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Asymptotic freedom and discontinuities in field theory models for $g \to 0$

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We prove that if perturbation theory is reliable and if the propagator of the field satisfies the Källén-Lehmann representation then asymptotic freedom is equivalent to having $Z_3(g \to 0) = 1$ in the $g\phi^4$ model. One may have $Z_3(g) \neq 0$ and nonasymptotic freedom, but then $Z_3(g \to 0) < 1$. More generally nonasymptotic freedom implies for any model that $Z_3(g \to 0) < 1$. The case $\beta(g) \equiv 0$ we have proposed before and in which perturbation theory breaks down is also examined and is shown to allow $Z_3(g \to 0) = 1$ when $Z_3(g) \not\equiv 0$ and to give an anomalous-dimension function of the coupling constant when $Z_3(g) \not\equiv 0$.

I. INTRODUCTION

It has been claimed[1] that if the wave-function renormalization constant $Z_3(g)$ of a field is a finite nonidentically vanishing function of the coupling constant then the field is necessarily asymptotically free. Since it is known that ordinary nongauge theories are not asymptotically free, the previous result would imply that for these models $Z_3(g) \equiv 0$. We shall discuss this point in detail here. In fact, we shall prove that, in the $g\phi^4$ model in four space-time dimensions, if (1) one has the Källén-Lehmann representation with positive weight function for the propagator, (2) g invertibility is satisfied and consequently one has renormalization-group (RG) representations, and (3) perturbation theory is reliable, then the necessary and sufficient condition for asymptotic freedom is $Z_3(g \to 0) = 1$. If $Z_3(g \to 0) < 1$ then the model is not asymptotically free and one has two possibilities: (a) $Z_3(g) \neq 0$ is a finite function of g or (b) $Z_3(g) \equiv 0$ for all g. These results are general for any model in which $\beta(g)$ starts at second order. For models in which $\beta(g)$ starts at third order one has $Z_3(g \to 0) = 0$ in all cases, and one can again have nonasymptotic freedom with $Z_3(g) \neq 0$.

In Sec. II we treat the $m = 0$ case and in Sec. III the massive case. In the $m \neq 0$ case we consider an explicit model of a situation in which perturbation theory is not reliable, and which can lead to $Z_3(g \to 0) = 1$ without asymptotic freedom when $Z_3(g) \neq 0$ and to an anomalous dimension depending on the coupling constant when $Z_3(g) \equiv 0$.

II. $m = 0$ CASE

A. Some useful formulas

Let us recall some useful formulas. We shall follow Ref. 2. If the postulate of g invertibility is satisfied we can write the RG equations. Let $\Gamma^{(2n)}(p_i; \theta, g_\theta)$ be the θ-normalized vertex functions of the model. We define the functions $z_i^0(p^2/\theta^2, g_\theta)$ in the following way:

$$\Gamma^{(2)}(p, -p; \theta, g_\theta) = p^2 z_3^0\left(\frac{p^2}{\theta^2}, g_\theta\right),$$
$$\Gamma^{(4)}(p_i; \theta, g_\theta)\big|_{\text{sym. point } p^2} = g_\theta z_1^0\left(\frac{p^2}{\theta^2}, g_\theta\right). \tag{1}$$

The normalization conditions are

$$z_i^0\left(\frac{p^2}{\theta^2}, g_\theta\right)\bigg|_{p^2=\theta^2} = 1, \quad i = 1, 3, \tag{2}$$

where $\theta^2 < 0$ is the spacelike normalization point. We also introduce the function z_5^0 by

$$z_5^0\left(\frac{p^2}{\theta^2}, g_\theta\right) = \frac{z_3^0(p^2/\theta^2, g_\theta)^2}{z_1^0(p^2/\theta^2, g_\theta)} \tag{3}$$

and the RG invariant $v^0(p^2/\theta^2, g_\theta)$ corresponding to the bound group $G(Z_1 Z_3{}^2)$, defined by

$$v^0\left(\frac{p^2}{\theta^2}, g_\theta\right) = \frac{g_\theta}{z_5^0(p^2/\theta^2, g_\theta)}. \tag{4}$$

The RG equations for these functions, which result from the change of variable

$$g_{\theta'} = \frac{g_\theta}{z_5^0(\theta'^2/\theta^2, g_\theta)} = v^0\left(\frac{\theta'^2}{\theta^2}, g_\theta\right), \tag{5}$$

are[2]

$$\left[\theta\frac{\partial}{\partial\theta} + \beta^0(g_\theta)\frac{\partial}{\partial g_\theta} - \gamma_i^0(g_\theta)\right] z_i^0\left(\frac{p^2}{\theta^2}, g_\theta\right) = 0, \quad i = 1, 3, 5, \tag{6}$$

where

$$\beta^0(g_\theta) = \theta\frac{\partial g_\theta}{\partial\theta}\bigg|_{g_{\theta'}} = \theta\frac{\partial}{\partial\theta}\frac{g_{\theta'}}{z_5^0(\theta^2/\theta'^2, g_{\theta'})}\bigg|_{g_{\theta'}},$$

$$\gamma_i^0(g_\theta) = -\frac{1}{z_i^0(\theta^2/\theta'^2, g_{\theta'})}\,\theta\frac{\partial}{\partial\theta}z_i^0\left(\frac{\theta^2}{\theta'^2}, g_{\theta'}\right)\Big|_{g_{\theta'}}, \tag{7}$$

with the derivatives $\theta(\partial/\partial\theta)$ taken keeping $g_{\theta'}$ fixed. We have the relation $\beta^0(g_\theta) = g_\theta\gamma_5^0(g_\theta) = g_\theta[2\gamma_3^0(g_\theta) - \gamma_1^0(g_\theta)]$, and a perturbation-theory calculation gives

$$\beta^0(g_\theta) = b_0 g_\theta^2 + \cdots, \quad b_0 > 0$$
$$\gamma_3^0(g) = c_0 g_\theta^2 + \cdots, \quad c_0 > 0. \tag{8}$$

The fact that $c_0 > 0$ is a consequence of unitarity, as we shall see, and is model independent.

