# Conformal properties of pseudoparticle configurations*

R. Jackiw, C. Nohl, and C. Rebbi

*Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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The known Euclidean Yang-Mills pseudoparticle solutions with Pontryagin index $n$ are parametrized by $5n$ constants describing the size and location of each pseudoparticle. By insisting on conformal covariance of the solutions, we show that more general solutions exist—they are parametrized by $5n+4$ constants. We further demonstrate that the additional degrees of freedom are not gauge artifacts and correspond to a new degeneracy of pseudoparticle configurations.

## I. INTRODUCTION

Recently Belavin, Polyakov, Schwartz, and Tyupkin[1] have shown that in the Euclidean domain the action functional of non-Abelian gauge theories possesses local minima different from the trivial absolute minimum corresponding to vanishing field strength $F^a_{\mu\nu}$. The implications of their discovery for the structure of the quantum theory are profound.[2]

The minima of the action are characterized by an integer $n$, the Pontryagin index, which labels topologically inequivalent classes of field configurations. Within each class, the action is bounded below by a constant multiple of $|n|$ and the bound is saturated by values of the potentials for which $F^a_{\mu\nu} = \pm {}^*F^a_{\mu\nu}$, where the dual of $F^a_{\mu\nu}$ is ${}^*F^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma}$.

A self-dual field configuration with unit Pontryagin index was exhibited in Ref. 1. This solution to the field equations is often called a pseudoparticle and depends on five parameters: the four coordinates of the pseudoparticle's position and a dimensional scale which measures the pseudoparticle's "size." The question of whether the bound on the action can be saturated also for values of the Pontryagin index different from unity was very recently answered in the affirmative by Witten,[3] who discovered a set of self-dual field configurations where arbitrary numbers of pseudoparticles appear aligned on a definite axis with arbitrary separations and sizes. Soon after, 't Hooft[4] was able to enlarge again the class of known exact solutions of the field equations by exhibiting self-dual field configurations with arbitrary $n$, described by $5n$ parameters, which may be interpreted as positions and sizes of the $n$ pseudoparticles. 't Hooft's solution makes use of a previously proposed ansatz which reduces the condition of self-duality to the Laplace equation for a scalar "superpotential," which can be singular.[5] The positions and residues of the singularities are the parameters that specify the solutions.

In this note we wish to investigate the behavior under conformal transformations of the class of solutions discovered by 't Hooft. Our main result is that, in order to satisfy conformal covariance, the general $n$-pseudoparticle solution must depend on $5n+4$ parameters, rather than the $5n$ parameters one might expect. We show that the dependence on the additional four parameters does not generally correspond to a gauge freedom; consequently they must be interpreted as having physical significance.

## II. SOLUTION OF SELF-DUALITY EQUATION

We begin by summarizing the construction of 't Hooft's solution. It is convenient to represent the potentials and field strengths as matrices in the space of infinitesimal generators of the internal-symmetry group. We consider an SU(2) gauge group and set

$$A_\mu = A^a_\mu \frac{\sigma^a}{2i}, \tag{2.1a}$$

$$F_{\mu\nu} = F^a_{\mu\nu}\frac{\sigma^a}{2i} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.1b}$$

where $\sigma^a$ are Pauli matrices. The action density $S$ and the Pontryagin density ${}^*S$ are

$$S = -\tfrac{1}{2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}, \qquad {}^*S = -\tfrac{1}{2}\operatorname{Tr} {}^*F_{\mu\nu}F^{\mu\nu}. \tag{2.2}$$

The Pontryagin index is given by

$$q = \frac{1}{8\pi^2}\int d^4x\, {}^*S. \tag{2.3}$$

It is useful to define a set of antisymmetric matrices $\bar\sigma_{\mu\nu}$ (Ref. 6)

$$\bar\sigma_{ij} = \frac{1}{4i}[\sigma_i, \sigma_j], \quad \bar\sigma_{i4} = -\tfrac{1}{2}\sigma^i, \qquad i,j = 1,2,3. \tag{2.4}$$

These matrices are anti-self-dual $\bar\sigma_{\mu\nu} = -{}^*\bar\sigma_{\mu\nu}$, and the ansatz for the gauge field is

$$A_\mu = i\bar{\sigma}_{\mu\nu} a^\nu, \tag{2.5}$$

where $a_\nu$ is a vector field that will be further specified below. We see that the three potentials $A^a_\mu$ are expressed in terms of the single potential $a_\nu$.

