mainly on the light-cone gauge (and the restrictions on $\dot{x}^+$). This leads precisely to the BBHP string (at least for fermions).

For string interactions we go to the 6-point function with quark trajectories moving in time as shown in Fig. 2. Here we make the gauge-invariant restriction that one quark turns around in time just *once* (at τ_0), and we integrate over τ_0. The detailed treatment of the 3-string vertex as well as N-point functions (Fig. 3) will be given elsewhere, though a brief sketch is offered near the end of the paper. The picture that emerges is very much like Mandelstam's[6] string formalism

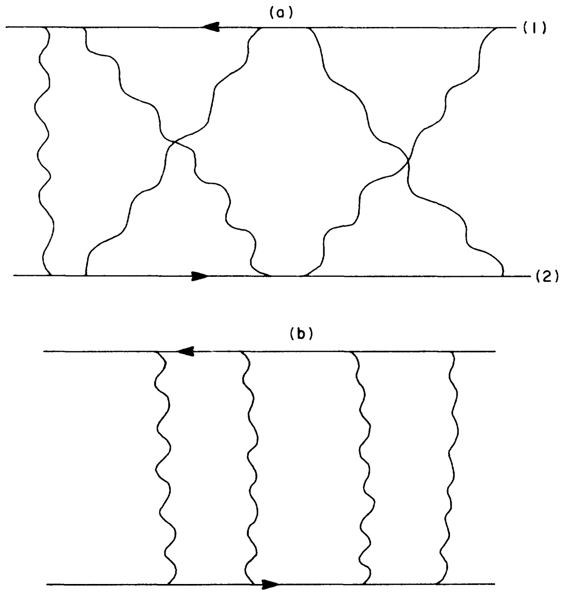

FIG. 1. (a) Quark-antiquark graph: $\dot{x}^+_{(1)}(\tau) > 0$, $\dot{x}^+_{(2)}(\tau) < 0$. (b) Remnant in light-cone gauge: $A^+ = 0$.

in 26 dimensions, with factorization properties of functional integrals being intimately related to hadronic factorization of the S matrix.

We begin with scalar particles, because Feynman did. Sections II and III are brief reviews of rewriting scalar electrodynamics in terms of classical Green's functions and Feynman's relativistic sum-over-classical-trajectories representation. We believe the quantum ordering relevant for this topic has not received such careful treatment elsewhere. In Sec. IV, we discuss our gauge-invariant "chopping" procedure (that is, planar in, say, the light-cone gauge). It turns out that light-cone scalar electrodynamics makes deviations from the light-cone BBHP string, due to quantum-ordering effects. The solution of this model is completed in Appendix B. In Sec. V, we make the transition to fermions (massive Schwinger model), and bring their parallel development up to the level of Sec. IV for bosons. In Sec. VI, the BBHP Hamiltonian

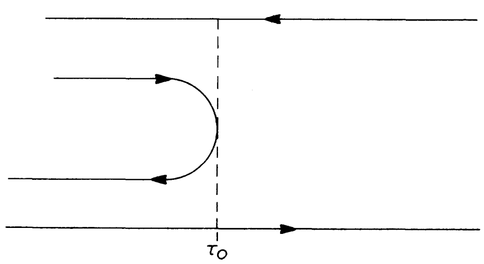

FIG. 2. Three-string vertex.

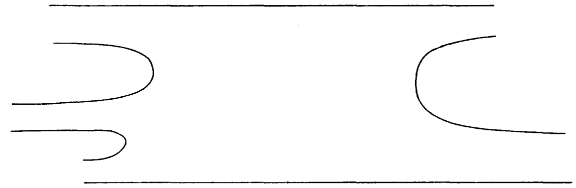

FIG. 3. N-point functions.

is obtained explicitly. Section VII is for remarks, including a brief sketch of the 3-string vertex. There is another appendix, Appendix A, in which we offer a toy classical field theory which is the BBHP string in an arbitrary gauge.

II. SCALAR ELECTRODYNAMICS AND REPRESENTATION IN TERMS OF CLASSICAL GREEN'S FUNCTIONAL

We begin with the generating functional for scalar electrodynamics[7]

$$\mathfrak{N} Z[J,J^*] = \int \mathfrak{D}\phi_1 \mathfrak{D}\phi_2 \mathfrak{D}A_\mu\, \delta[\chi(A)]\, \overline{\Delta}(A) \times \exp\left(i \int d^2x\, \mathfrak{L}_J\right), \tag{2.1}$$

where

$$\mathfrak{L}_J = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + (\partial^\mu - ieA^\mu)\phi^*(\partial_\mu + ieA_\mu)\phi - m^2\phi^*\phi + \phi^* J + J^*\phi\,. \tag{2.2}$$

Here $\chi(A)$ is the gauge condition and $\overline{\Delta}(A)$ the corresponding Faddeev-Popov determinant, $\phi = (1/\sqrt{2})(\phi_1 + i\phi_2)$, and $\mathfrak{N}$ is chosen so that $Z[0,0] = 1$. We will have no need for gluon sources. We have also made the simplifying assumption that there is no scalar potential. In two dimensions this is our option, for corrections to the assumption are calculable.

Integrating out the charged field, we express Z in terms of the classical Green's functional $G(x,y;A)$,

$$\mathfrak{N} Z[J,J^*] = \int \mathfrak{D}A_\mu\, \delta[\chi(A)]\, \overline{\Delta}(A) \det[G(x,y;A)] \times \exp\left(-\frac{1}{4}\int d^2x\, F_{\mu\nu}F^{\mu\nu}\right) \times \exp\left[-i\int d^2x\, d^2y\, J^*(x) \times G(x,y;A) J(y)\right]; \tag{2.3}$$

$$-[(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) + m^2] G(x,y;A) = \delta^{(2)}(x-y)\,. \tag{2.4}$$

Time-ordered boundary conditions are implied for

$G(x,y;A)$. As we shall detail in Sec. III, Feynman has shown us how to express $G(x,y;A)$ as a path integral over classical trajectories, and hence Z itself.

To calculate Green's functions, one takes appropriate functional derivatives with respect to J, J^*. In particular, we will have need for the four-point function shown in Fig. 4,

$$\langle 0|T(\phi^*(Z_1)\phi(Z_2)\phi(Z_3)\phi^*(Z_4))|0\rangle = G(Z_1,Z_2,Z_3,Z_4)$$
$$= \frac{1}{i^2}\int \mathfrak{D}A_\mu \delta[\chi(A)]\,\overline{\Delta}(A)\det[G(x,y;A)]\exp\left(-\frac{i}{4}\int d^2x\,F_{\mu\nu}F^{\mu\nu}\right)$$
$$\times[G(Z_2,Z_4;A)G(Z_3,Z_1;A)+G(Z_2,Z_1;A)G(Z_3,Z_4;A)]\,. \quad (2.5)$$

Here, of course, the first pair of G's contains the direct graphs, the second pair the annihilation graphs, and the determinant contains internal charged loops.

