Conformal-invariant finite quantum electrodynamics

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Quantum electrodynamics is modified by including exponential couplings of a massless scalar field so as to make the theory conformal invariant. Nonpolynomial Lagrangian techniques are employed to calculate the electron and photon self-energies to lowest order in the fine-structure constant and show that the usual ultraviolet infinities are suppressed, the minor coupling constant providing the cutoff. A proof of ultraviolet finiteness of the theory to arbitrary order is also presented.

I. INTRODUCTION

Ultraviolet infinities in field theories, especially electrodynamics, have worried theoreticians for a long time. These infinities persist[1] even in exact solutions of certain field theories (with polynomial interactions) in two and three space-time dimensions. This suggests that the fault probably does not lie in the perturbation expansion, but in the assumed nature of interaction. One is naturally inclined to look towards nonpolynomial interactions which, fortunately, are known to have remarkable convergence properties.[2] While *ad hoc* nonpolynomial modifications of quantum electrodynamics (there are innumerable possible varieties of them) could be made[3] to cure it of the ultraviolet infinities, it is clearly desirable that there should be a natural choice for such a modification. Now, the gravitational interaction is well known to be nonpolynomial and it has been conjectured by several theoreticians[4] that infinities in electrodynamics may be cured by taking into account the curvature of the space-time around the elctron and the photon. Isham, Salam, and Strathdee[5,6] have shown that quantum electrodynamics modified by Einstein's tensor gravity does indeed have a natural cutoff provided by the gravitational coupling constant.

Nonpolynomial Lagrangian theories have their own problems; in particular, they suffer from Borel ambiguities and ambiguities of distribution-theoretic origin.[7] Lehmann and Pohlmeyer[8] have shown that for nonpolynomial Lagrangians belonging to the localizable class (this includes, for example, Lagrangians of the exponential type but not of the rational type) these ambiguities may be removed by employing reasonable physical criteria on the large-momentum behavior of the superpropagators; moreover, such Lagrangians are free of Borel ambiguities and possess the desirable features of "good" field theories.[6,7] It is desirable, therefore, to work with localizable Lagrangians as is, indeed, done in Ref. 6 by adopting the exponential parametrization of the vierbein field.

While in principle it is possible to remove ultraviolet infinities in quantum field theory by harnessing gravity, the complications of tensor gravity make it very difficult to put it in practice; moreover, the infinities of quantized gravity itself have not been cured or even circumvented,[9] and one cannot claim to have achieved a consistent field theory with no ultraviolet divergences. The purpose of the present paper is to present a simpler alternative which does not have these disadvantages and which retains almost all advantages of gravity-modified field theories. Our prescription is to invoke conformal invariance, which can be achieved[10] by multiplying individual terms in a Poincaré-invariant Lagrangian by powers of $\exp[f\sigma(x)]$, where $\sigma(x)$ is a massless scalar field transforming inhomogeneously under conformal transformations. With this prescription, exponential couplings appear in those terms in the Lagrangian which have their original scale dimension different from -4; in particular, they appear in the mass terms. By applying appropriate field transformations the exponential couplings can be removed from the mass terms. After this transformation, there is generally an exponential factor in every interaction term in the Lagrangian. Now the absence of ultraviolet divergences can be demonstrated in the usual manner[5,6,7]; the minor coupling constant f provides the cutoff. In this paper we have carried out this program for quantum electrodynamics.

A lesson that one learns from the work of Isham, Salam, and Strathdee[5,6] is that, whereas the electromagnetic interaction alone cannot provide for stable matter, the electromagnetic and gravitational interactions together can. Now, the full glory of tensor gravity, while it may be needed to explain various gravitational phenomena, may not after all be essential for the stability problem; some simpler version of it may do as well. The present work is an example of this; indeed, the exponential interaction of the massless scalar fields is essentially scalar gravity.[11] The simplification so brought about can be exploited while constructing finite theories of other fundamental interactions. An extra bonus is that such theories, with built-in conformal invariance, will have highly desirable scaling properties.

The formalism of conformal-invariant quantum electrodynamics is presented in the next section. In Sec. III, lowest-order calculation of electron and photon self-energies is presented, and the ultraviolet divergences are shown to be absent. Section IV contains a discussion of gauge invariance. The proof of ultraviolet convergence of the theory in a general order is presented in the last section.