The self-duality condition $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduces to equations for $a_\nu$:

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu = -{}^*f_{\mu\nu}, \tag{2.6a}$$

$$\partial_\mu a^\mu + a_\mu a^\mu = 0. \tag{2.6b}$$

Equation (2.6a) may be satisfied if $a_\mu$ is derived from a scalar superpotential $\rho$:

$$a_\mu = \partial_\mu \ln\rho. \tag{2.7}$$

Then Eq. (2.6b) becomes

$$\frac{1}{\rho} \Box \rho = 0, \tag{2.8}$$

and the action density, which now is equal to the Pontryagin density, may be expressed in terms of $\rho$ (see also Ref. 7):

$$S = {}^*S = -\tfrac{1}{2} \Box\Box \ln\rho. \tag{2.9}$$

In order that $S$ be integrable, $\rho$ must never vanish, but singularities of the form $\rho(x) \approx \lambda^2/(x-y)^2$ are acceptable because $S$ remains regular at $x = y$.[7]

't Hooft takes for the solution of (2.8)

$$\rho(x) = 1 + \sum_{i=1}^{n} \frac{\lambda_i^2}{(x-y_i)^2}. \tag{2.10}$$

It is clear that a more general solution, which as we show below is conformally covariant [ a property not shared by (2.10)], can also be given by

$$\rho(x) = \sum_{i=1}^{N} \frac{\lambda_i^2}{(x-y_i)^2}, \tag{2.11}$$

and (2.10) may be regained for the case $y_N \to \infty$, $\lambda_N \to \infty$, with $\lambda_N^2/y_N^2 = 1$ and $n = N-1$.

It is important to evaluate the Pontryagin index. From (2.3) and (2.9) it follows that

$$q = -\frac{1}{16\pi^2} \int d^4x \Box\Box \ln\rho. \tag{2.12}$$

In (2.9) and (2.12) $\rho$ may be multiplied by a constant factor or by $(x-y)^2$ without changing $^*S$ or $q$.[7] Hence when $\rho$ is of the form (2.11), a manifestly nonsingular expression may be given for $^*S$:

$$^*S = -\tfrac{1}{2} \Box\Box \ln P_{2N-2}. \tag{2.13}$$

Here $P_{2N-2}$ is polynomial in $x$ of degree $2N-2$:

$$P_{2N-2}(x) = \Big[ \sum_{i=1}^{N} \lambda_i^2 \prod_{j \neq i} (x-y_i)^2 \Big] \Big( \sum_{i=1}^{N} \lambda_i^2 \Big)^{-1}. \tag{2.14}$$

From Eqs. (2.12), (2.13), and (2.14) we easily find the Pontryagin index of the solution (2.11):

$$\begin{aligned} q &= -\frac{1}{16\pi^2} \int d^4x \Box\Box \ln P_{2N-2} \\ &= -\lim_{R\to\infty} \int \frac{R^2 d\Omega}{16\pi^2} R_\mu \partial^\mu \Box \ln(R^{2N-2} + \cdots) \\ &= N-1. \end{aligned} \tag{2.15}$$

The use of Gauss's theorem in the evaluation of (2.15) is justified, since the integrand is nonsingular. Thus we see that although the superpotential $\rho$ depends on $N$ position parameters $y_i^\mu$, the field configuration has a Pontryagin index appropriate to $N-1$ pseudoparticles. If the action density displays $N-1$ maxima at all, there is no obvious simple relation between their positions and the parameters $y_i^\mu$, even in the limit of small $\lambda_i$'s.

A more direct relation between the $y_i^\mu$'s and the positions of the pseudoparticles can be obtained when $\rho$ is of the form (2.10). It is easy to show that for (2.10) the Pontryagin index is $n$, and the $y_i^\mu$'s and $\lambda_i$'s can be interpreted as positions and sizes of the pseudoparticles, in the sense that for small $\lambda_i^\mu$'s the maxima of the action density are centered about the $y_i^\mu$'s. The more general expression, Eq. (2.11), contains precisely four more relevant parameters (a common rescaling of the $\lambda_i$'s does not affect the expression of $A_\mu$) than the $5(N-1)$ coordinates and sizes of the pseudoparticles. One may ask, then, whether the additional four parameters are physical, specifying a further degeneracy of the solution when two or more pseudoparticles are put together, or whether the more restrictive form (2.10) completely exhausts the multipseudoparticle solutions. In what follows we shall answer the question; we find that our additional four parameters are truly present—they are neither gauge phantoms nor can they be incorporated in reparametrizing Eq. (2.10).