III. FEYNMAN'S SUM-OVER-CLASSICAL-TRAJECTORIES REPRESENTATION

Our task is to invert Eq. (2.4). This is best[8] done in an operator notation. We introduce the operator $\hat{G}$ and a set of states such that

$$\langle x|\hat{G}|y\rangle = G(x,y;A),\quad \langle x|y\rangle = \delta^2(x-y)\,. \quad (3.1)$$

We also introduce the operators $\hat{x}_\mu, \hat{P}_\mu$, with $[\hat{x}_\mu, \hat{P}_\nu] = ig_{\mu\nu}$, so that

$$\langle x|\hat{P}_\mu|y\rangle = -i\partial_\mu^{(x)}\delta^{(2)}(x-y)\,. \quad (3.2)$$

In this notation, Eq. (2.4) reads

$$\{[\hat{P}^\mu + eA^\mu(\hat{x})][\hat{P}_\mu + eA_\mu(\hat{x})] - m^2\}\hat{G} = \hat{1}\,. \quad (3.3)$$

The inversion is now accomplished via

$$\hat{G} = -\frac{i}{2}\int_0^\infty d\tau\,\exp\left(\frac{i\tau}{2}\{[\hat{P}^\mu + eA^\mu(\hat{x})][\hat{P}_\mu + eA_\mu(\hat{x})] - m^2\}\right) \quad (3.4)$$

and the choice of time-ordered boundary condition is now explicit. In the coordinate representation, we need express as a functional integral the quantity

$$G(x,y;A) = \langle x|\hat{G}|y\rangle$$
$$= -\frac{i}{2}\int_0^\infty d\tau\langle x|e^{-iH\tau}|y\rangle\,, \quad (3.5)$$
$$H = -\tfrac{1}{2}\{[\hat{P}^\mu + eA^\mu(\hat{x})][\hat{P}_\mu + eA_\mu(\hat{x})] - m^2\}\,.$$

The functional-integral form for the integrand is well known in quantum mechanics, and we obtain formally

$$G(x,y;A) = -\frac{i}{2}\int_0^\infty d\tau \int_{\substack{x(\tau)=x\\x(0)=y}} \mathfrak{D}x^\mu \mathfrak{D}P^\mu \exp\left(i\int_0^\tau d\tau'\{P\cdot\dot{x} + \tfrac{1}{2}[(P+eA)^2 - m^2]\}\right), \quad (3.6)$$

where $\dot{x}_\mu = (d/d\tau)x_\mu(\tau)$, and $A^\mu B_\mu \equiv A\cdot B$. Feynman's result can be obtained by integrating P^μ,

$$G(x,y;A) = -\frac{i}{2}\int_0^\infty d\tau \int_{\substack{x(\tau)=x\\x(0)=y}} \mathfrak{D}x^\mu \exp\left[i\int_0^\tau d\tau'(-\tfrac{1}{2}\dot{x}\cdot\dot{x} - e\dot{x}\cdot A - \tfrac{1}{2}m^2)\right]. \quad (3.7)$$

Thus the classical Green's functional is in terms of the dynamics of a point particle. The forms (3.6) and (3.7) are, however, quite formal. Because A_μ is a function of x_μ, quantum-ordering problems are serious, and of considerable interest in our approach. It will pay, therefore, to look closely at the lattice structure of the functional integrals.

We take our lattice as shown in Fig. 5. In the phase-space approach, one suspects that

$$A(\hat{x})\cdot\hat{P} + \hat{P}\cdot A(\hat{x}) \sim A(x_l)\cdot P_l + P_l\cdot A(x_{l-1}) \quad (3.8)$$

is the correct lattice ordering near each P. This can easily be verified in detail. Thus, the precise form of the functional integral in (3.6) is

$$\int_{\substack{x(\tau)=x\\x(0)=y}} \mathfrak{D}x^\mu \mathfrak{D}P^\mu \exp\left(i\int_0^\tau d\tau'\{P\cdot\dot{x} + \tfrac{1}{2}[(P+eA)^2 - m^2]\}\right)$$
$$\equiv \int \prod_{r=1}^{N-1} d^2x_r \prod_{t=1}^{N}\left(\frac{d^2P_t}{(2\pi)^2}\right)\exp\left(\frac{i\epsilon}{2}\sum_{l=1}^{N}\{[P_l + eA(x_l)]\cdot[P_l + eA(x_{l-1})] - m^2\}\right)\exp\left[i\sum_{l=1}^{N}P_l\cdot(x_l - x_{l-1})\right], \quad (3.9)$$

where ϵ is the lattice spacing ($\epsilon N=\tau$) and $x_0=y$, $x_N=x$.

Similarly, integration of P_l^μ in (3.9) gives the precise form of the functional integral in (3.7),

$$\int_{\substack{x(\tau)=x\\x(0)=y}} \mathfrak{D}x^\mu \exp\left[i\int_0^\tau d\tau'(-\tfrac{1}{2}\dot{x}^2 - e\dot{x}\cdot A - \tfrac{1}{2}m^2)\right]$$
$$\equiv \frac{1}{2\pi\epsilon}\int\prod_{r=1}^{N-1}\left(\frac{d^2x_r}{2\pi\epsilon}\right)\exp\left\{i\left[-\frac{1}{2\epsilon}\sum_{l=1}^{N}(x_l-x_{l-1})^2 - \frac{e}{2}\sum_{l=1}^{N}(x_l-x_{l-1})\cdot(A_l+A_{l-1}) - \frac{\epsilon}{2}\sum_{l=1}^{N}m^2\right]\right\}. \quad (3.10)$$

In what follows, we will generally work with the phase-space form.

IV. GAUGE-INVARIANT "CHOPPING" PROCEDURE AND "TOPOLOGICAL" EXPANSION

Consider the set of Green's functions defined by $Z[J,J^*]$ as in Eq. (2.3) [or, in particular, as in Eq. (2.5)]. Replace each $G(x,y;A)$ by the form (3.6). The $2N$-point Green's function involves functional integration over $\mathfrak{D}A_\mu\mathfrak{D}x^\mu_{(1)}\cdots\mathfrak{D}x^\mu_{(N)}$. In this form, it is not possible to perform the A_μ integration explicitly. It is our task to find a gauge-invariant approximation procedure that will show a string and string interactions. Our scheme is as follows.

Consider all the Green's functions,[9] and in particular the four-point function Eq. (2.5), Fig. 4, in the gauge-invariant approximation that det G is dropped (no closed charge-loops). (The A integration is now possible, but will not yield a BBHP-type string.) Consider the further approximation (as mentioned in the Introduction), that we keep only the first pair of G's (also gauge invariant), *and* that we consider *only* that region of $\mathfrak{D}x^\mu_{(1)}\mathfrak{D}x^\mu_{(2)}$ for which (say) $\dot{x}^+_{(1)}(\tau)>0$, $\dot{x}^+_{(2)}(\tau)<0$. We will show presently that such a "chopping" is gauge invariant, but this is to be expected in any case; we are only stating that the particles move uniformly in (proper) time: You cannot change a quark to an antiquark by a gauge transformation. The fact that, in this chopping, they never turn around is depicted in Fig. 1. As we shall see, this structure will be the BBHP-string propagator itself.

The next "chopping" is in the six-point function, where we require, as in Fig. 2, that one charged particle turns around exactly once in time [for that particle, sign$(\dot{x}^+(\tau))=\epsilon(\tau-\tau_0)$]. This will be the three-string vertex. In this way, we proceed to define the $2N$-point functions in the "tree" approximation, as in Fig. 3. This defines our "first" approximation. Factorization properties of the functional integrals will be useful in showing the hadronic factorization of these trees. "Sewing" trees back together again, in the manner common to dual models, will recapture all the original Feynman diagrams omitted in the first approximation.

Before mentioning other choppings (than $\dot{x}^+$), we offer a few qualitative details about the correspondence between the $\dot{x}^+$ chopping and Feynman diagrams. In a general gauge, this chopping (and no internal fermion loops), integrated over A_μ, contains crossed gluon diagrams [Fig. 1(a)], self-energies, and vertex corrections. If we go to the light-cone gauge, the combination of the monotonicity of each charged line and the instantaneous gluons suppresses the crossed gluon graphs. The presence or absence of self-energy graphs and vertex corrections depends, in the light-cone gauge, on whether or not we allow $\dot{x}^+=0$ on a given trajectory (see Fig. 6). To avoid such singularities (in the light-cone gauge), we will specify that $\dot{x}^+$ is strictly greater than or less than zero. This refinement is also a completely gauge-invariant notion, and will suppress self-energies and vertex corrections (only) in the light-cone gauge.[10]

Thus, (only) in the light-cone gauge, our chopping becomes the usual planar Feynman diagrams. Moreover, in the light-cone gauge, the general expansion discussed above becomes the familiar

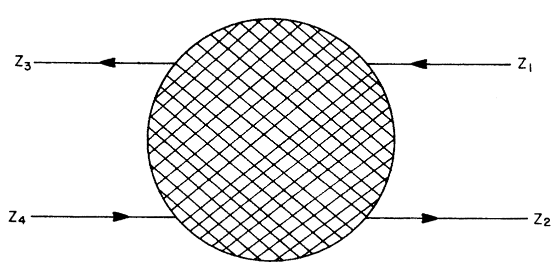

FIG. 4. Full four-point function.

