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New approach to the calculation of $F_1(\alpha)$ in massless quantum electrodynamics

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$F_1(\alpha)$ is defined as the contribution of the one-fermion-loop diagrams to the divergent part of the photon propagator in massless quantum electrodynamics. To sixth order, the perturbation expansion of $F_1(\alpha)$ has *rational* coefficients: $F_1(\alpha)=(2/3)(\alpha/2\pi)+(\alpha/2\pi)^2-(1/4)(\alpha/2\pi)^3+\cdots$. It is not known whether the next term in this series is a rational number; however, we propose a new method, which uses integration by parts, for evaluating Feynman integrals which give rational numbers. Using this method we easily rederive the first three terms in the series for $F_1(\alpha)$ and three other two-loop integrals, including the fourth-order correction to the vertex function $\Gamma^\mu(p,p)$. We believe that our new integration techniques are powerful enough to evaluate the fourth term in the series for $F_1(\alpha)$ if it is a rational number.

I. INTRODUCTION

The function $F_1(\alpha)$ is defined as the coefficient of the logarithmic divergence in the one-fermion-loop diagrams which contribute to the photon propagator in massless quantum electrodynamics. To date only the first three terms in the perturbation series for $F_1(\alpha)$ are known:

$$F_1(\alpha)=\frac{2}{3}\left(\frac{\alpha}{2\pi}\right)+1\left(\frac{\alpha}{2\pi}\right)^2-\frac{1}{4}\left(\frac{\alpha}{2\pi}\right)^3+\cdots . \quad (1)$$

The sixth-order contribution to $F_1(\alpha)$ was published by Rosner[1,2] in 1966. Rosner's result is remarkable because, although the transcendental number $\zeta(3)$ appears at intermediate stages of the calculation, it drops out at the end giving the *rational* number $\frac{1}{4}$.

The fact that the first three coefficients in the series (1) are rational has led to theoretical speculation that the other coefficients might be rational too, possibly as a profound consequence of the scale or conformal invariance of the underlying theory. The most direct way to resolve this question is, of course, to calculate the eighth-order contribution to $F_1(\alpha)$ and to see if the fourth coefficient in (1) is rational. Unfortunately, the eighth-order calculation involves doing formidable three-loop Feynman integrals for which until now no systematic procedure has existed.

In this paper we present a new, simple, and apparently very general technique for evaluating the Feynman integrals that arise in the $F_1(\alpha)$ calculation. *This technique is specifically designed to evaluate integrals which are rational numbers. Thus, if the fourth term in the series (1) has a rational coefficient, the technique we are proposing here should be sufficient to compute it.*

Our new integration technique uses integration by parts. To illustrate our computational approach we show how to evaluate in Euclidean space a simple one-loop integral which is both infrared and ultraviolet convergent:

$$I=\int d^4k\,\frac{p\cdot k}{k^4(p-k)^2} . \quad (2)$$

Note that the *integrand* depends on a 4-momentum p but that the *integral* I is a number independent of p.

To evaluate (2) we note that

$$\frac{p\cdot k}{k^4}=\Box\,\frac{p\cdot k}{-4k^2} , \quad (3)$$

where $\Box$ is the 4-dimensional Laplacian. If we substitute (3) into (2) and integrate by parts we obtain

$$I=\int d^4k\,\frac{p\cdot k}{-4k^2}\,\Box\,\frac{1}{(p-k)^2} . \quad (4)$$

Now we use the identity

$$\Box\,\frac{1}{(p-k)^2}=-4\pi^2\delta^4(p-k) \quad (5)$$

to reduce the integral in (4) to a triviality:

$$I=\int d^4k\,\frac{p\cdot k}{k^2}\,\pi^2\delta^4(p-k)=\pi^2 .$$

The appearance of a δ function in the integrand of (4) is a consequence of the masslessness of the theory.

Evaluating the integral in (2) in this way depends crucially on finding the derivative identity in (3). One may wonder whether or not such nice identities also exist for other integrals. So far, it appears that they always exist as long as the integral is a rational number, regardless of how many loop integrations are involved. In particular, we will show that *all* convergent one-loop integrals which have the form

$$\int d^4k \, \frac{f(p,k)}{k^4(p-k)^4} \ , \tag{6}$$

where f has degree 4, may be evaluated using derivative identities and integration by parts.