The group invariant $v^0(x,g)$ obeys the differential equation

$$x\frac{\partial v^0(x,g_\theta)}{\partial x} = \tfrac{1}{2}\beta^0(v^0(x,g_\theta)), \tag{9}$$

which can be integrated using the initial condition $v^0(1,g_\theta) = g_\theta$ to obtain the usual Gell-Mann–Low (GML) equation

$$\ln x = 2\int_{g_\theta}^{v^0(x,g_\theta)} \frac{dv}{\beta^0(v)}. \tag{10}$$

On the other hand, the functions $z_i^0(x,g)$ obey the differential equations

$$x\frac{\partial z_i^0(x,g_\theta)}{\partial x} = -\tfrac{1}{2}z_i^0(x,g_\theta)\gamma_i^0(v^0(x,g_\theta)). \tag{11}$$

We remark that all the equations we have written are consequences of the identity

$$z_i^0\left(\frac{p^2}{\theta'^2}, g_{\theta'}\right) = \frac{z_i^0(p^2/\theta^2, g_\theta)}{z_i^0(\theta'^2/\theta^2, g_\theta)}\Bigg|_{g_\theta \to g_{\theta'} = \frac{g_\theta}{z_5^0(\theta'^2/\theta^2, g_\theta)}}, \tag{12}$$

which is implied by the invertibility of the change of parameter $g_\theta \to g_{\theta'}$.

Let us call g_∞ the asymptotic value of the group invariant (invariant charge), i.e.,

$$v^0(x,g_\theta) \xrightarrow[x\to\infty]{} g_\infty. \tag{13}$$

By definition one says a model is asymptotically free if $g_\infty = 0$.

One has [see (8)] that $\beta^0(0) = \gamma_3^0(0) = 0$. The function $\beta^0(v)$ is positive near the origin, and for $v > 0$ either it has a zero at a finite value $\bar{g}_\infty$, $\beta^0(\bar{g}_\infty) = 0$, or it does not. In the first case one has from (10) that for $g_\theta > 0$ and small (precisely, $g_\theta < \bar{g}_\infty$) $v^0(x,g_\theta) \xrightarrow[x\to\infty]{} g_\infty = \bar{g}_\infty$, and in the second case one has $v^0(x,g_\theta) \xrightarrow[x\to\infty]{} g_\infty = \infty$. For $g_\theta < 0$ (Ref. 3) and small, Eq. (10) implies $v^0(x,g_\theta) \xrightarrow[x\to\infty]{} g_\infty = 0$, i.e., asymptotic freedom.

Consider now Eq. (11) for $i = 3$. The 2-point vertex function $\Gamma^{(2)} = \Delta^{-1}$, where Δ is the propagator of the field, is related to $z_3^0(x,g_\theta)$ by (1). Because Δ satisfies the Källén-Lehmann spectral representation resulting from the postulates of causality and unitarity, the function $z_3^0(x,g_\theta)$, which is normalized to 1 for $x = 1$, is decreasing in the interval $1 \le x < \infty$, and positive. We have then $0 < z_3^0(x,g_\theta) \le 1$ and $\partial z_3^0(x,g_\theta)/\partial x < 0$ for finite x, so that we conclude from (11) that

$$\gamma_3^0(g_\theta) > 0, \quad g_\theta < g_\infty, \quad g_\theta \neq 0. \tag{14}$$

For $g_\theta = g_\infty$ the function $\gamma_3^0(g_\theta)$ may vanish.

We define now the wave-function renormalization constant $Z_3^0(g_\theta)$ by

$$Z_3^0(g_\theta) = \lim_{x\to\infty} z_3^0(x,g_\theta), \quad g_\theta \neq 0, \tag{15}$$

and $Z_3^0(0)$ as the limit of $Z_3^0(g_\theta)$ when $g_\theta \to 0$, i.e.,

$$Z_3^0(0) = \lim_{g_\theta\to 0} Z_3^0(g_\theta). \tag{16}$$

From the properties of $z_3^0(x,g_\theta)$ implied by the Källén-Lehmann representation we must have $0 \le Z_3^0(g_\theta) \le 1$.