## III. EFFECTS OF CONFORMAL TRANSFORMATIONS

Let us first consider the behavior of the potentials $A_\mu$ under the infinitesimal special conformal transformation

$$x^\mu \to \bar{x}^\mu = x^\mu + 2\epsilon\cdot x x^\mu - \epsilon^\mu x^2. \tag{3.1}$$

A field $\varphi$ with scale dimension $d$ transforms covariantly if

$$\delta_c\varphi = (2\epsilon\cdot x x^\alpha - \epsilon^\alpha x^2)\partial_\alpha\varphi + 2\epsilon_\alpha x_\beta (g^{\alpha\beta} d - \Sigma^{\alpha\beta})\varphi, \tag{3.2}$$

where $\Sigma^{\alpha\beta}$ is the spin matrix of the field.[8] It is apparent from Eq. (2.5) that an infinitesimal conformal transformation of $A_\mu$ will not correspond to any simple transformation of $a_\nu$ (indeed, in general it will not be compatible with the ansatz), because the spin matrix $\Sigma^{\alpha\beta}$ in Eq. (3.2) operates on the free index $\mu$ of $\bar{\sigma}_{\mu\nu}$ and not on the index which is contracted with $a_\nu$. However, if together with the conformal transformation we perform a gauge transformation[6]

$$\delta A_\mu = \partial_\mu \chi - [\chi, A_\mu], \tag{3.3}$$

with

$$\chi = 2i\epsilon_\alpha x_\beta \bar{\sigma}^{\alpha\beta}, \tag{3.4}$$

then, because $\bar{\sigma}_{\mu\nu}$ and $\Sigma_{\mu\nu}$ have identical commutation relations, the net effect of the spin matrix in Eq. (3.2) and the commutator of the gauge transformation in Eq. (3.3) will be to transfer the action of the spin matrix from the free index to the second index of $\bar{\sigma}_{\mu\nu}$:

$$\begin{aligned} 2\epsilon_\alpha x_\beta (\Sigma^{\alpha\beta})_{\mu\nu}\, i\bar{\sigma}^{\nu\omega} a_\omega + 2\epsilon_\alpha x_\beta [i\bar{\sigma}^{\alpha\beta}, i\bar{\sigma}_{\mu\nu}] a^\nu &= -2\epsilon_\alpha x_\beta (\Sigma^{\alpha\beta})_{\nu\omega} i\bar{\sigma}_\mu{}^\omega a^\nu \\ &= i\bar{\sigma}_\mu{}^\nu 2\epsilon_\alpha x_\beta (\Sigma^{\alpha\beta})_{\nu\omega} a^\omega. \end{aligned} \tag{3.5}$$

Thus the variation of $A_\mu$ generated by a special conformal transformation followed by the gauge readjustment of Eqs. (3.3), (3.4) is identical to the variation induced by

$$\delta a_\mu = \delta_c a_\mu - 2\epsilon_\mu, \tag{3.6}$$

where the second term on the right-hand side comes from the gradient part of the gauge transformation. But this is precisely the variation of $a_\mu$ that follows from Eq. (2.7) if $\rho$ transforms as a scalar density of scale dimension $d=1$, since

$$\delta\rho = (2\epsilon\cdot x x^\alpha - \epsilon^\alpha x^2)\partial_\alpha \rho - 2\epsilon\cdot x \rho \tag{3.7}$$

implies

$$\begin{aligned} \delta a_\mu &= \delta\partial_\mu \ln\rho \\ &= \partial_\mu (2\epsilon\cdot x x^\alpha - \epsilon^\alpha x^2)\partial_\alpha \ln\rho - 2\epsilon_\mu \\ &= \delta_c a_\mu - 2\epsilon_\mu. \end{aligned} \tag{3.8}$$

Summarizing, a conformal transformation of $\rho$ as a scalar density of dimension 1 followed by a suitable gauge readjustment induces the correct conformal transformation of $A_\mu$.