Furthermore, we enumerate *all* two-loop integrals of the form

$$\int\int \frac{d^4k d^4l \, f(p,k,l)}{k^4 l^4 (p-k)^4 (p-l)^4 (k-l)^4} \ , \tag{7}$$

where f has degree 12, and show that there are only 15 independent integrals not related to one another by symmetry. Of these 15, 14 may be evaluated using derivative identities and integration by parts (twice in succession), while the remaining one gives $\zeta(3)$. As it turns out, the integrand Rosner used to compute the third term in the series for $F_1(\alpha)$ lies in the space composed of the 14 integrals which may be evaluated by integration by parts. Thus, the number may be efficiently computed without any intermediate stages in which $\zeta(3)$ appears.

Our method for evaluating integrals is in principle trivial. First we enumerate the most general derivative identity like that in (3). (This is easy; it involves writing down the most general homogeneous rational fraction of the appropriate degree and then taking its derivative.) From this we write down the most general integral which can be evaluated by integration by parts. To evaluate any particular integral by our method we fit the particular integrand to the general integrand by solving an overdetermined linear system of simultaneous equations. (This fitting procedure is drastically simplified if we first convert the integrands to a symmetric form. This symmetrization is described in Secs. II and III.) If the system has a solution then the particular integration is done by parts, a δ function appears, and the number of integration loops is lowered by one. This process is repeated until the integral is evaluated.

Thus, whenever the integration gives a rational number, the problem of evaluating the integral is reduced to the problem of solving a system of simultaneous linear equations. Since very large linear systems of equations can be solved by computer, we feel that there is a good chance of evaluating the eighth-order contribution to $F_1(\alpha)$ if it is a rational number. Moreover, if all the coefficients in (1) are rational, the method of integration by parts may even provide an iterative procedure for relating the higher coefficients to the lower ones.

Even if the integral is not a rational number, the method we are proposing here may still be very useful as a kind of "Occam's razor." It may serve to simplify the integrand by cutting away everything which gives a rational number and thereby reducing the integrand to its simplest possible form.

The remainder of this paper is organized as follows. In Sec. II we consider the simple case of one-loop integrals. In Sec. III we extend this discussion to the case of two-loop integrals. Finally, in Sec. IV we give a quick rederivation of Rosner's calculation of the sixth-order contribution to $F_1(\alpha)$ using the new integration techniques we have proposed in this paper. We also compute the fourth-order correction to the vertex function and obtain

$$\Gamma^\mu(p,p) = \gamma^\mu \left[1 + \frac{3}{4}\left(\frac{\alpha}{2\pi}\right) - \frac{3}{8}\left(\frac{\alpha}{2\pi}\right)^2\right] .$$

Finally, we evaluate two other two-loop integrals that arise in the $F_1(\alpha)$ calculation.

II. ONE-LOOP INTEGRALS

In this section we consider the general problem of evaluating one-loop integrals of the form in (6). This case will enable us to explore in a simple context the symmetry structure of the integrand.

The function $f(p,k)$ may be expressed as a homogeneous polynomial B of degree 2 in the three variables p^2, k^2, and $(p-k)^2$:

$$f(p,k) = B(k^2, p^2, (p-k)^2) \ .$$

We now rewrite (6) as

$$I[B(k^2,p^2,(p-k)^2)] = \int \frac{d^4k \, B(k^2,p^2,(p-k)^2)}{k^4(p-k)^4} \ . \tag{8}$$

The form in (8) is appropriate for studying the symmetries of I.

A. Symmetries of the general one-loop integral

One symmetry of I becomes evident if we replace p by $-p$ (which does not affect the integral) and then shift k by p:

$$\begin{aligned} I[B(k^2,p^2,(p-k)^2)] &= \int \frac{d^4k \, B(k^2,p^2,(p+k)^2)}{k^4(p+k)^4} \\ &= \int \frac{d^4k \, B((k-p)^2,p^2,k^2)}{k^4(k-p)^4} \\ &= I[B((p-k)^2,p^2,k^2)] \ . \end{aligned} \tag{9}$$