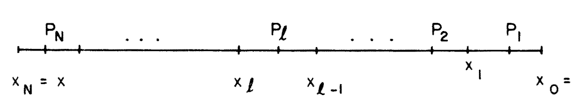

FIG. 5. Phase-space lattice.

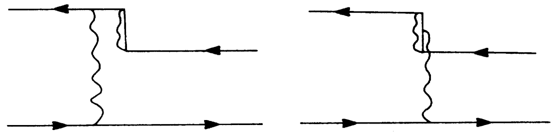

FIG. 6. Vertex and self-energy corrections. Because gluons are instantaneous, certain charged particles have $\dot{x}^+=0$.

"topological" expansion.[11] It is the organization that would be picked out by an N^{-1} expansion (later taken at $N=1$). Indeed, the light-cone gauge is a great simplification for the $\dot{x}^+$ chopping, and most of the paper will be written with that chopping-gauge combination.

There are in principle other possible gauge-invariant choppings, e.g., $\dot{x}^0_{(1)}>0$, $\dot{x}^0_{(2)}<0$ (for the four-point function). It is not obvious that this is equivalent to the $\dot{x}^+$ chopping. This gives a set of graphs which is apparently planar only in the axial gauge, and may give a string Hamiltonian like the BBHP Hamiltonian in their timelike gauge. We have not carried out this last step, and further investigation of this topic will be informative.

It remains to demonstrate our contention that the chopping procedure is gauge-invariant. Consider

$$G_S(x,y;A+\partial\Lambda)\equiv\frac{1}{2i}\int_0^\infty d\tau\int_{\substack{x(\tau)=x\\x(0)=y}}\mathfrak{D}x^\mu\mathfrak{D}P^\mu S[x]\exp\left(i\int_0^\tau d\tau'\{P\cdot\dot{x}+\tfrac{1}{2}[(P+eA+e\partial\Lambda)^2-m^2]\}\right),\tag{4.1}$$

where $S[x]$ may be any function of $x^\mu(\tau)$. For the full Green's functional, $S=1$; for the light-cone chopping, $S=\prod_{0<\tau'\le\tau}\theta(\pm\dot{x}^+)$, etc. If $G_S(x,y;A+\partial\Lambda)=\exp[-ie\Lambda(x)+ie\Lambda(y)]G_S(x,y;A)$, the chopping is gauge-invariant. To see that this is true for *arbitrary* $S(x)$, simply shift $P+e\partial\Lambda\equiv P'$. The resulting form is just the exponential at $\Lambda=0$ plus a term

$$\begin{aligned}i\int_0^\tau d\tau'(-e\partial\Lambda\cdot\dot{x})_{\tau'}&=-ie\int_0^\tau\frac{\partial\Lambda(x(\tau'))}{\partial\tau'd\tau'}\\&=-ie[\Lambda(x(\tau))-\Lambda(x(0))]\\&=-ie[\Lambda(x)-\Lambda(y)],\end{aligned}\tag{4.2}$$

which completes the demonstration. The chopping is gauge invariant for arbitrary S, including $\theta(\dot{x}^+)$, $\theta(\dot{x}^0)$; "strictly greater or less than" can be as easily employed $[\theta(0)=0]$.

We have studied our chopping by expanding (4.1) with $S=\theta(\dot{x}^+)$ in perturbation theory. In the light-cone gauge it corresponds to using the usual vertices and "chopped" propagators

$$\theta(p^+)\frac{1}{2p^+p^--m^2}.\tag{4.3}$$

Thus, it may be no surprise that our gauge-invariant approximation (planar in the light-cone gauge) to the massive Schwinger model (later) will yield the $N=1$ 't Hooft spectrum; essentially the same graphs are being summed. (The only difference is that we are summing "chopped" propagators, but 't Hooft's[11] integral equation "chops" itself in the process of solution.)

In other gauges, the $\dot{x}^+$ chopping is vastly more complicated: Factorized "vertices" are elusive because the natural vertices obtained from the functional integral pick up terms proportional to $\delta(x^+)$ times derivatives of propagators (and worse and worse in higher orders).

Thus, it is difficult to verify gauge invariance in perturbation theory. Indeed, in a technical sense, our previous demonstration of gauge invariance is only semiclassical, for one needs be careful to use a grid structure that maintains gauge invariance. We believe such (a rigorous gauge-covariant extension of the light-cone gauge) is not difficult to construct, but we will not pursue the subject further. Hereafter, we will use the $\dot{x}^+$ chopping and (then) the light-cone gauge.

We have brought our discussion of the four-point function to the form

$$\begin{aligned}G_C(Z_1Z_2Z_3Z_4)=\tfrac{1}{4}\int\mathfrak{D}A^+\mathfrak{D}A^-\delta(A^+)\exp\left(-\frac{i}{4}\int d^2x\,F_{\mu\nu}F^{\mu\nu}\right)\\\times\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int_{\substack{x_1(\tau_1)=Z_3,\ x_2(\tau_2)=Z_2\\x_1(0)=Z_1,\ x_2(0)=Z_4}}\mathfrak{D}x_1^+\mathfrak{D}x_1^-\mathfrak{D}P_1^+\mathfrak{D}P_1^-\mathfrak{D}x_2^+\mathfrak{D}x_2^-\mathfrak{D}P_2^+\mathfrak{D}P_2^-\\\times\prod_{0<\tau_1'\le\tau_1}\theta\left(+\frac{dx_1^+}{d\tau_1'}\right)\prod_{0<\tau_2'\le\tau_2}\theta\left(-\frac{dx_2^+}{d\tau_2'}\right)\prod_{i=1,2}\exp\left(i\int_0^{\tau_i}d\tau_i(P_i\cdot\dot{x}_i+\tfrac{1}{2}\{[P_i+eA(x_i)]^2-m^2\})_{\tau_i'}\right),\end{aligned}\tag{4.4}$$

where the notation G_C denotes "chopped." The range of the θ functions refers to the grid, where, because $\epsilon\dot{x}_l = x_l - x_{l-1}$, we must not require the θ function at $l=0$.

The last step, or steps, involves integration over $A^\pm$. Because of the quantum-ordering effects in the functional integral [see Eq. (3.9)], this is not as easy as it looks, and the result is not quite the light-cone BBHP string (except as $\hbar \to 0$). The calculation is completed in Appendix B, where it is compared with the integral equation approach to the Feynman graph summation. The two calculations agree.

Although scalar electrodynamics deviates (for $\hbar \neq 0$) from the BBHP string (and hence from the 't Hooft spectrum), we have learned enough now from the scalars to approach fermions. Indeed, in the massive Schwinger model, our approach will yield exactly the BBHP light-cone string.

V. FERMIONS AND THE MASSIVE SCHWINGER MODEL

In two dimensions, fermions are extremely simple and, by working in a certain subspace, we shall be able to take over our scalar results almost in toto.