Let us integrate (11) for $i = 3$, using the initial condition $z_3^0(1,g_\theta) = 1$; one obtains

$$z_3^0(x,g_\theta) = \exp\left[-\tfrac{1}{2}\int_1^x \frac{dt}{t}\gamma_3^0(v^0(t,g_\theta))\right]. \tag{17}$$

B. Connection between the function $Z_3^0(g)$ and asymptotic freedom

Let us study now the connection between the function $Z_3^0(g)$ and asymptotic freedom (AF). In (17) we make the change of variable $t \to v = v^0(t,g)$, invertible because of (9), and we obtain

$$z_3^0(x,g_\theta) = \exp\left[-\int_{g_\theta}^{v^0(x,g_\theta)} dv\,\frac{\gamma_3^0(v)}{\beta^0(v)}\right]. \tag{18}$$

Taking the limit $x \to \infty$ in (18) and using (13) and (14) one has

$$Z_3^0(g_\theta) = \exp\left[-\int_{g_\theta}^{g_\infty} dv\,\frac{\gamma_3^0(v)}{\beta^0(v)}\right]. \tag{19}$$

Consider now the case $g_\theta < 0$. Then one has $g_\infty = 0$, i.e., asymptotic freedom, and $Z_3^0(g_\theta) \neq 0$. Moreover (19) shows that $Z_3^0(0) = 1$.

Let us study now the case $g_\theta > 0$. Then $g_\infty \neq 0$ and we are in the situation of nonasymptotic freedom. Two cases are possible: (a) $\gamma_3^0(g_\infty) \neq 0$, in which case $Z_3^0(g_\theta) \equiv 0$, (b) $\gamma_3^0(g_\theta) = 0$ in such a way that the integral in (17) converges [this means that $\gamma_3^0(v^0(t,g_\theta)) < O((\ln t)^{-2}) \to 0$ for $t \to \infty$], in which case $Z_3^0(g_\theta) \neq 0$ will be a finite function of g_θ. But in case (b) we shall *not* have $Z_3^0(g_\theta \to 0) = 1$, as one is tempted to believe, since $\int_0^{g_\theta} dv\,\gamma_3^0(v)/\beta^0(v)$ is strictly positive and cannot vanish. We have rather

$$Z_3^0(0) = \exp\left[-\int_0^{g_\infty} dv\,\frac{\gamma_3^0(v)}{\beta^0(v)}\right] < 1. \tag{20}$$

This result just comes from the fact that in Eq. (17), which implies Eq. (19), the limits $x\to\infty$ and $g_\theta\to 0$ cannot be interchanged. This is clear since the integrand in (17) depends on $v^0(t,g_\theta)$, for which

$$0=\lim_{t\to\infty}\lim_{g_\theta\to 0} v^0(t,g)\neq \lim_{g_\theta\to 0}\lim_{t\to\infty} v^0(t,g_\theta)=g_\infty, \qquad (21)$$

where g_∞ is independent of g_θ. An alternative definition of Z_3^0 at $g_\theta=0$ by

$$\hat{Z}_3^0(0)=\lim_{x\to\infty}\lim_{g_\theta\to 0} z_3^0(x,g_\theta)$$

would give $\hat{Z}_3^0(0)=1$, as can be seen from (17), since $v^0(t,g_\theta)|_{g_\theta\to 0}=0$ and $\gamma_3^0(0)=0$. But then the function $Z_3^0(g_\theta)$ would be discontinuous at $g_\theta=0$.

Let us remark that $Z_3^0(0)<1$ is consistent with the Källén-Lehmann representation $Z_3^{-1}=1+\int_0^\infty \rho(s)ds$. Take as an example $\rho(s)=\alpha g_\theta^2 m^2/(g_\theta^2 s+m^2)^2$, then $\rho\to 0$ when $g_\theta\to 0$ at fixed s, but $Z_3^{-1}\to 1+\alpha$.

We see then that in the case of nonasymptotic freedom $Z_3^0(0)<1$. This result is in agreement with the remarks of Symanzik in Ref. 3, stating that if a solution exists for the model with $g_\theta<0$ (it can be argued on physical grounds that the solution should not exist[4]) it should be considered as a different mode and not as some kind of continuation of the solution for $g_\theta>0$. These discontinuities are still more striking if one looks at the vertex renormalization constant $Z_1^0(g_\theta)=z_1^0(x,g_\theta)|_{x\to\infty}$. For $g_\theta>0$, finite g_∞, and $Z_3^0(g_\theta)\neq 0$, one easily sees that $Z_1^0(g_\theta)\to\infty$ as g_θ^{-1} for $g_\theta\to 0$, while for $g_\theta<0$ one has $Z_1^0(g_\theta)\equiv 0$.

Let us comment now on the discussion of Ref. 1. The basic equation there is obtained from (6) for $i=3$ by taking the limit $x=p^2/\theta^2\to\infty$. One obtains, using (15) [and using $\theta(\partial/\partial\theta)z_3^0(p^2/\theta^2,g_\theta)\to 0$ because of the Källén-Lehmann representation],

$$\gamma_3^0(g_\theta)Z_3^0(g_\theta)-\beta^0(g_\theta)\frac{\partial Z_3^0}{\partial g_\theta}=0 \qquad (22)$$

if $Z_3^0(g_\theta)\neq 0$ [and just $0=0$ if $Z_3^0(g_\theta)\equiv 0$]. Integrating (22) one has

$$Z_3^0(g_\theta)=Z_3^0(0)\exp\left[\int_0^{g_\theta} dv\,\frac{\gamma_3^0(v)}{\beta^0(v)}\right], \qquad (23)$$

which is just Eq. (19), but there the initial condition is explicitly exhibited. The argument in Ref. 1 can be stated roughly in the following way: Assume $Z_3^0(0)=1$ in (23), then for $g_\theta>0$, as one knows that $\gamma_3^0(v)>0$ and $Z_3^0(g_\theta)<1$, one concludes that $\beta^0(v)$ must be negative near $v=0$, i.e., asymptotic freedom. This proof is certainly correct; what is misleading is to relate $Z_3^0(g_\theta)\neq 0$ to the property of AF, since Eq. (23) is consistent with nonasymptotic freedom.