To derive another symmetry we use

$$\int \frac{d\Omega_k}{2\pi^2} = 1$$

and

$$\frac{d^4k}{k^4} = \frac{dk}{|k|}\, d\Omega_k$$

to get

$$I = \int \frac{d\Omega_p}{2\pi^2} \int_0^\infty \frac{dk}{|k|} \int d\Omega_k \frac{B(k^2, p^2, (p-k)^2)}{(p-k)^4}$$

$$= \int_0^\infty \frac{dk}{|k|} \int\int \frac{d\Omega_k d\Omega_p}{2\pi^2} \frac{B(k^2, p^2, (p-k)^2)}{(p-k)^4}$$

$$\equiv \int_0^\infty \frac{dk}{|k|} g(|p|, |k|) \; .$$

But g is a dimensionless function of its arguments $|p|$ and $|k|$. Therefore

$$g(|p|, |k|) = g(|p|/|k|) \quad .$$

We can now replace the integration variable $|k|$ by $1/|k|$ and replace $|p|$ by $1/|p|$ without altering the value of the integral. We obtain

$$I = \int_0^\infty \frac{dk}{|k|} g(|k|, |p|) \; ,$$

where the arguments of g are *reversed*. Thus, another symmetry[3] of I is

$$I[B(k^2, p^2, (p-k)^2)] = I[B(p^2, k^2, (p-k)^2)] \; . \tag{10}$$

Combining the two symmetries (9) and (10) gives

$$I[B(p^2, k^2, (p-k)^2)] = I[B(\Pi(p^2, k^2, (p-k)^2))] \; , \tag{11}$$

where Π is an arbitrary permutation of three elements.

B. Evaluation of the general one-loop integral

To evaluate an integral of the form (8) we begin by symmetrizing the integrand with respect to the symmetry group. We do this by taking an average over all permutations:

$$I[B] = \tfrac{1}{6} I\left[\sum_{i=1}^{6} B(\Pi_i(p^2, k^2, (p-k)^2))\right] \; . \tag{12}$$

(Diagrammatically, this corresponds to averaging over all labelings of graphs which give the correct structure to the denominator.) For any polynomial B this gives

$$I[B] = \int \frac{d^4k}{k^4(p-k)^4} \{C_1[p^4 + k^4 + (p-k)^4] + C_2[p^2k^2 + p^2(p-k)^2 + k^2(p-k)^2]\} \; .$$

For $I[B]$ to exist, the integral in (8) must be infrared and ultraviolet convergent. This imposes the restriction that

$$C_2 = -2C_1 \; .$$

Hence, after symmetrization,

$$I[B] = C \int \frac{d^4k}{k^4(p-k)^4} \times \{p^4 + k^4 + (p-k)^4 - 2[p^2k^2 + p^2(p-k)^2 + k^2(p-k)^2]\} \; . \tag{13}$$

To evaluate the integral in (13) we compare it with the integral I in (2), whose value is π^2. Rewriting I in (2) in the form (8) gives

$$I = \int \frac{d^4k B}{k^4(p-k)^4} \quad ,$$

where $B = (p-k)^2[p^2 + k^2 - (p-k)^2]$.

Symmetrizing this integral, we obtain

$$\pi^2 = -\tfrac{1}{6} \int \frac{d^4k\{p^4 + k^4 + (p-k)^4 - 2[p^2k^2 + p^2(p-k)^2 + k^2(p-k)^2]\}}{k^4(p-k)^4} \quad .$$

Comparing this result with the formula in (12) gives

$$I[B] = -6C\pi^2 \; . \tag{14}$$

This simple formula is the result of evaluating the most general convergent one-loop integral.

C. Examples

To illustrate our method we evaluate two convergent one-loop integrals. The first integral,

$$I_1 = \int \frac{d^4k}{k^4(p-k)^4} [3(p^2 + k^2)p\cdot k - 4(p\cdot k)^2 - 2p^2k^2]$$

arises in the fourth-order $F_1(\alpha)$ calculation and appears explicitly in Ref. 2.

To evaluate I_1 we rewrite the integrand in terms of the variables k^2, p^2, and $(p-k)^2$:

$$B_1(k^2, p^2, (p-k)^2) = -(p-k)^4 + \tfrac{1}{2}k^4 + \tfrac{1}{2}p^4 + \tfrac{1}{2}(p-k)^2p^2 + \tfrac{1}{2}(p-k)^2k^2 - p^2k^2 \; .$$

Next, we symmetrize according to the formula in

(12). The symmetrized integrand is then 0. Thus,

$$I_1=0 \ .$$

The second integral we consider arises in the calculation of the vertex function in momentum space. It is

$$I_2=\int \frac{d^4k}{k^4(p-k)^4}\left[p^2k^2-(p\cdot k)^2\right] \ .$$

Rewriting the integrand in terms of k^2, p^2, and $(p-k)^2$ gives

$$B_2(k^2,p^2,(p-k)^2)=-\tfrac{1}{4}(p-k)^4-\tfrac{1}{4}p^4-\tfrac{1}{4}k^4+\tfrac{1}{2}p^2k^2+\tfrac{1}{2}(p-k)^2p^2+\tfrac{1}{2}(p-k)^2k^2 \ .$$