The Lagrangian density for the massive Schwinger model is

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}. \tag{5.1}$$

Following Kogut and Soper,[14] we introduce light-cone variables

$$A^\pm = \frac{A^0 \pm A^1}{\sqrt{2}}, \quad x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}$$
$$\gamma^\pm = \frac{\gamma^0 \pm \gamma^1}{\sqrt{2}}, \quad (\gamma^+)^2 = (\gamma^-)^2 = 0, \quad (\gamma^+, \gamma^-)_+ = 2 \tag{5.2}$$

and projectors

$$R_\pm = \tfrac{1}{2}\gamma^\mp\gamma^\pm, \quad R_+ + R_- = 1, \quad R_+R_- = 0 \tag{5.3}$$

and define

$$\psi_\pm = R_\pm \psi. \tag{5.4}$$

In these variables, the Lagrangian becomes

$$\mathcal{L} = \sqrt{2}(\psi_-)^\dagger(i\partial_- - eA^+)\psi_- + \sqrt{2}(\psi_+)^\dagger(i\partial_+ - eA^-)\psi_+ - \frac{m}{\sqrt{2}}[(\psi_-)^\dagger\gamma^+\psi_+ + (\psi_+)^\dagger\gamma^-\psi_-] + \tfrac{1}{2}F_{+-}{}^2, \tag{5.5}$$

where $F_{+-} = \partial_+ A^+ - \partial_- A^-$. The equation of motion for ψ_-,

$$\psi_- = \frac{m}{2}\frac{1}{i\partial_- - eA^+}\gamma^+\psi_+, \tag{5.6}$$

is a constraint. We shall be interested here only in the R_+ subspace, and thus we consider the generating functional

$$\mathfrak{N}Z[\xi, \xi^\dagger] = \int \mathfrak{D}A^+\mathfrak{D}A^-(\Delta\delta)\mathfrak{D}\psi_+\mathfrak{D}\psi_-\mathfrak{D}(\psi_+)^\dagger\mathfrak{D}(\psi_-)^\dagger \times e^{i\int d^2x\,\mathcal{L}_\xi}, \tag{5.7}$$

where

$$\mathcal{L}_\xi = \mathcal{L} + 2^{1/4}[\xi^\dagger\psi_+ + (\psi_+)^\dagger\xi], \tag{5.8}$$

and $(\Delta\delta)$ is an arbitrary gauge fixing term.

Integrating over ψ_-, $(\psi_-)^\dagger$, and rescaling $\psi_+, (\psi_+)^\dagger$ by $2^{-1/4}$, we obtain

$$\mathfrak{N}Z[\xi, \xi^\dagger] = \int \mathfrak{D}A^+\mathfrak{D}A^-(\Delta\delta)\mathfrak{D}\psi_+\mathfrak{D}(\psi_+)^\dagger e^{i\int \mathcal{L}_+ d^2x}, \tag{5.9}$$
$$\mathcal{L}_+ = \tfrac{1}{2}F_{+-}{}^2 + (\psi_+)^\dagger\left(i\partial_+ - eA^- - \frac{m^2}{2}\frac{1}{i\partial_- - eA^+}\right)\psi_+ + \xi^\dagger\psi_+ + (\psi_+)^\dagger\xi.$$

From the equal-x^+ commutation relations implied in a lightlike quantization of (5.9), together with the constraint equation (5.6), it is not hard to show that

$$[\psi(x), \psi(y)]_{+,+} = \tfrac{1}{2}\gamma^- + \frac{m}{2}\frac{1}{i\partial_- - eA^+} + \frac{m^2}{4}\frac{\gamma^+}{(i\partial_- - eA^+)^2}, \tag{5.10a}$$

$$(i\not{\partial} - m - e\not{A})\langle 0|T_+(\psi, \bar{\psi})|0\rangle = i\left(R_- + \frac{m}{2}\frac{1}{i\partial_- - eA^+}\gamma^+\right). \tag{5.10b}$$

Equation (5.10a) is the equal-x^+ anticommutator for the original ψ, and on its right-hand side $(\partial_-^{-1})_{xy} = \frac{1}{2}\epsilon(x^- - y^-)$. Equations (5.10b), following from the first, is the x^+-ordered Green's function for ψ; on its right-hand side $(\partial_-^{-1})_{xy} = \frac{1}{2}\epsilon(x^- - y^-)\delta(x^+ - y^+)$. Thus, time-ordered Green's functions do not agree with x^+-ordered Green's functions.[14] We are interested here, however, *only* in calculating the R_+ subspace. It is not hard to show that $R_+\langle 0|T_+(\psi, \bar{\psi})|0\rangle R_+ = R_+\langle 0|T(\psi, \bar{\psi})|0\rangle R_+$ (the latter is ordinary x^0 ordered, x^0 quantized) to all orders. The same is true for $\psi, \psi^\dagger$ Green's functions. We may proceed with confidence then to use x^+-ordered Green's functions.

In order to do the remaining fermonic integration in (5.9), we require the Green's functional for ψ_+:

$$\left(i\partial_+ - eA^- - \frac{m^2}{2}\frac{1}{i\partial_- - eA^+}\right)G_F[A] = \delta^{(2)}R_+. \tag{5.11}$$

Then

$$\mathfrak{N} Z(\xi,\xi^\dagger)=\int \mathfrak{D}A^+\mathfrak{D}A^-(\Delta\delta)\det(G_F^{-1})\exp\left[i\int d^2x\left(\tfrac{1}{2}F_{+-}{}^2-\xi^\dagger G_F\xi\right)\right]. \quad (5.12)$$

The four-point function of Fig. 4 is now

$$\begin{aligned}\langle 0|T(\psi_+(Z_2)(\psi_+)^\dagger(Z_1)\psi_+(Z_3)(\psi_+)^\dagger(Z_4))|0\rangle &= G(Z_2,Z_1,Z_3,Z_4)\\ &=\int \mathfrak{D}A^+\mathfrak{D}A^-(\Delta\delta)\det(G_F^{-1})\exp\left(i\int d^2x\,\frac{F_{+-}{}^2}{2}\right)\\ &\quad\times[G_F(Z_2,Z_4;A)G_F(Z_3,Z_1;A)-G_F(Z_3,Z_4;A)G_F(Z_2,Z_1;A)]. \end{aligned}\quad (5.13)$$

As for the scalar particles, we would like to express G_F as a functional integral over classical trajectories. Although such can be handled directly by the methods of the previous sections, it is more instructive to make close contact with our bosonic formula by defining yet another Green's functional $\overline{G}$:

$$G_F\equiv 2(i\partial_- - eA^+)\overline{G}, \quad (5.14)$$

The defining equation for $\overline{G}$ is then

$$[-2(\partial_+ + ieA^-)(\partial_- + ieA^+)-m^2]\overline{G}=\delta^{(2)}R_+. \quad (5.15)$$

This is extremely close, but not identical, to Eq. (2.4). The difference is a matter of quantum ordering, for in (2.4),

$$\begin{aligned}-(\partial_\mu+ieA_\mu)(\partial^\mu+ieA^\mu)=&-(\partial_+ + ieA^-)(\partial_- + ieA^+)\\ &-(\partial_- + ieA^+)(\partial_+ + ieA^-). \end{aligned}\quad (5.16)$$

An analysis parallel to that of the boson case then yields the desired functional-integral representation

$$\overline{G}(x,y;A)=-\frac{i}{2}R_+\int_0^\infty d\tau\int_{\substack{x(\tau)=x\\ x(0)=y}}\mathfrak{D}P^+\mathfrak{D}P^-\mathfrak{D}x^+\mathfrak{D}x^-\exp\left[i\int_0^\tau d\tau'\left(P^+\dot{x}^- + P^-\dot{x}^+ + (P^-+eA^-)(P^++eA^+)-\frac{m^2}{2}\right)\right]. \quad (5.17)$$

Again, we have paid careful attention to ordering on the grid with the result

$$\int_0^\tau d\tau'(P^-+eA^-)(P^++eA^+)\equiv\epsilon\sum_{r=1}^{N}(P_r^-P_r^+ + P_r^-A_{r-1}^+ + eP_r^+A_r^- + e^2A_r^+A_r^-). \quad (5.18)$$

We have brought the fermion discussion up to the level of Sec. III for the bosons. The discussion of Sec. IV is totally analogous. The chopping $\theta(\dot{x}^+)$ (etc.) is gauge invariant. *Then*, it is convenient to choose the light-cone gauge. In perturbation theory, our chopping turns out to be (for the light-cone gauge) the "usual" vertices and the chopped fermion propagator

$$\theta(q^+)\left(\gamma^+\frac{m^2}{2q^+}+\gamma^-q^+m\right)\frac{1}{2q^+q^- - m^2}\approx\theta(q^+)\frac{1}{\not{q}-m}. \quad (5.19)$$

The last equality is true for the R_+ subspace in the light-cone gauge. (Again, other gauges are vicious.) As mentioned above, then, the graphs we are summing (in the light-cone gauge) are (prechopped) planar ladders.