In a finite special conformal transformation

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2 x^2} \tag{3.9}$$

a scalar density of dimension 1 transforms as

$$\rho(x) \to \tilde{\rho}(x) = \frac{1}{1 - 2a\cdot x + a^2 x^2}\, \rho(\tilde{x}). \tag{3.10}$$

With a little algebra one finds that (2.11) transforms as

$$\tilde{\rho}(x) = \sum_{i=1}^{N} \frac{\tilde{\lambda}_i^2}{(x - \tilde{y}_i)^2}, \tag{3.11}$$

where

$$\tilde{\lambda}_i^2 = \frac{\lambda_i^2}{1 + 2a\cdot y_i + a^2 y_i^2}, \tag{3.12a}$$

$$\tilde{y}_i^\mu = \frac{y_i^\mu + a^\mu y_i^2}{1 + 2a\cdot y_i + a^2 y_i^2}. \tag{3.12b}$$

It is trivial to verify that Poincaré transformations and dilatations also leave invariant the form of Eq. (2.11). Thus, as anticipated, the class of solutions corresponding to Eqs. (2.5), (2.7), (2.8), and (2.11) is closed under the action of the full conformal group. On the other hand, conformal transformations take the solution (2.10) into the solution (2.11).

## IV. RESIDUAL GAUGE FREEDOM

The results obtained in Sec. III still leave open the possibility that some variations of the parameters in Eq. (2.11) correspond to a pure gauge transformation of the fields, so that not all of the $\lambda_i$'s and $y_i^\mu$'s would be physical parameters. We investigate in this section when such gauge transformations exist.

Under an infinitesimal gauge transformation, the gauge field transforms as

$$\delta A_\mu = \partial_\mu \chi - [\chi, A_\mu], \tag{4.1}$$

where $\chi$ is an anti-Hermitian matrix-valued field. We set

$$\chi = i\bar{\sigma}_{\alpha\beta}\omega^{\alpha\beta}, \tag{4.2}$$

with $\omega_{\alpha\beta}$ antisymmetric and anti-self-dual, and look for nontrivial solutions of the equation

$$\begin{aligned} \delta A_\mu &= i\bar{\sigma}_{\alpha\beta}\partial_\mu \omega^{\alpha\beta} + [\bar{\sigma}_{\alpha\beta}, \bar{\sigma}_{\mu\nu}]\omega^{\alpha\beta} a^\nu \\ &= i\bar{\sigma}_{\mu\nu}\delta a^\nu, \end{aligned} \tag{4.3}$$

where $\delta a^\nu = \delta(\partial^\nu \rho/\rho)$ is the variation of $a^\nu$ induced by an infinitesimal change of the parameters in Eq. (2.11).

Equation (4.3) implies

$$\bar{\sigma}_{\alpha\beta}(\partial_\mu \omega^{\alpha\beta} - a_\mu \omega^{\alpha\beta} + 4g^\beta{}_\mu \omega^{\alpha\nu} a_\nu + g^\beta{}_\mu \delta a^\alpha) = 0, \tag{4.4}$$

where we have used the fact that both $\bar{\sigma}_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are anti-self-dual to express

$$\bar{\sigma}^{\alpha\nu}\omega_{\mu\alpha} = -\bar{\sigma}^\alpha{}_\mu \omega^\nu{}_\alpha - \tfrac{1}{2} g^\nu{}_\mu \bar{\sigma}^{\alpha\beta}\omega_{\alpha\beta}. \tag{4.5}$$

It is convenient to define

$$\omega_{\alpha\beta} = \rho\tilde{\omega}_{\alpha\beta} \tag{4.6a}$$

and

$$4\omega_{\alpha\nu}a^\nu + \delta a_\alpha = \rho b_\alpha. \tag{4.6b}$$

Then Eq. (4.4) becomes

$$\bar{\sigma}_{\alpha\beta}(\partial_\mu \tilde{\omega}^{\alpha\beta} + g^\beta{}_\mu b^\alpha) = 0. \tag{4.7}$$

Multiplying with $\bar{\sigma}_{\alpha'\beta'}$ and taking the trace we find

$$4\partial_\mu \tilde{\omega}_{\alpha\beta} + g_{\beta\mu} b_\alpha - g_{\alpha\mu} b_\beta + \epsilon_{\alpha\beta\mu\gamma} b^\gamma = 0. \qquad (4.8)$$

This last equation poses formidable constraints on the possible forms of $\tilde{\omega}_{\alpha\beta}$ and $b_\alpha$. In particular, by using the symmetry properties of the various terms under permutations of the indices, one can show that $\tilde{\omega}_{\alpha\beta}$ must obey an equation where $b_\alpha$ does not appear:

$$2\partial_\mu \tilde{\omega}_{\alpha\beta} + \partial_\alpha \tilde{\omega}_{\mu\beta} + \partial_\beta \tilde{\omega}_{\alpha\mu} - g_{\alpha\mu}\partial_\gamma \tilde{\omega}^\gamma{}_\beta + g_{\beta\mu}\partial_\gamma \tilde{\omega}^\gamma{}_\alpha = 0. \qquad (4.9)$$