Note that this integrand is already in symmetrized form. Comparing with Eq. (13) gives $C=-\frac{1}{4}$. Thus, from Eq. (14), we have

$$I_2=-6C\pi^2=3\pi^2/2 \ .$$

In the Appendix we show how to evaluate a more complicated one-loop integral of the form

$$I=\int \frac{d^4k\, g(p,k)}{p^2k^6(p-k)^6} \ ,$$

where $g(p,k)$ has degree 10.

III. TWO-LOOP INTEGRALS

Two-loop integrals of the form in (7) may be written as

$$I(B)=\int\int \frac{d^4k}{k^4}\frac{d^4l}{l^4} \times \frac{B(l^2,(k-p)^2;k^2,(p-l)^2;p^2,(l-k)^2)}{(p-k)^4(p-l)^4(k-l)^4} \ , \tag{15}$$

where B is a homogeneous polynomial of degree 6.

A. Symmetries of the general two-loop integral

The symmetries of (15) are simple generalizations of the symmetries of the one-loop integral in (8). One symmetry, the generalization of that in (9), is obtained by replacing p by $-p$ and shifting k and l by p. The result is

$$I[B(l^2,(k-p)^2;k^2,(p-l)^2;p^2,(l-k)^2)] = I[B((p-l)^2,k^2;(k-p)^2,l^2;p^2,(l-k)^2)] \ . \tag{16}$$

The second symmetry is the generalization of that in (10). It is a consequence of the fact that the integral in (15) remains invariant if we integrate over the momenta p and k, or p and l, instead of k and l:

$$\int\int\frac{d^4k}{k^4}\frac{d^4l}{l^4}F(p,k,l)=\int\int\frac{d^4p}{p^4}\frac{d^4k}{k^4}F(p,k,l) =\int\int\frac{d^4p}{p^4}\frac{d^4l}{l^4}F(p,k,l) \ .$$

This equation is obtained by separating out the angular integrations, introducing a trivial integration over the angles of p, and performing inversions and scalings. This symmetry can be visualized in a simple way. If we consider B to be a function of ordered pairs of variables as in (15), then this symmetry implies that the integral is invariant under permutations of the three ordered pairs. The symmetry in (16) implies that the integral is invariant under the reversal of any two of the ordered pairs.

The order of the symmetry group $G=\{g_i\}$ for two-loop integrals is 24. Note that the elements of G send absolutely convergent integrals into absolutely convergent integrals. (An integral is absolutely convergent if it is both infrared and ultraviolet convergent, whether or not the angular integrations are performed symmetrically.) To totally symmetrize the integrand with respect to G we replace the polynomial B by

$$\mathrm{sym}[B]=\tfrac{1}{24}\sum_{i=1}^{24} g_i B \ .$$

B. Representation of a symmetrized polynomial

To represent the function B in (15) we have used the basis $p^2, k^2, l^2, (p-k)^2, (p-l)^2, (k-l)^2$ because any element of the symmetry group G maps monomials into monomials. The group of symmetries G therefore divides the set of monomials into equivalence classes of monomials. If two monomials, m_1 and m_2, are in the same equivalence class then $\mathrm{sym}[m_1]=\mathrm{sym}[m_2]$. Let C be a set of monomials formed by taking one monomial out of each equivalence class. All symmetrized polynomials can be formed uniquely by symmetrizing a linear combination of the monomials in C. For example, for one-loop integrals there are two equivalence classes: $\{p^4, k^4, (p-k)^4\}$ and $\{p^2k^2, p^2(p-k)^2, k^2(p-k)^2\}$. If we let $C=\{k^4, p^2k^2\}$ then all symmetrized polynomials have the form $\mathrm{sym}[c_1k^4+c_2p^2k^2]$.

The dimension of the space of symmetrized polynomials is just the number of equivalence classes. This number is precisely the number of linear equations which must be solved to determine the value of the integral. The most general polynomial B in (15) which has six arguments and has degree 6, is a sum of 462 monomials. The number of equivalence classes is only 32, which is a drastic reduction in the complexity in the system of linear equations. Furthermore, the dimension of the space of symmetric polynomials which give absolutely convergent integrals is 15. Finally, the dimension of the space of symmetric polynomials giving absolutely convergent integrals which are rational numbers is 14.