II. CONFORMAL-INVARIANT QUANTUM ELECTRODYNAMICS

The conformal group contains, in addition to the transformations of the Poincaré group, the dilatations

$$x^\mu \to x'^\mu = e^\lambda x^\mu, \quad \lambda \text{ real} \tag{2.1}$$

and the special conformal transformations

$$x^\mu \to x'^\mu = \frac{x^\mu + \beta^\mu x^2}{1+2\beta\cdot x+\beta^2 x^2} . \tag{2.2}$$

A convenient method of constructing representations of the conformal group from those of the homogeneous Lorentz group is based on the observation[10] that all transformations of the conformal group have the following property:

$$\frac{\partial x'^\mu}{\partial x^\rho}\,\frac{\partial x'^\nu}{\partial x^\sigma}\,g_{\mu\nu} = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{1/2} g_{\rho\sigma} . \tag{2.3}$$

The transformations of the Poincaré group and the dilatations (2.1) obviously satisfy (2.3); that the special conformal transformations (2.2) also satisfy (2.3) can be easily verified. It follows from Eq. (2.3) that if $x^\mu \to x'^\mu$ is a transformation of the conformal group, then

$$\Lambda^\mu{}_\nu(x) = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{-1/4} \frac{\partial x'^\mu}{\partial x^\nu} \tag{2.4}$$

is a Lorentz matrix. Now, if a set of fields $\phi_\alpha(x)$ transform under the homogeneous Lorentz transformation $x\to x'=Lx$ as

$$\phi'_\alpha(x') = [D(L)]_{\alpha\beta}\,\phi_\beta(x) , \tag{2.5}$$

then the transformations

$$\phi'_\alpha(x') = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{l_\phi/4} [D(\Lambda(x))]_{\alpha\beta}\,\phi_\beta(x) \tag{2.6}$$

clearly provide a representation of the conformal group. The number l_ϕ is the scale dimension of the field. We will assume, for simplicity, that the scale dimensions have their canonical values, that is, $l_\phi = -1$ for scalar and vector fields and $l_\phi = -\frac{3}{2}$ for the spinor fields.

To construct conformal-invariant Lagrangians, we introduce a scalar field $\sigma(x)$ which transforms[10] as

$$\sigma'(x') = \sigma(x) + \frac{1}{4f}\,\ln\left|\det\left(\frac{\partial x'}{\partial x}\right)\right| . \tag{2.7}$$

For Poincaré transformations, the inhomogeneous term clearly vanishes; for the scale transformations (2.1), Eq. (2.7) simplifies to

$$\sigma'(x') = \sigma(x) + \lambda/f . \tag{2.8}$$

The transformation law (2.7) can also be written in the following useful form:

$$\exp[f\sigma'(x')] = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{1/4} \exp[f\sigma(x)] . \tag{2.9}$$

The space-time dependence of the transformation coefficients in Eq. (2.6) implies that the derivatives of fields will not have a linear homogeneous transformation law; this property is enjoyed by the covariant derivatives[10]

$$\Delta_\mu\phi = \partial_\mu\phi - f(l_\phi g_{\mu\nu} - iS_{\mu\nu})(\partial^\nu\sigma)\phi , \tag{2.10}$$

which have a transformation law similar to (2.6) with $l_\phi \to l_\phi - 1$. Here $S_{\mu\nu}$ are matrices representing the Lorentz generators in the representation of the ϕ fields. For spin $\frac{1}{2}$ and spin 1, they have the following forms:

$$\begin{aligned} &\text{spin } \tfrac{1}{2}: S_{\mu\nu} = \tfrac{1}{4} i[\gamma_\mu, \gamma_\nu] , \\ &\text{spin } 1: (S_{\mu\nu}A)_\rho = i(g_{\mu\rho}A_\nu - g_{\nu\rho}A_\mu) . \end{aligned} \tag{2.11}$$

To ensure that the action is conformal invariant, we need the following transformation property for the Lagrangian:

$$\mathcal{L}'(x') = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{-1} [\mathcal{L}(x) + \text{four-divergence}] . \tag{2.12}$$

When ordinary derivatives have been replaced by covariant derivatives in a Poincaré-invariant Lagrangian, each term in the resulting Lagrangian will be conformal invariant except for powers of $|\det(\partial x'/\partial x)|$; this deficiency can clearly be corrected by multiplying each term by appropriate power of $\exp(f\sigma)$. To this Lagrangian we must add the kinetic-energy term for the σ field. In this connection, we mote that the covariant derivatives (2.10) of the fields $\exp(\pm f\sigma)$ vanish identically; however, we can employ the Lagrangian

$$\mathcal{L}_\sigma = \tfrac{1}{2}\,\partial_\mu\sigma\,\partial^\mu\sigma\exp(-2f\sigma) , \tag{2.13}$$

which is conformal invariant up to a four-divergence.[10]

We are now prepared to construct a conformal-invariant Lagrangian for quantum electrodynamics. Noting that

$$\Delta_\mu A_\nu - \Delta_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv F_{\mu\nu}, \tag{2.14}$$

we obtain the following Lagrangian:

$$\mathcal{L} = \tfrac{1}{2} i[\bar{\Psi}\gamma^\mu D_\mu \Psi - (D_\mu \bar{\Psi})\gamma^\mu \Psi] - m\bar{\Psi}\Psi \exp(-f\sigma) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial^\mu\sigma\partial_\mu\sigma \exp(-2f\sigma), \tag{2.15}$$

where

$$D_\mu \Psi = (\partial_\mu + ieA_\mu)\Psi + f(\tfrac{3}{2}g_{\mu\nu} + iS_{\mu\nu})(\partial^\nu\sigma)\Psi. \tag{2.16}$$