Putting together all of our previous discussion, and noting that $2i\partial_-$ brings down $-2P^+$ for each G_F, we arrive at the "chopped" four-point function,

$$\begin{aligned}G_C^F(Z_2,Z_1,Z_3,Z_4)=&(R_+)_{(1)}(R_+)_{(2)}\int\mathfrak{D}A^+\mathfrak{D}A^-\delta(A^+)\exp\left(-\frac{i}{4}\int d^2x\,F_{\mu\nu}F^{\mu\nu}\right)\\ &\times\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int\mathfrak{D}x_1^+\mathfrak{D}x_1^-\mathfrak{D}P_1^+\mathfrak{D}P_1^-\mathfrak{D}x_2^+\mathfrak{D}x_2^-\mathfrak{D}P_2^+\mathfrak{D}P_2^-\\ &\times\prod_{0<\tau_1'\le\tau_1}\theta(\dot{x}_1^+)\prod_{0<\tau_2'\le\tau_2}\theta(-\dot{x}_2^+)P_1^+(\tau_1)P_2^+(\tau_2)\\ &\times\prod_{i=1,2}\exp\left(i\int_0^{\tau_i}d\tau_i(P_i\cdot\dot{x}_i+\tfrac{1}{2}\{[P_i+eA(x_i)]^2-m^2\})_{\tau_i'}\right), \end{aligned}\quad (5.20)$$

$$x_1(0)=Z_1,\quad x_2(0)=Z_4,\quad x(\tau_1)=Z_3,\quad x_2(\tau_2)=z_2$$

and the proviso, Eq. (5.18). This is to be compared with Eq. (4.3). As promised, we have brought the fermions up to the level of the bosonic discussion in Sec. IV. We are now ready to find the string. In what follows, we suppress the simple factors $(R_+)_{(1)}, (R_+)_{(2)}$.

VI. BRINGING OUT THE STRING

We can do the functional integration over the gauge field in Eq. (5.19), and, in this sense, our dynamics will be entirely in terms of the geometrical variables $x_i(\tau)$. However, the most efficient way to see the BBHP string is somewhat intricate.

We start with one of the two Green's functionals in (5.19):

$$G_F^C(Z_3,Z_1;A)=i\int_0^\infty d\tau\int_{\substack{x(\tau)=Z_3\\ x(0)=Z_1}}\mathfrak{D}x^\mu\mathfrak{D}P^\mu\prod_{0<\tau'\leq\tau}\theta\left(\frac{dx^+}{d\tau'}\right)P^+(\tau)\exp\left(i\int_0^\tau d\tau'\{P\cdot\dot{x}+\tfrac{1}{2}[(P+eA)^2-m^2]\}\right). \tag{6.1}$$

Because we are in the gauge $A^+=0$, we can perform the P^- integration. The result is

$$\prod_{l=1}^N\int\frac{dP_l^-}{2\pi}\exp\left(i\epsilon\sum_{l=1}^N P_l^-P_l^++i\sum_{l=1}^N(x_l^+-x_{l-1}^+)P_l^-\right)=\prod_{l=1}^N\delta(\epsilon P_l^++x_l^+-x_{l-1}^+)\equiv\delta(\epsilon_{\tau'}(P^++\dot{x}^+)). \tag{6.2}$$

We have written $\epsilon_{\tau'}=\epsilon$ in preparation for a change of variable to a nonuniform lattice. Thus

$$G_F^C(Z_3,Z_1;A)=i\int_0^\infty d\tau\exp\left(-i\int_0^\tau\frac{m^2}{2}d\tau'\right)\int_{\substack{x(\tau)=Z_3\\ x(0)=Z_1}}\mathfrak{D}x^-\mathfrak{D}x^+\mathfrak{D}P^+\theta(-P^+)\delta(\epsilon_{\tau'}(\dot{x}^++P^+))P^+(\tau)$$
$$\times\exp\left[i\int_0^\tau(P^+\dot{x}^-+eP^+A^-)d\tau'\right]. \tag{6.3}$$

There are two related problems with this form. First, the $d\tau$ integration prevents us from putting the path integral in any standard form; second, the $\delta(\dot{x}^++P^+)$ is in a mixed form, neither phase space nor action formalism. Both of these may be solved by the following change of variable on the lattice.

We define new variables of integration λ,λ' by

$$\tau'=-\int_0^{\lambda'}\frac{d\bar{\lambda}}{\bar{P}^+(\bar{\lambda})},\quad \tau=-\int_0^{\lambda}\frac{d\bar{\lambda}}{\bar{P}^+(\bar{\lambda})}. \tag{6.4}$$

Here, we have defined $P^+(\tau')=\bar{P}^+(\lambda')$, and we will take $x^\pm(\tau')=\bar{x}^\pm(\lambda')$. Formally, then, our "constraint" $dx^++P^+d\tau'=0$ becomes just $d\bar{x}^+=d\lambda'$, i.e., $\bar{x}^\pm\cong\lambda'$. The constraint is a *gauge choice* for the string. Note also that this transformation is *only* well defined *because* of $\theta(-P^+)$. In particular, it is only well defined for $P^+\neq 0$. The assumption that $P^+<0$, equivalent to defining $\theta(0)=0$, is where we drop self-energy and vertex corrections.

Let us examine this in greater detail. For each δ function in our δ functional

$$\epsilon_{\tau'}\left(\frac{x_l^+-x_{l-1}^+}{\epsilon_\tau}+P_l^+\right)=\epsilon_{\lambda'}^l\left(\frac{x_l^+-x_{l-1}^+}{\epsilon_{\lambda'}^l}-1\right)$$
$$=\epsilon_{\lambda'}^l\left(\frac{\bar{x}_l^+-\bar{x}_{l-1}^+}{\epsilon_{\lambda'}^l}-1\right). \tag{6.5}$$

In the first step we have used $d\lambda'=-P^+(\tau')d\tau'$ $[\epsilon_{\lambda'}^k=-P_k^+\epsilon_{\tau'}]$, and we have noted that, since $\epsilon_{\tau'}$ is uniform, $\epsilon_{\lambda'}^l$ depends on the location in the lattice. The second step involves just the statement $x^+(\tau')=\bar{x}^+(\lambda')$. Thus

$$\delta(\epsilon_{\tau'}(\dot{x}^++P^+))=\prod_{l=1}^N\delta(\bar{x}_l^+-\bar{x}_{l-1}^+-\epsilon_{\lambda'}^l)$$
$$=\delta(Z_3^+-Z_1^++\lambda)\prod_{0<\lambda'<\lambda}\delta(\bar{x}^+(\lambda')-\lambda'-Z_1^+). \tag{6.6}$$

In the last step, we have reorganized the N δ functions, remembering that $\bar{x}_0^+=x_0^+=x^+(0)=Z_1$, $\bar{x}_N^+=x_N^+=x^+(\tau)=Z_3^+$, and $\sum_{l=1}^l\epsilon_{\lambda'}^l=Z_1^++\lambda'$. There are $N-1$ δ functions in the last product. This is just the right number to do the integrations over λ and $\bar{x}^+$. The result is

$$G_F^C(Z_3,Z_1;A)=-i\theta(Z_3^+-Z_1^+)$$
$$\times\int_{\substack{\bar{x}^-(Z_3^+-Z_1^+)=Z_3^-\\ \bar{x}^-(0)=Z_1^-}}\mathfrak{D}\bar{P}^+\mathfrak{D}\bar{x}^-\theta(-\bar{P}^+)e^{iS},$$

$$S=\int_0^{Z_3^+-Z_1^+}d\lambda'\left[\frac{m^2}{2\bar{P}^+}-eA^-(\lambda'+Z_1^+,\bar{x}^-(\lambda'))+\bar{P}^+\dot{\bar{x}}^-\right]. \tag{6.7}$$