C. Derivative identities

Only one derivative identity is required to find all 14 convergent two-loop integrals. Although the identity may be expressed in terms of a 4-dimensional Laplacian, it is more convenient to use a divergence; to wit, we consider the most general vector function $f^\alpha(k,l,p)$ and use the divergence theorem to evaluate $\int d^4k\, \partial_\alpha f^\alpha$. If we explicitly calculate $\partial_\alpha f^\alpha$ and symmetrize the numerator of the resulting integrand, we can then evaluate any convergent integral by comparing the symmetrized integrands. This comparison consists of solving a system of 32 simultaneous linear equations.

In the next section, we illustrate this procedure by presenting some intermediate results in the calculation of the third coefficient of the expansion of $F_1(\alpha)$ in (1).

IV. CALCULATION OF $F_1(\alpha)$

In this section we outline the calculation of $F_1(\alpha)$ to third order in α.

The Feynman rules in Euclidean momentum space are the following:

1. For each integration the differential element is $i\,d^4k(2\pi)^{-4}$. We integrate over all except one loop.
2. For each vertex insert γ^α.
3. For each pair of vertices insert $ie_0^{\,2}$.
4. For each electron propagator insert $1/i\not{p}$.
5. For each differentiated electron line, indicated on Fig. 1 by a tick mark, insert γ^α (see Ref. 4).
6. Insert one minus sign if the tick marks are on different sides of the electron loop.
7. For each photon propagator insert $(p^2 g^{\mu\nu} - \lambda p^\mu p^\nu)/p^4$, where $\lambda = 1 + O(\alpha)$ is chosen to make the vertex function $\Gamma^\mu(p,q)$ finite; when $p=q$, $\Gamma^\mu(p,p)$ takes the form (see Refs. 2 and 4)

$$\Gamma^\mu(p,p) = \gamma^\mu \sum_{n=0}^{\infty} B^{(n)} .$$

8. We define $\alpha = e_0^{\,2}/4\pi$.

After simplifying using Ward's identity, the contribution to $F_1(\alpha)$ to sixth order may be represented diagrammatically as in Fig. 1. The diagrams have been grouped together so that each pair of parentheses represents an absolutely convergent integral.

Upon taking traces we find that the integral which represents the sixth-order contribution to $F_1(\alpha)$ is

$$\left(\frac{\alpha}{2\pi}\right)^3 \int\int \frac{d^4k\,d^4l\,f(p,k,l)}{48\pi^4 k^4 l^4 (p-k)^4 (p-l)^4 (k-l)^4} ,$$

where

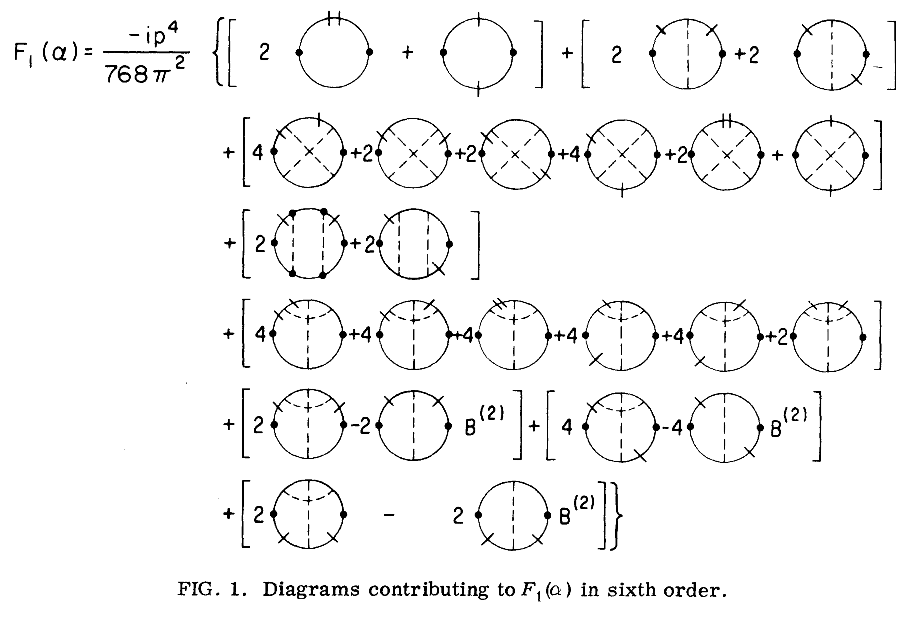

FIG. 1. Diagrams contributing to $F_1(\alpha)$ in sixth order.