Applying the field transformation

$$\Psi(x) = \psi(x)\exp[\tfrac{1}{2}f\sigma(x)] \tag{2.17}$$

and after some simplifications, we have

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm em} + \mathcal{L}', \tag{2.18}$$

where

$$\mathcal{L}_0 = \tfrac{1}{2} i[\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma, \tag{2.19}$$

$$\mathcal{L}_{\rm em} = -\tfrac{1}{2}e(\gamma^\mu)_{\alpha\beta}[\bar{\psi}_\alpha, \psi_\beta]A_\mu \exp(f\sigma), \tag{2.20}$$

and

$$\mathcal{L}' = \tfrac{1}{2} i[\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi](e^{f\sigma} - 1) + \tfrac{1}{2}\partial^\mu\sigma\partial_\mu\sigma(e^{-2f\sigma} - 1). \tag{2.21}$$

We shall employ the exponentially modified electromagnetic-interaction Lagrangian $\mathcal{L}_{\rm em}$ to calculate the electron and photon self-energies in the following section. The coupling constant f is assumed to be very small ($|fm| \ll e$). In this connection we note[11] that the field $\exp(-2f\sigma)$ couples to the trace of the energy-momentum tensor [a simple way to see this is that $\exp(f\sigma)$ terms are inserted in only those terms in the initial Poincaré-invariant Lagrangian which have the scale dimension $l \neq -4$; these are precisely the terms that contribute to the trace of the energy-momentum tensor]; the additional interaction is, therefore, scalar gravity.

III. SUPPRESSION OF ULTRAVIOLET DIVERGENCES IN LOWEST ORDER

In this section we shall apply the techniques similar to those in Refs. 5 and 3 to the interaction Lagrangian of the previous section to calculate the electron and photon self-energies to the lowest order in e. The diagrams for these processes are the conventional ones with an additional super-propagator[12] between the two vertices.[3,5]

A. Electron self-energy

In lowest order, the matrix element for electron self-energy is

$$\Sigma(p) = ie^2 \int d^4x\, e^{ip\cdot x}\gamma^\mu S(x)\gamma_\mu D(x)\exp[f^2 D(x)], \tag{3.1}$$

where

$$D(x) = \langle 0 | T(\sigma(x)\sigma(0)) | 0\rangle = -\frac{1}{4\pi^2(x^2 - i\epsilon)} \tag{3.2}$$

and

$$S(x) = \langle 0 | T(\psi(x)\bar{\psi}(0)) | 0\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon}. \tag{3.3}$$

Now

$$D(x)\exp[f^2 D(x)] = \sum_{n=0}^\infty \frac{(f^2)^n}{n!}[D(x)]^{n+1} = \frac{1}{2\pi i}\int_C dz\, \Gamma(-z)(-f^2)^z [D(x)]^{z+1}, \tag{3.4}$$

where C is the contour enclosing the positive real axis clockwise. The contour can clearly be opened up to lie parallel to the imaginary axis with $-1 < \text{Re}z < 0$; we call this later contour C'. Now, making use of the Gel'fand-Shilov formula[13]

$$[D(x)]^z = -i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \times \frac{(4\pi)^{-2(z-1)}(-k^2)^{z-2}\Gamma(2-z)}{\Gamma(z)}, \quad 0 < \text{Re}z < 2 \tag{3.5}$$

we obtain

$$\Sigma(p) = \frac{1}{2\pi i}\int_{C'} dz\, \Gamma(-z)\Sigma(p, z), \tag{3.6}$$

with

$$\Sigma(p, z) = F(z)\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{\gamma\cdot(p-k) + m}{(p-k)^2 - m^2 + i\epsilon} \times \gamma_\mu(-k^2)^{z-1} \tag{3.7}$$

and

$$F(z) = -ie^2(-f^2)^z \frac{(4\pi)^{-2z}\Gamma(-z)}{\Gamma(z)}. \tag{3.8}$$

We note at this point that the contour C' in (3.6) can be shifted to the left as much as we like. For sufficiently negative Rez, the convergence of the integral in (3.7) is assured; moreover, as we shall presently see, no new singularities appear for Re$z < 0$ after the momentum integration, so that the contour can be taken to be C' again. Now we have, after simple manipulations,

$$\Sigma(p,z)=\gamma\cdot pA(p^2,z)+mB(p^2,z)\,, \tag{3.9}$$

$$B(p^2,z)=\frac{1}{4m}\,\mathrm{Tr}[\Sigma(p,z)]=4F(z)I(p^2,z)\,, \tag{3.10}$$

and

$$A(p^2,z)=\frac{1}{4p^2}\,\mathrm{Tr}[\gamma\cdot p\Sigma(p,z)]$$
$$=-2F(z)\left[I(p^2,z)+\frac{1}{z}\frac{\partial}{\partial p^2}I(p^2,z+1)\right], \tag{3.11}$$

where

$$I(p^2,z)=\int\frac{d^4k}{(2\pi)^4}\,\frac{(-k^2)^{z-1}}{(p-k)^2-m^2+i\epsilon}\,. \tag{3.12}$$