A final simple change of variable

$$\lambda' + Z_1^+ = \lambda_1, \quad \bar{P}^+(\lambda') = -P_1^+(\lambda_1), \quad \bar{x}^-(\lambda') = x_1^-(\lambda_1) \tag{6.8}$$

brings us to the useful form

$$G_F^C(Z_3,Z_1;A) = -i\theta(Z_3^+ - Z_1^+) \times \int_{\substack{x_1^-(z_3^+)=z_3^- \\ x_1^-(z_1^+)=z_1^-}} \mathfrak{D}P_1^+ \mathfrak{D}x_1^- \theta(+P_1^+)\, e^{iS_1},$$
$$S_1 = \int_{z_1^+}^{z_3^+} d\lambda_1 \left[-P_1^+ \dot{x}_1^- - \frac{m^2}{2P_1^+} - eA^-(\lambda_1, x_1^-(\lambda_1)) \right]. \tag{6.9}$$

This is the simplest representation for the Green's functional. It clearly displays a single particle moving in a field A^- with Hamiltonian $H_1 = (m^2/2P_1^+) + eA^-$.

Similarly, for the other Green's functional [chopped with $\theta(-\dot{x}^+)$], we find

$$G_F^C(Z_2,Z_4;A) = i\theta(Z_4^+ - Z_2^+) \times \int_{\substack{x_2^-(z_4^+)=z_4^- \\ x_2(z_2^+)=z_2^-}} \mathfrak{D}x_2^- \mathfrak{D}P_2^+ \theta(+P_2^+)\, e^{iS_2},$$
$$S_2 = \int_{z_2^+}^{z_4^+} d\lambda_2 \left[-P_2^+ \dot{x}_2^- - \frac{m^2}{2P_2^+} + eA^-(\lambda_2, x_2^-(\lambda_2)) \right]. \tag{6.10}$$

We insert these back into the four-point function

$$G^C(Z_2,Z_1,Z_3,Z_4) = \int \mathfrak{D}A^+ \mathfrak{D}A^- \delta(A^+) \exp\left(-\frac{i}{4} \int d^2x\, F_{\mu\nu}F^{\mu\nu} \right) G_F^C(Z_2,Z_4;A)\, G_F^C(Z_3,Z_1;A) \ . \tag{6.11}$$

Because the interaction is instantaneous, the quarks fail to interact when one pulls ahead of the other (in proper time), therefore we will lose no dynamics by assuming $Z_4^+ = Z_3^+$, $Z_1^+ = Z_2^+$. (This is not necessary, merely convenient.) In this form the final functional integration over A^- is particularly simple:

$$\int \mathfrak{D}A^- \exp\left(i \int_{z_1^+}^{z_3^+} d^2x \left[\tfrac{1}{2}(\partial_- A^-)^2 + JA^- \right] \right) = \exp\left(\tfrac{1}{2} \int d^2x\, d^2y\, J(x) (\partial_-)_{xy}^{-2} J(y) \right). \tag{6.12}$$

Further[15] $(\partial_-)_{xy}^{-2} = \frac{1}{2}|x^- - y^-|\, \delta(x^+ - y^+)$, and here $J = e[\delta(x^- - x_2^-) - \delta(x - x_1^-)]$. Our final result is

$$G^C(Z_2,Z_1,Z_3,Z_4)\Big|_{\substack{z_3^+=z_4^+ \\ z_1^+=z_2^+}} = \int_{\substack{x_1^-(z_3^+)=z_3^-;\; x_1^-(z_1^+)=z_1^- \\ x_2^-(z_3^+)=z_4^-;\; x_2^-(z_1^+)=z_2^-}} \mathfrak{D}x_1^- \mathfrak{D}P_1^+ \mathfrak{D}x_2^- \mathfrak{D}P_2^+ \theta(+P_1^+)\, \theta(+P_2^+)\, e^{iS},$$
$$S = \int_{z_1^+}^{z_3^+} d\tau (-P_1^+ \dot{x}_1^- - P_2^+ \dot{x}_2^- - H), \quad H = \frac{m^2}{2P_1^+} + \frac{m^2}{2P_2^+} + \frac{e^2}{2}|x_1^- - x_2^-|. \tag{6.13}$$

Thus we have shown that the chopped equal-time four-point Green's function of the massive Schwinger model is the transition amplitude for the (no-fold) BBHP string.

Confidentially, we worried for some time whether we had made a sign error: Our result says (for free particles)

$$H = \frac{m^2}{2P^+}, \quad P^+ > 0, \quad (x^-, P^+) = -i$$

$$\left[\text{or } H = -\frac{m^2}{2P^+}, \quad P^+ < 0, \quad (x^-, P^+) = i \right].$$

The reader is invited to check for himself that this is precisely correct for $L = -m(\dot{x}^2)^{1/2}$ in the $x^+ = \tau$ gauge.

For those readers interested in studying the string model from the point of view of a general constrained Hamiltonian path integral, we remark that our result can also be put in the form

$$\int \mathfrak{D}P_1^\mu \mathfrak{D}P_2^\mu \mathfrak{D}x_1^\mu \mathfrak{D}x_2^\mu \mathfrak{D}P^\mu(\sigma,\tau) \mathfrak{D}z^\mu(\sigma,\tau)\, \theta(-P_1^+)\, \theta(-P_2^+)\, \delta\left(P^+ - \frac{e^2}{2} z'^+ (z'^-)\right) \delta\left(P^- + \frac{e^2}{2}|z'^-|\right)$$
$$\times |2P_1^+ 2P_2^+|\, \delta(P_1{}^2 - m^2)\, \delta(P_2{}^2 - m^2)\, \delta(x_1^+ - \tau)\, \delta(x_2^+ - \tau)\, \delta(z^+ - \tau)\, \delta(z^- - a(\tau) - b(\tau)\sigma)$$
$$\times \exp\left(i \int P_1 \cdot \dot{x}_1 d\tau \right) \exp\left(i \int P_2 \cdot \dot{x}_2 d\tau \right) \exp\left(i \int P(\sigma,\tau) \cdot \dot{z}(\sigma,\tau)\, d\sigma d\tau \right),$$

with the understanding that $\lim_{\sigma\to\sigma_{1,2}(\tau)} z^\mu(\sigma,\tau) = x^\mu_{1,2}(\tau)$, and $z' = dz/d\sigma$, etc. This can also be derived fairly directly from the string action itself. We have tried Faddeev-Popov tricks to change to timelike gauge. Formally, it works; the Faddeev-Popov method is, however, sensitive only to the low order in $\hbar$.

We remind the reader that it is no surprise to find the massive Schwinger model in our gauge-invariant approximation (which is planar in the light-cone gauge) yielding the $N=1$ 't Hooft spectrum. We have simply found the functional form of the same graphs. Of course, in the absence of damping factors from N^{-1}, we might expect *inaccuracy*. The approximation is presumably accurate for high-lying states (because these correspond to $\hbar\to 0$). The approximation is, however, grossly inaccurate for the low-lying states: The full massive-Schwinger model is well known not to have a Goldstone boson for small mass, yet the 't Hooft spectrum (and our approximation) does.[16]

Another point worth making is clear from our work. At least in this approximation, the BBHP string apparently does not know whether it came from Abelian or non-Abelian dynamics. (The only difference is $\alpha' = e^2 \to e^2 N$.)

VII. REMARKS

We have also used our methods to construct the three-string vertex. The method, which strongly resembles Mandelstam's[6] methods in dual string models, yields the known answer of Callan, Coote, and Gross[1] and Bars.[2] The details will be reported elsewhere. Here we confine ourselves to a brief sketch.