The discussion we have done applies without changes to any one-charge model in which the series for $\beta^0(g_\theta)$ starts at second order. This is because the basic equation we use is (19), which is model independent, but we need there $\gamma_3^0(v)/\beta^0(v)$ to be integrable at the origin $v=0$, and this is the case if

$$\beta^0(g_\theta)=b_0 g_\theta^2+O(g_\theta^3),\ \gamma_3^0(g_\theta)=c_0 g_\theta^2+O(g_\theta^3).$$

The other one-charge models are such that $\beta^0(g_\theta)$ starts at third order (for instance $g_\theta\bar\psi\gamma_5\psi\phi$, $e_\theta\bar\psi\gamma_\mu\psi A^\mu$, etc.), and $\gamma_3^0(g_\theta)$ at second order. Put $\beta^0(v)=bv^3+O(v^4)$, $\gamma_3^0(v)=cv^2+O(v^3)$, with $c>0$ because of the Källén-Lehmann representation. The integrand of Eq. (19) is now singular near $v=0$, where it behaves as

$$\frac{\gamma_3^0(v)}{\beta^0(v)}=\frac{c}{bv}[1+O(v)]. \qquad (24)$$

For $b>0$ one has nonasymptotic freedom, and for $b<0$ one has AF. We consider now AF ($b>0$). One has from (19) that $Z_3^0(g_\theta)\equiv 0$ if $\gamma_3^0(g_\infty)\neq 0$ and $Z_3^0(g_\theta)\neq 0$ if $\gamma_3^0(g_\infty)=0$. In this second case we obtain from (22) for small g_θ, when integrating between ϵ and g_θ,

$$Z_3^0(g_\theta)=Z_3^0(\epsilon)\left(\frac{g_\theta^2}{\epsilon^2}\right)^{c/2b}, \qquad (25)$$

and we see that $Z_3^0(g_\theta\to 0)=0$. As is well known, this is the situation in finite QED in the GML case [$\exists e_\infty<\infty$ such that $\beta^0(e_\infty)=0$], since there one has $\gamma_3^0(e_\theta)/\beta^0(e_\theta)=2/e_\theta$ so that $\gamma_3^0(e_\infty)=0$, and from (22) one obtains

$$Z_3^0(e_\theta)=\frac{e_\theta^2}{e_\infty^2}.$$

We consider now AF ($b<0$). One has from (9) that $v^0(x,g_\theta)\xrightarrow[x\to\infty]{} g_\infty=0$, more precisely, $v^0(x,g_\theta)^2$ vanishes as $-(b\ln x)^{-1}$. Let us calculate now $Z_3^0(g_\theta)$ from (17), taking there the limit $x\to\infty$. We replace in the integrand $\gamma_3^0(v)=cv^2$, which is certainly allowed for small g_θ since $v^0(1,g_\theta)=g_\theta$ and $v^0(x,g_\theta)\to 0$ when x increases. Using $v^0(x,g_\theta)^2=O(-(b\ln x)^{-1})$ we obtain from (17) that $Z_3^0(g_\theta)=z_3^0(x,g_\theta)|_{x\to\infty}$ vanishes as $(\ln x)^{c/2b}$. We see then that when $\beta^0(g_\theta)$ starts at third order one can have $Z_3^0(g_\theta)\neq 0$ without AF, and that in this case $Z_3^0(g_\theta\to 0)=0$. Finally, we conclude then that nonasymptotic freedom is consistent with $Z_3^0(g_\theta)\neq 0$, and that nonasymptotic freedom implies always $Z_3^0(g_\theta\to 0)<1$.

C. Asymptotic behavior of general vertex functions $\Gamma^{(2n)}(p_i;\theta,g_0)$

Let us look now briefly at the asymptotic behavior of general vertex functions $\Gamma^{(2n)}(p_i;\theta,g_\theta)$. From the identity

$$\Gamma^{(2n)}(p_i;\theta,g_\theta)=z_3^0\left(\frac{\theta'^2}{\theta^2},g_\theta\right)^n\Gamma^{(2n)}(p_i;\theta',g_{\theta'}),\tag{26}$$

with $g_{\theta'}$ given by (5), one obtains (putting $\lambda^2=\theta'^2/\theta^2$)

$$\Gamma^{(2n)}(\lambda p_i;\theta,g_\theta)=\lambda^{4-2n}z_3^0(\lambda^2,g_\theta)^n\Gamma^{(2n)}(p_i;\theta,v^0(\lambda^2,g_\theta)),\tag{27}$$

and using (17) for $z_3^0(\lambda^2,g_\theta)$ one has

$$\Gamma^{(2n)}(\lambda p_i;\theta,g_\theta)=\lambda^{4-2n}\exp\left[-\frac{n}{2}\int_1^{\lambda^2}\frac{dt}{t}\gamma_3^0(v^0(t,g_\theta))\right]\times\Gamma^{(2n)}(p_i;\theta,v^0(\lambda^2,g)).\tag{28}$$

The dominant term for $\lambda^2\to\infty$ will be $[v^0(t,g_\theta)\xrightarrow[t\to\infty]{} g_\infty]$

$$\lambda^{4-2n[1+\frac{1}{2}\gamma_3^0(g_\infty)]}\Gamma^{(2n)}(p_i;\theta,g_\infty).\tag{29}$$