Proceeding as in Ref. 6, we obtain

$$I(p^2,z)=\frac{-i}{16\pi^2}\,\Gamma(-z)\Gamma(1+z)m^{2z}$$
$$\times F(1-z,-z;2;p^2/m^2)\,. \tag{3.13}$$

This gives, after a simple calculation,

$$A(p^2,z)=-\frac{\alpha}{2\pi}\left(-\frac{f^2m^2}{16\pi^2}\right)^z\Gamma(1-z)\Gamma(-z)$$
$$\times[F(1-z,-z;2;\,p^2/m^2)$$
$$-\tfrac{1}{2}(z+1)F(1-z,-z;3;\,p^2/m^2)]\,, \tag{3.14}$$

$$B(p^2,z)=\frac{\alpha}{\pi}\left(-\frac{f^2m^2}{16\pi^2}\right)^z\Gamma(1-z)\Gamma(-z)$$
$$\times F(1-z,-z,2;p^2/m^2)\,. \tag{3.15}$$

The contour C' in (3.6) can now be replaced by C. When the z integration is carried out, contributions are obtained from the double pole at $z=0$ and triple poles at $z=1,2,\ldots$. The contribution from $z=n$ has terms proportional to $f^{2n}\ln(fm)$ and to f^{2n}; it therefore represents the (regularized) n-graviton exchange to the second-order electron self-energy. Considering the double pole at $z=0$ only and writing

$$\Sigma(p)=\gamma\cdot pA(p^2)+mB(p^2)\,,$$

we obtain

$$A(p^2)=-\frac{\alpha}{2\pi}\left\{\ln\left(\frac{4\pi}{mf}\right)+\frac{m^2-p^2}{2p^2}\left[1+\frac{m^2+p^2}{p^2}\ln\left(\frac{m^2-p^2}{m^2}\right)\right]-\tfrac{3}{2}\gamma+\tfrac{5}{4}\right\}+O(f^2\ln f)\,, \tag{3.16}$$

$$B(p^2)=\frac{2\alpha}{\pi}\left[\ln\left(\frac{4\pi}{mf}\right)+\frac{m^2-p^2}{2p^2}\ln\left(\frac{m^2-p^2}{m^2}\right)-\tfrac{3}{2}\gamma+\tfrac{1}{2}\right]+O(f^2\ln f)\,. \tag{3.17}$$

This gives, for the electron self-mass,

$$\delta m=m[A(m^2)+B(m^2)]$$
$$=\frac{\alpha m}{2\pi}\left[3\ln\left(\frac{4\pi}{mf}\right)-\tfrac{9}{2}\gamma+\tfrac{3}{4}\right]+O(f^2\ln f), \tag{3.18}$$

and, for the renormalization constant,

$$Z_2^{-1}=1-A(m^2)-2m^2[A'(m^2)+B'(m^2)]\,, \tag{3.19}$$

where γ is the Euler constant. [Equation (3.19) contains the usual infrared divergence.] As expected, the minor coupling constant f provides a natural cutoff; the terms with $\ln(4\pi/mf)$ in the above formulas are reminiscent of the logarithmic infinity in the $f=0$ theory.

B. Photon self-energy

The lowest-order matrix element for the photon self-energy is

$$\Pi_{\mu\nu}(k)=ie^2\int d^4x\,e^{ik\cdot x}\,\mathrm{Tr}[\gamma_\mu S(x)\gamma_\nu S(-x)]\exp[f^2D(x)]\,. \tag{3.20}$$

Proceeding as before, we have

$$\Pi_{\mu\nu}(k)=\frac{1}{2\pi i}\int_{C'}dz\,\Gamma(-z)\Pi_{\mu\nu}(k,z) \tag{3.21}$$

with

$$\Pi_{\mu\nu}(k,z)=ie^2(-f^2)^z\int d^4x\,e^{ik\cdot x}\,\mathrm{Tr}[\gamma_\mu S(x)\gamma_\nu S(-x)][D(x)]^z\,. \tag{3.22}$$