Figure 7 is a factorized form of Fig. 2. Referring now to Fig. 7, the external regions are calculated in a relatively simple manner, as they correspond just to 3 free-strong propagators, as analyzed in the text. (For example, one of the two chopping factors of the Z_3, Z_4 string is $\theta(-\dot{Z}^+)$, while for the Z_5, Z_6 string it is $\theta(\dot{\bar{Z}}^+)$. The three strings are joined functionally at the infinitesimal vertex about $\bar{x}^+$ [and integrated $(1/\epsilon)\int d\bar{x}^+$]. The "joining" condition is that $Z^-(\bar{x}^+) = \bar{Z}^-(\bar{x}^+)$ (in the infinitesimal region). Thus, at $\bar{x}^+$, one large string splits into two.)

Similar calculations can presumably be made for N-point functions, and basic vertices (3-point, 4-point?) isolated by factorization. The program, reminiscent of dual functional constructions, doubtless also has a Kaku-Kikkawa[17] formulation.

As mentioned in the Introduction, extensions of these ideas to non-Abelian gluons in four dimensions can be studied. We have little doubt, e.g., that, following our ideas, a geometrical picture

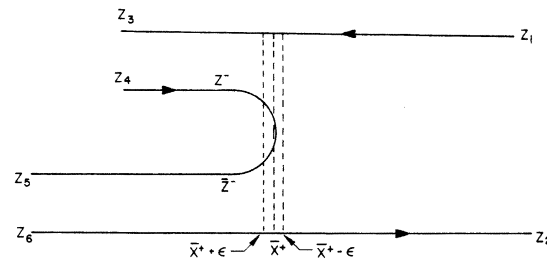

FIG. 7. Factorized form of three-string vertex.

(particle variables rather than field variables) can be constructed for the planar diagrams in the N^{-1} expansion. In four dimensions, however, dynamics must have more to do with renormalization, and it will be interesting to see if our stringlike language will shed any light on confinement.

A final remark on our gauge-covariant "chopping" procedure. We recall that the "other" gauges (besides the light-cone gauge) are extremely complicated. In a practical sense then, the gauge-covariant form of the chopping may be considered to be primarily a gauge-covariant continuation of the light-cone gauge result. The covariant form does, however, appear to evade here the nonperturbative phenomena of Refs. 12 and 13. The ideas presented here (that each "gauge" defines by choice of variables[12,13] a theory, whose Green's functions can be covariantly extended to *all* gauges, and/or possible equivalence of gauges if done uniformly in light-cone quantization) may be of interest in the non-Abelian context.

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APPENDIX A: TOY CLASSICAL FIELD THEORY WHICH IS THE STRING IN AN ARBITRARY GAUGE

In the text, we analyzed the string content of scalar electrodynamics and the massive Schwinger model in the light-cone gauge. Although they differ to higher order in $\hbar$, classically they are both the BBHP string. In this appendix we want to present a simple field theory which, at least classically, is the BBHP string in *every* gauge. The result is exact.

The action for the model is

$$s=-m\int d\tau[(\dot{x}_1{}^2)^{1/2}+(\dot{x}_2{}^2)^{1/2}]$$
$$+\int d^2x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-A^\nu J_\nu), \quad \text{(A1)}$$

where

$$J^\nu(x)=-e\int\frac{dx_1^\nu}{d\tau}\delta^{(2)}(x-x_1(\tau))d\tau$$
$$+e\int\frac{dx_2^\nu}{d\tau}\delta^{(2)}(x-x_2(\tau))d\tau. \quad \text{(A2)}$$

Physically, we are discussing two oppositely-charged particles interacting with the electromagnetic field. We choose to introduce (spurious) string variables into the model by Dirac's trick[18]

$$J^\nu=\partial_\mu G^{\mu\nu},$$
$$G^{\mu\nu}(x)=e\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma(\mathring{z}^\mu z'^\nu-\mathring{z}^\nu z'^\mu)\delta^{(2)}(x-z(\sigma,\tau)). \quad \text{(A3)}$$

Here, $\mathring{z}=\partial z/\partial\tau$, $z'=\partial z/\partial\sigma$, and $z^\mu(\sigma,\tau)$ is the world sheet defined by the world lines of the two particles,

$$z^\mu(\sigma_1(\tau),\tau)=x_1^\mu(\tau),\quad z^\mu(\sigma_2(\tau),\tau)=x_2^\mu(\tau). \quad \text{(A4)}$$

Using the action with $J^\nu=\partial_\mu G^{\mu\nu}$, we extract equations of motion by subjecting S to an infinitesimal variation with respect to all variables

$$\delta S=-m\int\left(\frac{\mathring{x}_1\cdot\partial_\tau\delta x_1}{(\mathring{x}_1{}^2)^{1/2}}+\frac{\mathring{x}_2\cdot\partial_\tau\delta x_2}{(\mathring{x}_2{}^2)^{1/2}}\right)d\tau+\int d^2x(\partial_\mu F^{\mu\nu}-\partial_\mu G^{\mu\nu})\delta A_\nu$$
$$+e\int d^2x\,F_{\mu\nu}(x)\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma[-\partial_\lambda^x\delta^{(2)}(x-z(\sigma,\tau))\delta z^\lambda\mathring{z}^\mu z'^\nu]$$
$$+\int d^2x\,F_{\mu\nu}e\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}\delta^{(2)}(x-z(\sigma,\tau))d\sigma[(\partial_\tau\delta z^\mu)z'^\nu-(\partial_\sigma\delta z^\nu)\mathring{z}^\mu]. \quad \text{(A5)}$$

The gluon equations are immediate,

$$\partial_\mu F^{\mu\nu}=\partial_\mu G^{\mu\nu}, \quad \text{(A6)}$$

to which the most general solution is

$$F^{\mu\nu}(x)=G^{\mu\nu}(x)+a\epsilon^{\mu\nu}. \quad \text{(A7)}$$

Here $\epsilon^{\mu\nu}$ is the antisymmetric symbol, $\epsilon^{01}=+1$, and a is an arbitrary constant.[15] Using the identity

$$\delta^{(2)}(z(\sigma,\tau)-z(\sigma',\tau'))=\frac{\delta^2(\sigma-\sigma')\delta(\tau-\tau')}{\sqrt{-g}}, \quad \text{(A8)}$$

where $\sqrt{-g}=|\mathring{z}_0z_1'-\mathring{z}_1z_0'|=[(\mathring{z}z')^2-\mathring{z}^2z'^2]^{1/2}$, it is easy to see that on the world sheet

$$F_{\mu\nu}(z(\sigma,\tau))=\frac{e(\mathring{z}_\mu z_\nu'-\mathring{z}_\nu z_\mu')}{\xi(\sigma,\tau)\sqrt{-g}}+a\epsilon_{\mu\nu}, \quad \text{(A9)}$$

where $\xi=1$ in the world sheet, and $\xi=2$ at its boundary. (This is the relation between electromagnetic and string variables.) Using (A9), another useful identity is immediate,

$$F_{\mu\nu}(z)F^{\mu\nu}(z)\big|_{a=0}=-\frac{2e^2}{\xi^2(\sigma,\tau)}. \quad \text{(A10)}$$

Consider next the third term in the variation Eq. (A5). We shall show that its contribution to the equations of motion vanishes. We manipulate the integrand of that term by the identity

$$-\partial_\lambda^x\delta^{(2)}(x-z(\sigma,\tau))=\xi(\sigma,\tau)\int du\,dv\frac{\delta}{\delta z^\lambda(u,v)}$$
$$\times\delta^{(2)}(x-z(\sigma,\tau)). \quad \text{(A11)}$$

In this form, it is easy to do the d^2x integration to obtain a form proportional to

$$(\mathring{z}_\mu z_\nu'-\mathring{z}_\nu z_\mu')\frac{\delta}{\delta z^\lambda}F^{\mu\nu}(z)=\tfrac{1}{2}\sqrt{-g}\,\frac{\delta}{\delta z^\lambda}[F_{\mu\nu}(z)F^{\mu\nu}(z)]\bigg|_{a=0}$$
$$=0. \quad \text{(A12)}$$