Consider nonasymptotic freedom. In the case $Z_3^0(g_\theta)\equiv 0$ one has the anomalous dimension $\gamma_3^0(g_\infty)\neq 0$, but when $Z_3^0(g_\theta)\not\equiv 0$ one has $\gamma_3^0(g_\infty)=0$ and one obtains canonical scaling in λ^{4-2n}. Of course, for Green's functions containing composite operators θ_l, canonical behavior is not obtained since in general $\gamma_{\theta_l}^0(g_\infty)\neq 0$. Consider now AF. Here $g_\infty=0$ and consequently $\gamma_3^0(g_\infty)=\gamma_{\theta_l}^0(g_\infty)=0$, and one has canonical scaling for all Green's functions [up to logarithms when $\beta^0(g_\theta)$ starts at order three, since then $z_3^0(\lambda^2,g_\theta)|_{\overrightarrow{\lambda^2\to\infty}}(L\lambda^2)^{c/2b}$].

III. MASSIVE CASE

A. Our formalism

We introduce here our formalism following Ref. 2. Let $\Gamma^{(2n)}(p_i;m^2,g)$ be the renormalized vertex functions, and define the dimensionless functions $d_1(x,g)$ and $d_3(x,g)$ by

$$\Gamma^{(4)}(p_i;m^2,g)|_{\text{sym. point } p^2}=gd_1\left(\frac{p^2}{m^2},g\right),$$

$$\Gamma^{(2)}(p,-p;m^2,g)=(p^2-m^2)d_3\left(\frac{p^2}{m^2},g\right).$$

The normalization conditions are $d_3(1,g)=d_1(1,g)=1$. We also introduce the functions $d_5(x,g)\equiv d_3(x,g)^2/d_1(x,g)$ and $v_R(x,g)\equiv g/d_5(x,g)$. If g invertibility is satisfied for the change of variable $g_\theta=g/d_5(\theta^2/m^2,g)\equiv v_R(\theta^2/m^2,g)$, $\theta^2\leq m^2$, then we can write the RG representations for the $G(Z_1Z_3{}^2)$ group[2] in the form

$$d_i(x,g)=\exp\left[-\int_1^x\frac{dt}{t}\phi_i\left(\frac{1}{t},v_R(t,g)\right)\right],\quad i=1,3,5\tag{30}$$

$$v_R(x,g)=g+\int_1^x\frac{dt}{t}v_R(t,g)^2F_5\left(\frac{1}{t},v_R(t,g)\right),\tag{31}$$

where the functions $\phi_i(1/x,v)$ are related by

$$F_5\left(\frac{1}{x},v\right)=\frac{1}{v}\phi_5\left(\frac{1}{x},v\right)=\frac{1}{v}\left[2\phi_3\left(\frac{1}{x},v\right)-\phi_1\left(\frac{1}{x},v\right)\right].$$

The θ-normalized vertex functions $\Gamma^{(2n)}(p_i;\theta^2,m^2,g_\theta)$ can be defined by

$$\Gamma^{(2n)}(p_i;\theta^2,m^2,g_\theta)\equiv d_3\left(\frac{\theta^2}{m^2},g\right)^{-n}\Gamma^{(2n)}(p_i;m^2,g)|_{g\to g_\theta}.\tag{32}$$

We also introduce the functions

$$z_i\left(\frac{p^2}{\theta^2},\frac{m^2}{\theta^2},g_\theta\right)\equiv\frac{d_i(p^2/m^2,g)}{d_i(\theta^2/m^2,g)}\Bigg|_{g\to g_\theta=\frac{g}{d_5(\theta^2/m^2,g)}},\quad i=1,3,5,\tag{33}$$

normalized at $p^2=\theta^2$, i.e., $z_i(1,m^2/\theta^2,g_\theta)=1$. Two coupling constants g_θ and $g_{\theta'}$ are related by

$$g_{\theta'}=\frac{g_\theta}{z_5(\theta'^2/\theta^2,m^2/\theta^2,g_\theta)}\equiv v\left(\frac{\theta'^2}{\theta^2},\frac{m^2}{\theta^2},g_\theta\right)=v_R\left(\frac{\theta'^2}{m^2},g\right)=\frac{g}{d_5(\theta'^2/m^2,g)}.\tag{34}$$

The quantity $g_{\theta'}=v(\theta'^2/\theta^2,m^2/\theta^2,g_\theta)$ is the group invariant (invariant charge). For $\theta^2=m^2$ the θ-normalized theory becomes the initial renormalized theory,

$$\Gamma^{(2n)}(p_i;\theta^2,m^2,g_\theta)|_{\theta^2=m^2}=\Gamma^{(2n)}(p_i;m^2,g),$$

and for $\theta^2\to\infty$ the θ-normalized vertex functions go into the bare vertex functions (only mass renormalized).