(Notice the similarity to the equation satisfied by a Killing vector.) Once Eq. (4.9) is satisfied, Eq. (4.8) reduces to the condition

$$b_\alpha + \tfrac{4}{3}\partial^\beta \tilde{\omega}_{\alpha\beta} = 0. \qquad (4.10)$$

From the integrability conditions which follow from Eq. (4.9), after nontrivial algebra, one proves that the most general solution to Eq. (4.9) is

$$\tilde{\omega}_{\alpha\beta}(x) = 2x_\alpha A_{\beta\gamma} x^\gamma - 2x_\beta A_{\alpha\gamma} x^\gamma + x^2 A_{\alpha\beta} + B_\alpha x_\beta - B_\beta x_\alpha - \epsilon_{\alpha\beta\gamma\delta} B^\gamma x^\delta + C_{\alpha\beta}, \qquad (4.11)$$

with constant $A_{\alpha\beta}$, $B_\alpha$, and $C_{\alpha\beta}$. Furthermore $A_{\alpha\beta}$ ($C_{\alpha\beta}$) must be anti-symmetric and self-dual (anti-self-dual).

Equations (4.11) and (4.6) now give

$$4(2x_\alpha A_{\beta\gamma} x^\gamma - 2x_\beta A_{\alpha\gamma} x^\gamma + x^2 A_{\alpha\beta} + B_\alpha x_\beta - B_\beta x_\alpha - \epsilon_{\alpha\beta\gamma\delta} B^\gamma x^\delta + C_{\alpha\beta})\partial^\beta \rho(x) + \delta a_\alpha(x) - 4\rho(x)(2A_{\alpha\beta} x^\beta - B_\alpha) = 0. \qquad (4.12)$$

The left-hand side of this equation develops singularities when $x^\mu$ approaches any of the $y_i^\mu$. Requiring that the residues vanish, we find the conditions

$$2y_{i\alpha} A_{\beta\gamma} y_i^\gamma - 2y_{i\beta} A_{\alpha\gamma} y_i^\gamma + y_i^2 A_{\alpha\beta} + B_\alpha y_{i\beta} - B_\beta y_{i\alpha} - \epsilon_{\alpha\beta\gamma\delta} B^\gamma y_i^\delta + C_{\alpha\beta} = 0, \quad i = 1, \ldots, N, \qquad (4.13)$$

and

$$\delta y_{i\alpha} + \lambda_i^2 (4A_{\alpha\gamma} y_i^\gamma - 2B_\alpha) = 0, \quad i = 1, \ldots, N. \qquad (4.14)$$

Equations (4.13) represent a constraint on the positions of the singularities in $\rho$ for the existence of a nontrivial gauge transformation preserving the ansatz of Eq. (2.5). For a general configuration of singularities the set of Eqs. (4.13) ($3N$ homogeneous linear equations for the ten independent components of $A_{\alpha\beta}, B_\alpha, C_{\alpha\beta}$) will admit no nonzero solutions and the ansatz completely fixes the gauge. If Eqs. (4.13) are compatible, then Eq. (4.14) specifies the changes of the position parameters that can be achieved with a gauge transformation.

A simple way to understand the geometrical meaning of conditions (4.13) and (4.14) is to observe that both equations are invariant under the fifteen-parameter group of general conformal transformations. With a rather straightforward computation one verifies that the conformally transformed variables $\tilde{y}_i^\mu$ and $\delta\tilde{y}_i^\mu$ satisfy equations analogous to (4.13) and (4.14), where $A_{\alpha\beta}$, $B_\alpha$, and $C_{\alpha\beta}$ are replaced by new tensors $\tilde{A}_{\alpha\beta}$, $\tilde{B}_\alpha$, and $\tilde{C}_{\alpha\beta}$. If Eqs. (4.13) admit a nontrivial solution, then by a suitable conformal transformation we can make $\tilde{A}_{\alpha\beta} = \tilde{C}_{\alpha\beta} = 0$, and the equations reduce to

$$\tilde{B}_\alpha \tilde{y}_{i\beta} - \tilde{B}_\beta \tilde{y}_{i\alpha} - \epsilon_{\alpha\beta\gamma\delta}\tilde{B}^\gamma \tilde{y}_i^\delta = 0, \quad i = 1, \ldots, N. \qquad (4.15)$$