The z equations of motion then can be obtained from the last term of Eq. (A5). We integrate by parts in σ,τ, using the identities

$$0=\int d\tau\frac{\partial}{\partial\tau}\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\,F_{\mu\nu}(z(\sigma,\tau))\delta z^\mu z'^\nu$$
$$=\int d\tau\,F_{\mu\nu}(z(\sigma,\tau))\delta z^\mu z'^\nu\Big|_{\sigma=\sigma_2(\tau)}\mathring{\sigma}_2(\tau)$$
$$-\int d\tau\,F_{\mu\nu}(z(\sigma,\tau))\delta z^\mu z'^\nu\Big|_{\sigma=\sigma_1(\tau)}\mathring{\sigma}_1(\tau)$$
$$+\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\,\delta z^\mu\frac{\partial}{\partial\tau}(F_{\mu\nu}z'^\nu)$$
$$+\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\left(\frac{\partial}{\partial\tau}\delta z^\mu\right)F_{\mu\nu}z'^\nu, \quad \text{(A13a)}$$

$$\frac{dx_1^\lambda(\tau)}{dt}=\frac{d}{d\tau}z^\lambda(\sigma_1(\tau),\tau)$$
$$=\mathring{z}^\lambda+z'^\lambda\mathring{\sigma}_1(\tau)\big|_{\sigma=\sigma_1(\tau)}. \quad \text{(A13b)}$$

The term in question takes the final form

$$-e\int\delta z^\nu\left\{\frac{\partial}{\partial\sigma}[\dot z^\mu F_{\mu\nu}(z)]-\frac{\partial}{\partial\tau}[z'^\mu F_{\mu\nu}(z)]\right\}d\sigma d\tau$$

$$-e\int d\tau\,\delta x_2^\mu F_{\mu\nu}(x_2(\tau))\dot x_2^\nu(\tau)$$

$$+e\int d\tau\,\delta x_1^\mu F_{\mu\nu}(x_1(\tau))\dot x_1^\nu(\tau). \quad (A14)$$

Thus, taken together with Eq. (A9), our equations of motion are

$$m\frac{d}{d\tau}\left(\frac{\dot x_1^\lambda}{(\dot x_1^{\,2})^{1/2}}\right)-e\dot x_{1\mu}F^{\mu\lambda}(z(\sigma_1(\tau),\tau))=0,$$

$$m\frac{d}{d\tau}\left(\frac{\dot x_1^\lambda}{(\dot x_1^{\,2})^{1/2}}\right)+e\dot x_{2\mu}F^{\mu\lambda}(z(\sigma_2(\tau),\tau))=0, \quad (A15)$$

$$\frac{\partial}{\partial\sigma}(\dot z^\mu F_{\mu\nu})=\frac{\partial}{\partial\tau}(z'^\mu F_{\mu\nu}).$$

The reader may verify for himself that the same equations follow from the effective action

$$s=-m\int d\tau[(\dot x_1^{\,2})^{1/2}+(\dot x_2^{\,2})^{1/2}]$$

$$-\frac{e^2}{2}\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\sqrt{-g}$$

$$+\tfrac{1}{2}ea\int d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\,\epsilon^{\mu\nu}(\dot z_\mu z'_\nu-z'_\mu\dot z_\nu). \quad (A16)$$

At $a=0$, this is precisely the BBHP action. With $a\neq 0$ this is a generalization (reparametrization invariance is still guaranteed for transformations in which the Jacobian does not change sign in the range). In the light-cone gauge, the potential is modified to

$$\frac{e^2}{2}|x_2^-(\tau)-x_1^-(\tau)|-ea[x_2^-(\tau)-x_1^-(\tau)], \quad (A17)$$

so the generalization is interesting as long as $\frac{1}{2}e^2>|ea|$.

It is always amusing to see what one gets by plugging equations of motion back into the action directly:

$$\int d^2x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-A^\nu\partial^\mu G_{\mu\nu})$$

$$=\int d^2x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}F^{\mu\nu}G_{\mu\nu})$$

$$=\int d^2x(\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}-\tfrac{1}{4}a^2\epsilon_{\mu\nu}\epsilon^{\mu\nu}), \quad (A18)$$

where we have used Eq. (A7). It is another line or so of algebra, including an application of Eq. (A8) to verify that the "plug-in" works easily for $a=0$, and not for $a\neq 0$. With more care for surface terms during integration by parts, we believe the plug-in works for all a.

We should finally make the point that we have only proven the equivalence of this classical model with the no-fold string. It is doubtful that an analog of (A3) is valid for a string with folds.

APPENDIX B: STRING HAMILTONIAN FOR SCALAR ELECTRODYNAMICS

In the text, we broke off the development for charged scalars after Sec. IV. In this section, we will finish the calculation. In fact, we shall offer only a sketch here—taking for granted that the reader has understood our subsequent discussion for fermions.

The two cases differ essentially only in quantum-ordering effects [compare Eqs. (3.9) and (5.18)]. For the bosons, the crucial terms are, on the lattice,

$$+\frac{e\epsilon}{2}\sum_l P_l^+[A^-(x_l)+A^-(x_{l-1})]$$

$$\approx +e\epsilon\sum_l \tfrac{1}{2}(P_l^++P_{l+1}^+)A^-(x_l), \quad (B1)$$

where we have dropped certain boundary terms for simplicity. On the transformed lattice

$$\tilde\epsilon_l=-P_l^+\epsilon \quad (B2)$$

these terms become

$$-\frac{e}{2}\sum_l\tilde\epsilon_l\left(1+\frac{P_{l+1}^+}{P_l^+}A^-(x_l)\right) \quad (B3)$$

and similarly for the oppositely-charged boson. The integration over the gluon field may now be done, yielding the effective interaction

$$V=\frac{e^2}{2}\frac{1}{4}\sum_l\tilde\epsilon_l\left(1+\frac{P_{l+1}^{+(1)}}{P_l^{+(1)}}\right)|x_l^{(1)}-x_l^{(2)}|\left(1+\frac{P_{l+1}^{+(2)}}{P_l^{+(2)}}\right). \quad (B4)$$

It is not hard to convince oneself that this interaction becomes, in operator notation,

$$V_{\rm op}=\frac{e^2}{2}\frac{1}{4}\left(|x_1^--x_2^-|+P^{+(1)}|x_1^--x_2^-|\frac{1}{P^{+(1)}}\right.$$

$$+P^{+(2)}|x_1^--x_2^-|\frac{1}{P^{+(2)}}$$

$$\left.+P^{+(1)}P^{+(2)}|x_1-x_2|\frac{1}{P^{+(1)}P^{+(2)}}\right). \quad (B5)$$

It requires straightforward but somewhat more involved effort to show that the spectrum of this Hamiltonian is described by the integral equation

$$\tilde{\mu}^2\phi(x)=\left(\frac{\gamma}{x}+\frac{\gamma}{1-x}\right)\phi(x) - \int_0^1 dy\left\{\frac{1}{4}\left[\frac{(y+x)[2-(y+x)]}{y(1-y)}\right]\right\} \times\frac{1}{(x-y)^2}\phi(y), \tag{B6}$$

$$\gamma=\frac{m^2\pi^2}{e^2},\quad \tilde{\mu}^2=\frac{\mu^2\pi^2}{e^2}$$

and μ^2 is the invariant mass squared of the system. The integral equation differs from 't Hooft's[9] equation by the factor in curly brackets. Because that factor goes to 1 as $x\to y$, the high-lying spectrum ($\hbar$ small) is the same as that of BBHP and 't Hooft, in agreement with our previous remarks. The kernel can easily be symmetrized, and an Hermitian inner product established. The spectrum is discrete and real. The same equation was found by Bardeen and Pearson[19] in a different context.

Not surprisingly, this is also the integral equation one gets in summing light-cone planar diagrams[20] in the large-N limit of non-Abelian scalar electrodynamics (with $g^2\to g^2N$).

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[8]We thank A. Jevicki for helping us streamline the derivation of Feynman's result.

[9]In this paper, we omit discussion of trapping at the two-point–function level.

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