We have the identities ($i=1,3,5$)

$$\left[m\frac{\partial}{\partial m}+\beta\left(\frac{m^2}{\theta^2},g\right)\frac{\partial}{\partial g}-\gamma_i\left(\frac{m^2}{\theta^2},g\right)\right]d_i\left(\frac{p^2}{m^2},g\right)=\Delta_i\left(\frac{p^2}{m^2},\frac{\theta^2}{m^2},g\right),\tag{35}$$

with

$$\beta\left(\frac{m^2}{\theta^2},g\right)\equiv m\frac{\partial g}{\partial m}\Bigg|_{g_\theta},$$

$$\gamma_i\left(\frac{m^2}{\theta^2},g\right)\equiv m\frac{\partial}{\partial m}d_i\left(\frac{\theta^2}{m^2},g\right)\Bigg|_{g_\theta},$$

$$\Delta_i\equiv d_i\left(\frac{\theta^2}{m^2},g\right)m\frac{\partial}{\partial m}z_i\left(\frac{p^2}{\theta^2},\frac{m^2}{\theta^2},g_\theta\right)\Bigg|_{g_\theta},$$

where all derivatives are taken keeping $g_\theta = v_R(\theta^2/m^2, g)$ constant. One has $\beta = g\gamma_5 = g(2\gamma_3 - \gamma_1)$. For $\theta^2 \to -\infty$ Eqs. (32) are the usual Callan-Symanzik (CS) equations, and the functions $\beta(m^2/\theta^2, g)$ and $\gamma_3(m^2/\theta^2, g)$, when perturbatively expanded and for $\theta^2 \to -\infty$, become the usual CS functions $\beta(g)$ and $\gamma_3(g)$,

$$\beta(g) = b_0 g^2 + \cdots, \quad b_0 > 0$$
$$\gamma_3(g) = c_0 g^2 + \cdots, \quad c_0 > 0, \tag{36}$$

where b_0 and c_0 are the same as in (8) (the series becomes different at higher orders). From the definition of the functions β and γ_i we obtain ($x = \theta^2/m^2$)

$$\beta\left(\frac{1}{x}, g\right) = 2\frac{x\partial v_R(x,g)/\partial x}{\partial v_R(x,g)/\partial g}$$
$$= 2\frac{v_R(x,g)^2 F_5(1/x, v_R(x,g))}{\partial v_R(x,g)\partial g}, \tag{37}$$

$$\gamma_i\left(\frac{1}{x}, g\right) d_i(x,g) - \beta\left(\frac{1}{x}, g\right)\frac{\partial d_i(x,g)}{\partial g}$$
$$= -2x\frac{\partial d_i(x,g)}{\partial x}$$
$$= 2d_i(x,g)\phi_i\left(\frac{1}{x}, v_R(x,g)\right). \tag{38}$$

Because of the Källén-Lehmann representation for the propagator we have $d_3(x,g) \le 1$, $\partial d_3/\partial x > 0$, $x \le 1$, so that we conclude from (30) that $(1/x)\phi_3(1/x, v_R(x,g)) < 0$, $x \le 1$. We define $Z_3(g)$ as in the $m=0$ case by

$$Z_3(g) \equiv d_3(x,g)\big|_{x\to-\infty}, \quad g \ne 0, \tag{39}$$

and $Z_3(0) = Z_3(g\to 0)$. Two cases are possible now; either

$$\phi_3\left(\frac{1}{x}, v_R(x,g)\right) \le O((\ln x)^{-2}), \quad x \to -\infty, \tag{40}$$

and then $Z_3(g) < 1$ is a finite function of g because the integral in (30) converges, or

$$\phi_3\left(\frac{1}{x}, v_R(x,g)\right) > O((\ln x)^{-2}) \tag{41}$$

and $Z_3(g) = 0$ because the integral in (30) diverges.

The perturbative properties of the function $\phi_n(1/x, v)$ are as follows. Expand

$$\phi_i\left(\frac{1}{x}, v\right) = \sum_1^\infty \phi_{in}\left(\frac{1}{x}\right) v^n.$$
$$F_5\left(\frac{1}{x}, v\right) = \sum_0^\infty \psi_n\left(\frac{1}{x}\right) v^n, \tag{42}$$

where $\psi_n(1/x) = \phi_{5,n+1}(1/x)$. One proves[2] that $\phi_{in}(1/x) \underset{x\to-\infty}{\longrightarrow} \phi_{in}(0) < \infty$.

We define the bare coupling constant g_B, which is the effective coupling at high energy, by

$$g_B = \lim_{\theta^2\to-\infty} g_\theta = \lim_{\theta^2\to-\infty} v_R\left(\frac{\theta^2}{m^2}, g\right). \tag{43}$$

Asymptotic freedom is defined as the case in which $g_B = 0$.

We shall distinguish in what follows between two possibilities: (1) Perturbation theory is reliable in the sense that the functions $\phi_i(0, v)$ are well represented by the series

$$\phi_i(0, v) = \sum_1^\infty \phi_{in}(0) v^n, \tag{44}$$

which exist because $\phi_{in}(0) < \infty$, and (2) perturbation theory is not reliable.