$$
\begin{aligned}
f(p,k,l) = {} & e^2b^4 + \{10e^3 + (-2f+10c-10a)e^2 + [-2f^2 + (2d-4c+3af-cd+2c^2-2ac]\,e\}\,b^3 \\
& +(5e^4 + (-10f-4d-10c-14a)e^3 + [-3f^2 + (17d-38c+27a)f + 3d^2 + (9c+2a)d - 15c^2 + 3ac + 9a^2]\,e^2 \\
& \quad + \{4f^3 + (-6d+12c-3a)f^2 + [2d^2 + (3a-25c)d - 12c^2 + 31ac - 15a^2]\,f \\
& \quad\quad + 9cd^2 + (5c^2-3ac)d - 4c^3 + 2ac^2 + 2a^2c\}\,e + (4c-2a)f^3 + [(4a-8c)d + 8c^2 - 10ac + 2a^2]\,f^2 \\
& \quad\quad + [(4c-2a)d^2 + (-6c^2+9ac-3a^2)d - 4c^3 + 8ac^2 - 4a^2c]\,f + (3a^2-3ac)d^2 + (2c^3 - 4ac^2 + 2a^2c)d)b^2 \\
& +((-8f+4d-12c)e^4 + [-2f^2 + (19d+12c+26a)f - 4d^2 + (6c-4a)d + 6c^2 + 15ac]\,e^3 \\
& \quad + \{4f^3 + (-19d+24c-2a)f^2 + [-21d^2 + (-45c-27a)d + 20c^2 - 17ac - 26a^2]\,f \\
& \quad\quad + 26cd^2 + (-6c^2-9ac)d + 4c^3 + 13ac^2 - 15a^2c\}\,e^2 \\
& \quad + \{-2f^4 + (2d-2a)f^3 + [4d^2 + (18c+8a)d + 40c^2 - 42ac + 4a^2]\,f^2 \\
& \quad\quad + [-4d^3 + (33c-7a)d^2 + (17a^2-2c^2)d + 24c^3 - 70ac^2 + 33a^2c + 8a^3]\,f \\
& \quad\quad + (-20c-4a)d^3 + (-20c^2-5ac)d^2 + (-2c^3 - 8ac^2 + 9a^2c)d + 2c^4 + 2ac^3 - 4a^2c^2\}\,e + (2a-8c)f^4 \\
& \quad + [(24c-6a)d - 24c^2 + 20ac - 2a^2]\,f^3 + [(6a-24c)d^2 + (46c^2 - 70ac + 19a^2)d - 8c^3 + 18ac^2 - 10a^2c]\,f^2 \\
& \quad + [(8c-2a)d^3 + (-48c^2 + 79ac - 17a^2)d^2 + (4c^3 - 46ac^2 + 55a^2c - 13a^3)d + 8c^4 - 16ac^3 + 8a^2c^2]\,f \\
& \quad + (20c^2 - 17ac - 6a^2)d^3 + (-2c^3 + 15ac^2 - 15a^2c + 2a^3)d^2 + (-2c^4 + 2ac^3 + 2a^2c^2 - 2a^3c)d)b + (8c-8d)fe^4 \\
& + [(8d-8c)f^2 + (4d^2 + 16ad - 4c^2 - 16ac)f - 4ad^2 + 8acd - 4ac^2]\,e^3 \\
& + \{(12c-2d)f^3 + [22d^2 + (-14c-12a)d + 16c^2 - 8ac]\,f^2 \\
& \quad + [-19ad^2 + (-8c^2 + 21ac - 10a^2)d + 8c^3 - 18ac^2 + 20a^2c]\,f \\
& \quad + 4ad^3 + (a^2 - 14ac)d^2 + (16ac^2 - 15a^2c)d - 6ac^3 + 6a^2c^2\}\,e^2 \\
& + \{(2d-8c)f^4 + [-6d^2 + (20c-2a)d - 40c^2 + 8ac]\,f^3 \\
& \quad + [4d^3 + (3a-58c)d^2 + 50c^2 - 26ac + 6a^2)d - 40c^3 + 62ac^2]\,f^2 \\
& \quad + [(24c+a)d^3 + (-32c^2 + 49ac - 9a^2)d^2 + (16c^3 - 50ac^2 + 7a^2c + 2a^3)d - 8c^4 + 36ac^3 - 18a^2c^2 - 8a^3c]\,f \\
& \quad + 2acd^3 + (-6ac^2 + 11a^2c + 8a^3)d^2 + (6ac^3 - 5a^2c^2 - a^3c)d - 2ac^4 + 2a^2c^3\}\,e + 4cf^5 \\
& + [(-16c-2a)d + 16c^2 - 8ac]\,f^4 + [(20c+8a)d^2 + (-48c^2 + 48ac + 4a^2)d + 16c^3 - 24ac^2 + 4a^2c]\,f^3 \\
& + [(-8c-10a)d^3 + (60c^2 - 42ac - 29a^2)d^2 + (-32c^3 + 104ac^2 - 44a^2c - 2a^3)d - 8ac^3 + 8a^2c^2]\,f^2 \\
& + [4ad^4 + (-24c^2 - 10ac + 26a^2)d^3 + (28c^3 - 62ac^2 - 20a^2c + 26a^3)d^2 + (40ac^3 - 52a^2c^2 + 12a^3c)d - 4c^5 \\
& \quad + 8ac^4 - 4a^2c^3]\,f + (4ac+3a^2)d^4 + (12a^2c - 6ac^2)d^3 + (-11a^2c^2 + 16a^3c - 5a^4)d^2 + (2ac^4 - 4a^2c^3 + 2a^3c^2)d\,.
\end{aligned}
\tag{17}
$$