To be able to apply the formula (3.5), we write

$$\Pi_{\mu\nu}(k) = \Pi_{\mu\nu}(k,0) + \frac{1}{2\pi i}\int_{C''} dz\, \Gamma(-z)\Pi_{\mu\nu}(k,z)\,, \tag{3.23}$$

where the contour C'' is parallel to C' with $0<\mathrm{Re}z<1$. In the integrand in Eq. (3.23), we can now use Eq. (3.5) and obtain

$$\Pi_{\mu\nu}(k,z) = K(z)\int\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}(-q^2-i\epsilon)^{z-2}\,\mathrm{Tr}\left[\gamma_\mu \frac{\gamma\cdot p+m}{p^2-m^2+i\epsilon}\gamma_\nu \frac{\gamma\cdot(p+q-k)+m}{(p+q-k)^2-m^2+i\epsilon}\right], \tag{3.24}$$

where

$$K(z) = -e^2(-f^2)^z(4\pi)^{-2z+2}\frac{\Gamma(2-z)}{\Gamma(z)}\,. \tag{3.25}$$

To secure convergence of the integral in (3.24), we shift the contour in Eq. (3.23) to the left. In the process we encounter a simple pole at $z=0$ whose contribution cancels the $\Pi_{\mu\nu}(k,0)$ term in Eq. (3.23) and then no other singularities so that we can write

$$\Pi_{\mu\nu}(k) = \frac{1}{2\pi i}\int_{C'''} dz\, \Gamma(-z)\Pi_{\mu\nu}(k,z)\,, \tag{3.26}$$

where $\Pi_{\mu\nu}(k,z)$ is still given by (3.24) and the contour C''' is parallel to the imaginary axis with $-2<\mathrm{Re}z<-1$. We now write

$$\Pi_{\mu\nu}(k,z) = (k^2 g_{\mu\nu} - k_\mu k_\nu)C(k^2,z) + g_{\mu\nu}D(k^2,z)\,. \tag{3.27}$$

Proceeding as in Ref. 5, we obtain

$$C(k^2,z) = \frac{\alpha}{24\sqrt{\pi}}\left(-\frac{f^2m^2}{4\pi^2}\right)^z \frac{\Gamma(-z)\Gamma(2-z)\Gamma(4-z)}{\Gamma(\frac{5}{2}-z)}\,{}_3F_2\left(4-z, 2-z, -z; 4, \tfrac{5}{2}-z; \frac{k^2}{4m^2}\right) \tag{3.28}$$

and

$$D(k^2,z) = -\frac{3\alpha}{4\sqrt{\pi}}\left(-\frac{f^2m^2}{4\pi^2}\right)^z \frac{m^2}{z+1}\frac{[\Gamma(1-z)]^2\Gamma(2-z)}{\Gamma(\frac{3}{2}-z)}\,{}_3F_2\left(2-z, 1-z, -1-z; 3, \tfrac{3}{2}-z; \frac{k^2}{4m^2}\right). \tag{3.29}$$

We now fold the contour C''' on the positive real axis. Writing the decomposition analogous to (3.27) for $\Pi_{\mu\nu}(k)$, we calculate the contribution of the double pole at $z=0$ for $C(k^2)$ and of the simple poles at $z=-1$ and $z=0$ for $D(k^2)$. We obtain

$$C(k^2) = \frac{\alpha}{3\pi}\left[\ln\left(\frac{4\pi^2}{m^2f^2}\right) - \frac{2\lambda-1}{2\lambda}\left(\frac{\lambda+1}{\lambda}\right)^{1/2}\ln\frac{(1+\lambda)^{1/2}+\sqrt{\lambda}}{(1+\lambda)^{1/2}-\sqrt{\lambda}} + \frac{1}{6} - \frac{1}{\lambda} - 3\gamma + \ln 4\right] + O(f^2\ln f)\,, \tag{3.30}$$

$$D(k^2) = -\frac{8\pi\alpha}{f^2} - \frac{3\alpha}{2\pi}\left(m^2 - \frac{1}{9}k^2\right) + O(f^2\ln f)\,, \tag{3.31}$$

where $\lambda = -k^2/4m^2$. This gives, for the photon wave-function renormalization constant,

$$\begin{aligned} Z_3 &= 1 - C(0) \\ &= 1 - \frac{\alpha}{3\pi}\left[\ln\left(\frac{4\pi^2}{m^2f^2}\right) + \frac{1}{6} - 3\gamma + \ln 4\right] + O(f^2\ln f)\,. \end{aligned} \tag{3.32}$$

Once again, the minor coupling costant provides a natural cutoff.

IV. GAUGE INVARIANCE

A nonzero value of $D(k^2)$ clearly implies lack of gauge invariance. This is not quite unexpected because effectively we are working with the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{em}$ which is not gauge invariant; only the total Lagrangian $\mathcal{L}_0 + \mathcal{L}_{em} + \mathcal{L}'$ is gauge invariant. From the structure of $\mathcal{L}'$ one might expect that its inclusion would affect only terms of order $f^2\ln f$ and above; however, the mechanism which produced f^{-2} term in $D(k^2)$ might operate here as well.