B. Behavior of $Z_3(g)$ in relation to asymptotic freedom

Let us study now the behavior of $Z_3(g)$ in relation to asymptotic freedom. We start with case (1) where the analysis is the same as in the $m=0$ case. We want to study the limit $g_B = v_R(t,g)\big|_{t\to-\infty}$. From (31) we obtain

$$x\frac{\partial v_R(x,g)}{\partial x} = v_R(x,g)^2 F_5\left(\frac{1}{x}, v_R(x,g)\right). \tag{45}$$

The function $v_R(x,g)$ will have the same asymptotic behavor as $v_R^\infty(x,g)$ solution of the GML equation

$$x\frac{\partial v_R^\infty(x,g)}{\partial x} = \psi_+(v_R^\infty(x,g)), \tag{46}$$

with the GML function defined by

$$\psi_+(x) = v^2 F_5(0, v) = v^2\sum_0^\infty \psi_n(0) v^n. \tag{47}$$

The asumptotic invariant $v_R^\infty(x,g) = g + O(g^2)$ is obtained in perturbation theory by dropping from the perturbation series of $v_R(x,g)$ all the terms that vanish when $x \to -\infty$. Integrating (46) between -1 and $x < -1$ one obtains [putting $q(g) = v_R^\infty(-1, g) = g + O(g^2)$]

$$\ln|x| = \int_{q(g)}^{v_R^\infty(x,g)} \frac{dv}{\psi_+(v)}, \tag{48}$$

which is the GML equation. This equation replaces Eq. (10) of the $m=0$ case and can be analyzed in the same way. One has

$$\beta(g) = 2g^2[\psi_0(0) + g\psi_1(0) + \cdots],$$
$$\psi_+(v) = v^2 F_5(0, v) = v^2[\psi_0(0) + v\psi_1(0) + \cdots]. \tag{49}$$

Note that $\beta(g) \ne 2\psi_+(g)$ [they are equal up to third order as can be seen in (46)]. From (49) and (36) we see that $b_0 = 2\psi_0(0) > 0$. We can now repeat the analysis of Eq. (10) to conclude that for $g<0$ one has asymptotic freedom, while for $g>0$ one has $g_B=\bar{g}_B$ if $\psi_+(\bar{g}_B)=0$, $\bar{g}_B<\infty$, or $g_B=\infty$ if $\psi_+(g_B)$ has no finite zero.

Let us take now the limit $x\to-\infty$ of (38) for $i=3$. We obtain ($x\partial d_3/\partial x\to 0$)

$$\gamma_3(g)Z_3(g)-\beta(g)\frac{\partial Z_3}{\partial g}=0 \qquad (50)$$

if $Z_3(g)\neq 0$, and $0=0$ if $Z_3(g)\equiv 0$. Integrating (50) one has

$$Z_3(g)=Z_3(0)\exp\left[\int_0^g dv\,\frac{\gamma_3(v)}{\beta(v)}\right], \qquad (51)$$

which replaces Eq. (23). The analysis is done in the same way and we conclude again that asymptotic freedom is equivalent to $Z_3(0)=1$. In the case of nonasymptotic freedom one can have $Z_3(g)\neq 0$ with $Z_3(0)<1$, and the anomalous dimension of the field which is here $\phi_3(0,g_B)$ will vanish because of (40). The study of models in which $\beta(g)$ starts at third order is done again by the same techniques and leads us to the same conclusions as in the $m=0$ situation.

We consider now case (2) in which perturbation theory is not reliable in the sense that the functions $\phi_i(0,v)$ are not represented by the series (44). Among the variety of cases (see Ref. 5) one is particularly appealing; it is the situation in which g_B is a finite function $g_B(g)$ of g. We have given in Ref. 2 an explicit example that realizes this case reproducing all the known information from perturbation theory. Let us give some general features of this case. From (37) we see that $\beta(g)\equiv 0$ because $\partial v_R(x,g)/\partial g|_{x\to-\infty}=\partial g_B/\partial g\neq 0$ and $x\partial v_R/\partial x\to 0$ [we assume only that $x\partial v_R/\partial x$ is not an oscillating function so that $v_R(x,g)\to g_B<\infty$, $x\to-\infty$, implies $x\partial v_R/\partial x\to 0$, a condition that is verified in the example given in Ref. 2]. Note that (37) implies that if g_B is finite, then $\beta(g)$ can be different from zero only if $\partial g_B/\partial g=0$, i.e., if g_B is independent of g. Coming back to our example we see that $F_5(1/x,v)\xrightarrow[x\to-\infty]{}0$ for fixed v belonging to some domain $\mathfrak{D}$. The points $v\in\mathfrak{D}$ are the set of values that g_B can take. But from the relation

$$F_5\left(\frac{1}{x},v\right)=\frac{1}{v}\phi_5\left(\frac{1}{x},v\right)=\frac{1}{v}\left[2\phi_3\left(\frac{1}{x},v\right)-\phi_1\left(\frac{1}{x},v\right)\right] \qquad (52)$$

we see that we must have

$$2\phi_3\left(\frac{1}{x},v\right)\Big|_{x\to-\infty}=\phi_1\left(\frac{1}{x},v\right)\Big|_{x\to-\infty},\quad v \text{ fixed},\ v\in\mathfrak{D}. \qquad (53)$$

Now, from (30) for $i=3$ we find that either the integral converges if $\phi_3(1/t,v_R(t,g))\leq O((\ln t)^{-2})$ and $d_3(x,g)\xrightarrow[x\to-\infty]{}Z_3(g)\neq 0$ or the integral diverges and $Z_3(g)\equiv 0$. In the first case, using (43) and (53) we see that $\phi_3(1/x,g_B)\to 0$, $\phi_1(1/x,g_B)\to 0$, $x\to-\infty$, $g_B\in\mathfrak{D}$, so that neither of these two functions if of the GML type [$\phi_i(0,v_R(\infty))=0$ does not fix a value of $v_R(\infty)$; see Ref. 5]. Moreover, from (50) we see that $\gamma_3(g)Z_3(g)\equiv 0$ [$\beta(g)\equiv 0$ here] so that $\gamma_3(g)\equiv 0$. One can see by studying explicit models of this kind of situation[6] that one may now have $Z_3(0)=1$ without asymptotic freedom. The second case, $Z_3(g)\equiv 0$, will give an anomalous-dimension function of the coupling constant as we shall see below.