This implies

$$\tilde{y}_{i\alpha} = K_i \tilde{B}_\alpha, \quad i = 1, \ldots, N \qquad (4.16)$$

and we see that the images $\tilde{y}_i^\mu$ of the points $y_i^\mu$ lie on a straight line through the origin. We conclude that a nontrivial gauge transformation preserving the ansatz of Eq. (2.5) may exist only if the singularities in Eq. (2.11) lie on a circle (or on a straight line, as a circle through the point at infinity). If Eq. (4.16) is satisfied, the condition

$$\delta\tilde{y}_{i\alpha} = 2\lambda_i^2 \tilde{B}_\alpha, \quad i = 1, \ldots, N \qquad (4.17)$$

further specifies that the effect of the gauge transformation may only be to move the images $\tilde{y}_i^\mu$ on the same straight line, and therefore the points $y_i^\mu$ on the circle they determine, in a one-parameter group of transformations. If the points are on a straight line it is easy to prove that the gauge transformation actually exists, and by conformal covariance this extends also to the case where the points lie on a circle.

Summarizing, if the $N$ points $y_i^\mu$ in Eq. (2.11) do not lie on a circle, the ansatz of Eqs. (2.5) and (2.7) completely fixes the gauge, and all the $5n+4$ parameters that specify a field configuration with Pontryagin index $n = N-1$ are relevant. For $N = 3$ (three points always lie on a circle) the two-pseudoparticle field configuration is characterized by thirteen physical parameters, since one of the fourteen parameters corresponds to a gauge transformation which moves the pseudoparticle around the circle. If $N = 2$ there is a threefold variety of circles passing through $y_1^\mu$ and $y_2^\mu$, so that only five of the nine parameters in Eq. (2.11) are physical; thus a single pseudoparticle is characterized by position and size only.

It is intriguing that when two or more pseudoparticles are put together more parameters are necessary to specify the field configuration than the positions and sizes of the pseudoparticles. The results we have obtained call for a physical explanation of this additional degeneracy; one wonders whether there exist still further solutions to the self-duality equations beyond those discussed here. These problems are currently under investigation.

*Note added in proof.* Recently it has been possible to show that for an SU(2) gauge theory the general $n$-pseudoparticle solution is specified by at least $8n-3$ parameters.[9] The interpretation of the additional $3n-3$ parameters is that they describe the relative orientations of the pseudoparticles in group space.

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[5]F. Wilczek, Princeton University report (unpublished); F. Corrigan and D. Fairlie, Durham University report (unpublished).

[6]We use the notations and conventions adopted in our investigation of the conformal properties of the single pseudoparticle solution; R. Jackiw and C. Rebbi, Phys. Rev. D 14, 517 (1976). The anti-self-dual matrix $\bar{\sigma}_{\mu\nu}$ is appropriate to the self-dual field configurations which are discussed in the text. A parallel discussion may be given for anti-self-dual field configurations; for these the self-dual matrix $\sigma_{\mu\nu}$ is appropriate ($\sigma_{ij}=\bar{\sigma}_{ij}$, $\sigma_{i4}=-\bar{\sigma}_{i4}$).

[7]The equality $-\frac{1}{2}\,\mathrm{Tr}F^{\mu\nu}F_{\mu\nu}=-\frac{1}{2}\,\mathrm{Tr}\ {}^*F^{\mu\nu}F_{\mu\nu}=-\frac{1}{2}\Box\Box\ln\rho$ holds away from the singularities of $\rho$. At a singularity $\lambda^2/(x-y)^2$ the action density is regular but $\Box\Box\ln\rho$ acquires a $\delta^4(x-y)$ contribution. We avoid this difficulty by excluding from the integration of the action and Pontryagin densities infinitesimal neighborhoods around each singularity — a legitimate procedure since $S$ and ${}^*S$ are nonsingular and continuous there. More simply we may define $\Box\Box\ln(x-y)^2=0$, a procedure we adopt henceforth.

[8]For a summary of the conformal group see S. Treiman, R. Jackiw, and D. Gross, *Lectures on Current Algebra and Its Applications* (Princeton Univ. Press, Princeton, New Jersey, 1972), p. 97.

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