We note in this connection that the f^{-2} term in $D(k^2)$ is reminiscent of the quadratic infinity in this quantity encountered in a naive perturbation-theoretic calculation[14] in conventional quantum electrodynamics (which is based on a formally gauge-invariant Lagrangian); in this latter situation a careful calculation employing a well-defined gauge-invariant current operator[15,16] or appropriate gauge-invariant regularization[17] does ensure $D=0$. This would suggest that the fault probably does not (entirely) lie with the omission of $\mathcal{L}'$ and that even its inclusion may not, after all, ensure gauge invariance.

The need for a modification of the current operator is there in our formalism as well, because the product of field operators at the same space-time point is singular as usual. We will now show that a calculation along the lines of Ref. 16 employing the modified current operator does ensure gauge invariance to zeroth order in the minor coupling constant.

If we replace $\bar{\psi}\gamma^\mu\psi$ in $\mathcal{L}_{\rm em}$ by the strictly gauge-invariant current operator

$$\bar{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[-ie\int_{x-\epsilon/2}^{x+\epsilon/2}d\xi^\nu A_\nu(\xi)\right] \tag{4.1}$$

we have to consider the additional interaction

$$\mathcal{L}'_{\rm em}=ie^2\bar{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right)A_\mu(x) \times\exp[f\sigma(x)]\int_{x-\epsilon/2}^{x+\epsilon/2}A_\nu(\xi)d\xi^\nu\,. \tag{4.2}$$

To obtain the additional contribution $\Pi'_{\mu\nu}(k)$ to the polarization tensor, we must express the first-order S matrix due to $\mathcal{L}'_{\rm em}$ in the form[18]

$$S_1=i\int\mathcal{L}'_{\rm em}(x)dx =-i:\int d^4k\,A^\mu(k)\Pi'_{\mu\nu}(k)A^\nu(k):+\cdots\,, \tag{4.3}$$

where the dots indicate other terms in the normal-product expansion, and

$$A_\mu(k)=\frac{1}{(2\pi)^2}\int d^4x\,e^{ik\cdot x}A_\mu(x)\,.$$

The ξ intergration can be carried out along a straight-line path by writing

$$\xi^\nu=x^\nu+\tfrac{1}{2}s\epsilon^\nu\,,\quad -1\le s\le 1\,.$$

Noting that

$$\langle 0\,|\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma_\mu\psi(x-\tfrac{1}{2}\epsilon)\,|0\rangle=-\mathrm{Tr}[\gamma_\mu S(-\epsilon)]\,,$$

we obtain,[16] after a straightforward calculation,

$$\Pi'_{\mu\nu}(k)=ie^2\epsilon_\nu\mathrm{Tr}[\gamma_\mu S(-\epsilon)]\frac{2}{k\cdot\epsilon}\sin\left(\frac{k\cdot\epsilon}{2}\right)\,. \tag{4.4}$$

This gives, in the limit $\epsilon^\nu\to 0$, a quadratically divergent contribution to $D(k^2)$ which cancels the usual quadratic infinity in $D(k^2)$.

Calling the contribution to $D(k^2)$ from (4.4) as $D'(k^2)$ and using Eqs. (3.21) and (3.27), we have

$$\begin{aligned}[D(k^2)]_{\rm total}&=D'(k^2)+\frac{1}{2\pi i}\int_{C'}dz\,\Gamma(-z)D(k^2,z)\\&=D'(k^2)+D(k^2,0)\\&\quad+\frac{1}{2\pi i}\int_{C''}dz\,\Gamma(-z)D(k^2,z)\,.\end{aligned} \tag{4.5}$$

Now, $D(k^2,0)$ is nothing but the usual quadratically divergent quantity and, as mentioned above,

$$\lim_{\epsilon^\nu\to 0}[D'(k^2)+D(k^2,0)]=0\,. \tag{4.6}$$

We have, therefore

$$[D(k^2)]_{\rm total}=\frac{1}{2\pi i}\int_{C''}dz\,\Gamma(-z)D(k^2,z)\,. \tag{4.7}$$

Now, the contour C'' parallel to the imaginary axis with $0<\mathrm{Re}z<1$, Eq. (4.7), on folding the contour to the right, will (formally) give terms of order $f^2\ln f$ and higher. Had the quantity on the right of Eq. (4.7) been finite as such, our demonstration of gauge invariance to order f^0 would have been complete. Unfortunately, this is not the case, as we have seen in the previous section (the contour had to be shifted to the left). We will now show that inclusion of $\mathcal{L}'$, which also gives (formally) contributions of the same lowest order in f, makes the right-hand side of Eq. (4.7) convergent without shifting the contour to the left.