C. Asymptotic behavior in the massive case

Let us study now asymptotic behavior in the massive case. From (32) we obtain the identity

$$d_3\left(\frac{\theta^2}{m^2},g\right)^n\Gamma^{(2n)}(p_i;\theta^2,m^2,g_\theta)=d_3\left(\frac{\theta^2}{m^2},g\right)^n\Gamma^{(2n)}(p_i;\theta'^2,m^2,g_{\theta'}). \qquad (54)$$

Putting $\lambda^2=\theta'^2/\theta^2$ and after some manipulations (for details see Ref. 7) we obtain the identity

$$\Gamma^{(2n)}(\lambda p_i;\theta^2,m^2,g_\theta)=\lambda^{4-2n}\exp\left[-n\int_1^{\lambda^2}\frac{dt}{t}\,\phi_3\left(\frac{m^2}{\theta^2 t},v\left(t,\frac{m^2}{\theta^2},g_\theta\right)\right)\right]\times\Gamma^{(2n)}\left(p_i;\theta^2,\frac{m^2}{\lambda^2},v\left(\lambda^2,\frac{m^2}{\theta^2},g_\theta\right)\right), \qquad (55)$$

a consequence only of g invertibility. From (34) we see that $v(t,m^2/\theta^2,g_\theta)\xrightarrow[t\to\infty]{}g_B$. On the other hand, the exponential in (55) is just equal to $z_3(\lambda^2,m^2/\theta^2,g_\theta)^n$; this is because we have used to obtain (52) the RG representation

$$z_i\left(\frac{p^2}{\theta^2},\frac{m^2}{\theta^2},g_\theta\right)=\exp\left[-\int_1^{p^2/\theta^2}\frac{dt}{t}\,\phi_i\left(\frac{m^2}{\theta^2 t},v\left(t,\frac{m^2}{\theta^2},g_\theta\right)\right)\right], \qquad (56)$$

which reduces to (30) for $\theta^2=m^2$. We see then from (33) that for $Z_3(g)\neq 0$ the exponential tends to $[Z_3(g)/d_3(\theta^2/m^2,g)]^n$. The vertex function on the right-hand side of (52) goes into a zero-mass-limit vertex function evaluated at the value g_B of the coupling when $\lambda^2\to\infty$, so that the p_i must be Euclidean nonexceptional momentum for which we know the $m^2\to 0$ limit to exist for the θ-normalized theory in perturbation theory.

Let us consider case (1). When $g_B=0$ (AF) one has $Z_3(g)\neq 0$ [when $\beta(g)$ starts at second order] and (52) implies

$$\lambda^{4-2n}\left(\frac{Z_3(g)}{d_3(\theta^2/m^2,g)}\right)^n \Gamma^{(2n)}(p_i;\theta^2,0,0). \qquad (57)$$

When $\beta(g)$ starts at third order, one has canonical scaling up to logarithms as we have explained for the $m=0$ models.

In the case of nonasymptotic freedom with $Z_3(g) \neq 0$ the result is

$$\lambda^{4-2n}\left(\frac{Z_3(g)}{d_3(\theta^2/m^2,g)}\right)^n \Gamma^{(2n)}(p_i;\theta^2,0,g_B), \qquad (58)$$

that is still canonical scaling provided the zero-mass vertex functions exist for $g=g_B\neq 0$ (same remark for Green's functions of composite operators as in the $m=0$ case). Finally, when $Z_3(g) \equiv 0$ the field obtains the anomalous dimension $\phi_3(0,g_B)\neq 0$, and the dominant term is

$$\lambda^{4-2n[1+\phi_3(0,g_B)]}\Gamma^{(2n)}(p_i;\theta^2,0,g_B). \qquad (59)$$

We consider now case (2) and, more specifically, the situation we have explained when g_B is a finite function $g_B(g)$. If $Z_3(g)\neq 0$ we shall obtain (57). But when $Z_3(g)\equiv 0$ an interesting possibility arises, since one obtains again (56) but now the anomalous dimension $\phi_3(0,g_B(g))\neq 0$ becomes a function of the coupling constant g.

Let us finally remark that we have constantly used in our discussion the RG representations; i.e., we have assumed the validity of g invertibility, a postulate that may be false,[8] as it is demonstrated to happen in an explicit model of field theory[9] where the violation of the RG representations is explicitly exhibited.

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[1]A. Hosoya, Phys. Rev. D 10, 3937 (1974).
[2]E. Tirapegui, Nuovo Cimento 29A, 153 (1975).
[3]K. Symanzik, Lett. Nuovo Cimento 6, 77 (1973).
[4]D. Gross, 1975 Les Houches lectures (unpublished).
[5]B. Jouvet, Nuovo Cimento 23A, 621 (1974).
[6]B. Jouvet and G. Bordes, Nuovo Cimento 28A, 273 (1975).
[7]E. Tirapegui, lectures at the ICTP, Trieste Report No. ICTP 75/60 (unpublished).
[8]B. Jouvet, Nuovo Cimento 23A, 521 (1974).
[9]P. Coullet and E. Tirapegui, Lett. Nuovo Cimento 13, 107 (1975).