In (17) we have used the notation

$$a \equiv k^2,\quad b \equiv p^2,\quad c \equiv l^2,\quad d \equiv (p-l)^2,\quad e \equiv (k-l)^2,\quad f \equiv (k-p)^2\ .$$

After symmetrization, the function $f(p,k,l)$ in (17) simplifies to

$$
\begin{aligned}
\operatorname{sym}[f(p,k,l)] = 2\operatorname{sym}[& 10e^2b^4 - 20aeb^4 - 4afb^4 + 4adb^4 + 13e^3b^3 - 90ae^2b^3 + 43afeb^3 + 57adeb^3 + 32aceb^3 \\
& + 28a^2eb^3 + 8a^2fb^3 + 8acdb^3 - 16a^2db^3 + 44ade^2b^2 + 101ace^2b^2 + 25a^2e^2b^2 - 118a^2feb^2 \\
& - 88acdeb^2 - 78a^2deb^2 - 44a^2ceb^2 + 15a^2f^2b^2 + 48acdfb^2 + 4a^2cfb^2]\ .
\end{aligned}
\tag{18}
$$

Matching the symmetrized version of f in (18) to the symmetrized integrand obtained from a derivative identity allows us to perform the one-loop integration. The resulting one-loop integral is

$$-\left(\frac{\alpha}{2\pi}\right)^3 \int \frac{d^4k\, g(p,k)}{24\pi^2 k^6 p^2 (p-k)^4}\ , \tag{19}$$

where

$$
\begin{aligned}
g(p,k) = {} & (197k^2p\cdot k - 90k^4)p^4 \\
& + [112(p\cdot k)^3 - 528k^2(p\cdot k)^2 + 197k^4p\cdot k]\,p^2 \\
& + 112k^2(p\circ k)^3\ .
\end{aligned}
$$

To evaluate the integral in (19) we symmetrize its integrand and match it to the integrand in (13). The final result is

$$-\frac{1}{4}\left(\frac{\alpha}{2\pi}\right)^3 , \tag{20}$$

which agrees with Rosner's calculation in Refs. 1 and 2.

It is worthwhile noting that the diagrams in Fig. 1 represent the divergent part of the (gauge-invariant) vacuum polarization tensor $\Pi_{\mu\nu}$ only in the finite gauge; that is, only when λ in the photon propagator is chosen to be

$$\lambda = 1 - \frac{3}{4}\left(\frac{\alpha}{2\pi}\right) + O(\alpha^2) . \tag{21}$$

For other choices of λ, the diagrams in Fig. 1 represent the divergent part of a gauge-dependent quantity, which Rosner refers to as $\Pi_{\mu\nu}^{(p)}$ [see Eq. (24) in Ref. 2]. As a further test of the computational techniques proposed here we have calculated the sixth-order contribution to the divergent part of $\Pi_{\mu\nu}^{(p)}$ for arbitrary λ. The result is a quadratic polynomial in λ whose three coefficients are all two-loop integrals.[5] All three integrals give rational numbers:

$$\text{sixth-order contribution to coefficient of logarithmic divergence of } \Pi_{\mu\nu}^{(p)} = (\tfrac{1}{4}\lambda^2 - \tfrac{3}{2}\lambda + 1)\left(\frac{\alpha}{2\pi}\right)^3 . \tag{22}$$

Note that if we take $\lambda = 1$ (finite gauge) then (22) reduces to (20).