To include the effects of $\mathcal{L}'$, it is sufficient[6] for our purpose to consider the diagram shown in Fig. 1. Dashed lines represent the superpropagators. This diagram represents the sum of diagrams with and without the indicated modifications due to $\mathcal{L}'$ so that each of the superpropagator lines with momenta q', q'', and q''' represents the sum of zero, one, two, ... graviton exchanges (and not merely one, two, ... graviton exchanges as the

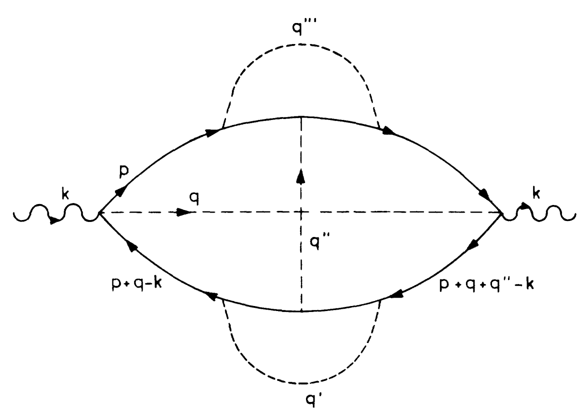

FIG. 1. Photon self-energy supergraph.

structure of $\mathcal{L}'$ might suggest); in the language of Ref. 6, "cradling" in the above-mentioned three superpropagator lines is understood.

It will be useful to first consider the modified electron-electron-graviton vertex shown in Fig. 2. (Remember that the dashed lines are superpropagator lines.) The matrix element for this diagram is

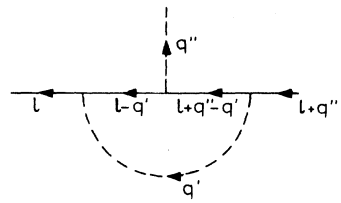

FIG. 2. Electron-electron-graviton "supervertex."

$$M(l,q'') = \int_{C'} dz_1 h(z_1) \int d^4q' \gamma\cdot q' \frac{1}{\gamma\cdot(l-q')-m} \gamma\cdot q'' \frac{1}{\gamma\cdot(l-q'+q'')-m} \gamma\cdot q'(-q'^2)^{z_1-2} . \tag{4.8}$$

The function $h(z)$ includes the z-dependent factors other than $(-q^2)^{z-2}$ in Eq. (3.5) and the various constants. The diagram of Fig. 1 gives

$$k^2D(k^2) \sim \int_{C''} dz \int_{C'} dz_2 h(z)h(z_2) \int\int d^4q\, d^4q'' \times \mathrm{Tr}\left[\gamma\cdot k \frac{1}{\gamma\cdot(p+q-k)-m} M(p+q-k,q'') \frac{1}{\gamma\cdot(p+q+q''-k)-m} \gamma\cdot k \frac{1}{\gamma\cdot(p+q'')-m} \times M(p+q'',-q'') \frac{1}{\gamma\cdot p-m}\right] (-q^2)^{z-2}(-q''^2)^{z_2-2} , \tag{4.9}$$

which is to replace Eq. (4.7).

The q' integral in Eq. (4.8) is convergent without shifting the contour C'. Moreover,

$$\lim_{|q''|\to\infty} M(l,q'') \lesssim |q''| (\ln|q''|)^{n_1} , \quad \lim_{|l|\to\infty} M(l,q'') \lesssim (\ln|l|)^{n_2} , \tag{4.10}$$

where n_1 and n_2 are some positive integers. Now a look at the integrand in Eq. (4.9) shows that, taking (4.10) into account, both q'' and q integrations are convergent without shifting the contours C' and C''. This implies that $D(k^2)$ vanishes up to terms of zeroth order in f.

V. SUPPRESSION OF ULTRAVIOLET DIVERGENCES TO ALL ORDERS

We will now present a simple proof to the effect that ultraviolet infinities are absent in the theory formulated in Sec. II in any order in e. We do not hope to keep the level of mathematical rigor of, for example, Ref. 19 (where finiteness of a simple exponential coupling theory has been demonstrated); however, we believe our proof can stand the stress of full mathematical rigor.

Consider a general graph in an arbitrary order in e. It has, between any two vertices, a superpropagator, at most one photon line, and at most two fermion lines. Working, as in Sec. III, in the Feynman gauge, the photon propagator can always be absorbed in the superpropagator as was done in the electron self-energy calculation (in a general gauge the same can be done after using some calculus of derivatives). Now, ignoring the term $\mathcal{L}'$ in the Lagrangian for the time being, the matrix element for a general diagram can be written in the symbolic form

$$M \sim \int \prod_{i<j} dz_{ij} d^4q_{ij} \Gamma(-z_{ij}) \left(-\frac{f^2}{16\pi^2}\right)^{z_{ij}} \frac{\Gamma(2-z_{ij}-\alpha_{ij})}{\Gamma(z_{ij}+\alpha_{ij})} (-q_{ij}{}^2)^{z_{ij}+\alpha_{ij}-2} \times \prod_a d^4p_a \frac{1}{\gamma\cdot p_a - m} \prod_K (P_K - \sum p - \sum q) , \tag{5.1}$$

where p's and q's represent the momenta of internal electron and superpropagator lines and P_K is the sum of external momenta at the Kth vertex. The symbol α_{ij} (=0 or 1) represents the number of photon lines between the ith and the jth vertices. All inessential factors have been omitted. The contours for the z_{ij} integrations lie parallel to the imaginary axis, with $-1 < \mathrm{Re}\, z_{ij} < 0$. It has been assumed that, whenever necessary, the contours have been shifted to the right to make the exponents $z_{ij}+\alpha_{ij}$ satisfy the condition $0 < \mathrm{Re}(z_{ij}+\alpha_{ij}) < 2$ and, after substituting the Gel'fand-Shilov formula (3.5), have been brought back to the original position as was done in the photon self-energy calculation [see Eq. (3.23)]; this causes no problems.

Now, one can proceed to apply the power-counting arguments, as in Ref. 20; however, a simple observation, which is based on our experience with the photon self-energy calculation in Sec. III, simplifies the proof tremendously. This is that the variables z_{ij} appear with a positive sign in the exponents of q_{ij}^2 and in the Γ functions in the denominator and with a negative sign in the Γ functions in the numerator; this circumstance can be exploited to shift the contours for z_{ij} to the left as much as necessary to secure convergence of q_{ij} integrations, without encountering singularities. Now, after the momentum-conserving δ functions have been used, we can have one of the following two situations:

(i) All the independent loop momenta can be chosen to be the superpropagator momenta; these can be made ultraviolet convergent as explained above.

(ii) Some of the electron momenta have to be chosen as independent loop momenta; this happens when there is an electron loop as in the $\Pi_{\mu\nu}$ calculation. In such situations, at least one of the electron propagators will involve a linear combination of the loop momentum and one of the superpropagator momenta [see Eq. (3.24)]. This fact, combined with the fact that the q_{ij} integrations can be rendered as much convergent as we please, ensures the convergence of all p integrations.

After the momentum integrations, we have

$$M \sim \int \prod_{i<j} \frac{\Gamma(-z_{ij})\Gamma(2-z_{ij}-\alpha_{ij})}{\Gamma(z_{ij}+\alpha_{ij})}\left(-\frac{f^2}{16\pi^2}\right)^{z_{ij}} \times f(P,z)\,, \tag{5.2}$$

where $f(P,z)$ is some function of the external momenta and the z_{ij}'s. Now, it is not difficult to see that $f(P,z)$ is a bounded function of the z_{ij}'s on their contours. The convergence of the integrals in Eq. (5.2) is, therefore, governed by the gamma functions. Now $\Gamma(z)$, with Rez fixed and $|\text{Im} z|$ taking large values, goes to zero very fast, and since there are two Γ functions in the numerator a and one in the denominator, the integrals in (5.2) are convergent. More precisely,[21]

$$\Gamma(az+b) \sim \sqrt{2\pi}\; e^{-az}(az)^{az+b-1/2}\,, \qquad |\arg z| < \pi,\quad a>0. \tag{5.3}$$

Writing $z = x+iy = re^{i\theta}$, Eq. (5.3) gives

$$|\Gamma(az+b)| \sim \text{const} \times e^{-ax}(ar)^{ax+b-1/2}e^{-ay\theta}\,. \tag{5.4}$$

Remembering that y and θ alwyas have the same sign, the assertions made above are easily verified.

Since the above arguments apply to any graph as well as to all its subgraphs, the proof of ultraviolet convergence of any graph computed with the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{em}$ is complete.

Now we consider the inclusion of $\mathcal{L}'$. This brings about the following modifications:

(a) Two new types of vertices are introduced: those involving electron and σ lines only and those involving σ lines only. This fact by itself does not affect our arguments above because nowhere have we made any crucial use of the fact that all vertices should be of the type of $\mathcal{L}_{em}$ only.

(b) The couplings in $\mathcal{L}'$ will give rise to some momentum factors at the vertices. This also does not affect our arguments above because our convergence argument (based on shifting the contours to the left) is powerful enough to take care of any such additional factors.

(c) What could possibly create trouble is the fact that the z_{ij} contours for the superpropagator lines arising from the couplings in $\mathcal{L}'$ will lie in the region $0 < \text{Re} z_{ij} < 1$, so that when one of these contours is shifted to the left the factor $\Gamma(-z_{ij})$ gives a singularity at $z_{ij} = 0$. However, this is not troublesome because the additional contribution from the pole at $z_{ij} = 0$ is finite. To see this, we note that the pole term corresponds to replacing the vertices of $\mathcal{L}'$ involved in such a superpropagator by appropriate "kinetic energy kinks"[6] in an electron or σ line. This clearly does not create any convergence problems.

It follows that the arguments presented above remain valid when the additional term $\mathcal{L}'$ in the Lagrangian is included. In fact, one could start with the whole Lagrangian in the beginning and construct a finiteness proof along similar lines. However, we have found it simpler to present it as above.

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