The result in (22) is useful because part of the contribution to $F_1(\alpha)$ in eighth order comes from evaluating the diagrams in Fig. 1, with λ given in (21) and identifying the terms proportional to $(\alpha/2\pi)^4$. If we do this, we obtain

$$\frac{3}{4}\left(\frac{\alpha}{2\pi}\right)^4 .$$

As a final test of our calculational procedures we have evaluated the two-loop integral which gives the fourth-order correction to the vertex function $\Gamma^\mu(p,p)$. We find that it too is a rational number[6]:

$$\Gamma^\mu(p,p) = \gamma^\mu\left[1 + \frac{3}{4}\left(\frac{\alpha}{2\pi}\right) - \frac{3}{8}\left(\frac{\alpha}{2\pi}\right)^2 + \cdots\right] . \tag{23}$$

In general, we have found that the trace calculations required to produce a two-loop integrand like that in (17) requires about one minute of computer time. Symmetrizing the integrand to give a formula like that in (18) also requires about one minute. Solving the system of simultaneous equations and performing the first loop integration takes an additional two seconds of computer time. The second loop integration takes less than one second.

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APPENDIX

To further illustrate our integration technique, we show how to evaluate one-loop integrals of the form

$$I = \int \frac{d^4k\, g(p,k)}{p^2k^6(p-k)^6} , \tag{A1}$$

where $g(p,k)$ has degree 10.

After symmetrization g will have the form

$$\text{sym}[g] = \text{sym}[C_1k^{10} + C_2k^8p^2 + C_3k^6p^4 + C_4k^6p^2(p-k)^2 + C_5k^4p^4(p-k)^2] ,$$

where sym represents averaging over all six permutations of $k^2, p^2, (p-k)^2$.

Imposing the requirement that the integral (A1) be absolutely convergent reduces the number of constants to two:

$$\text{sym}[g] = \text{sym}[C_1[k^{10} - 6k^8p^2 + 4k^6p^4 + 6k^4p^4(p-k)^2] + C_2[k^6p^2(p-k)^2 - 2k^4p^4(p-k)^2]] . \tag{A2}$$

When C_1 is zero, one can simplify the integral by dividing both numerator and denominator by $p^2k^2(p-k)^2$; the resulting integral,

$$I = \int \frac{d^4k\, C_2\, \text{sym}[k^4 - 2p^2k^2]}{k^4(p-k)^4} = -2\pi^2 C_2 , \tag{A3}$$

is evaluated using (14).

When C_1 and C_2 are arbitrary we need a new derivative identity:

$$\int \frac{d^4k}{p^2k^2} \Box_k \frac{(p\cdot k - p^2)^3}{(p-k)^4} = 4\pi^2 . \tag{A4}$$

The integral is easily evaluated using integration by parts.

Performing the indicated differentiation in (A4) and symmetrizing the integrand gives

$$4\pi^2 = \int \frac{d^4k \,\mathrm{sym}\left[-4k^{10} + 24k^8p^2 - 16k^6p^4 - 21k^6p^2(p-k)^2 + 18k^4p^4(p-k)^2\right]}{2p^2k^6(p-k)^6} \ .$$

A glance at (A2) shows that for this integrand $C_1 = -2$ and $C_2 = -21/2$. Thus, recalling (A3), we have

$$\int \frac{d^4k g(p,k)}{p^2k^6(p-k)^6} = (\tfrac{17}{2}C_1 - 2C_2)\pi^2 \ . \tag{A5}$$

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[1]J. Rosner, Phys. Rev. Lett. 17, 1190 (1966).

[2]J. Rosner, Ann. Phys. (N.Y.) 44, 11 (1967).

[3]This symmetry is discussed by Rosner in Ref. 2.

[4]See K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967).

[5]It is an interesting accident that the fourth-order contribution to the divergence of $\Pi_{\mu\nu}^{(p)}$ is independent of the gauge constant λ. See Eq. (56) in Ref. 2.

[6]The general form of $\Gamma^\mu(p,p)$ in (23) is predicted in Ref. 4.