PHYSICAL REVIEW D VOLUME 15, NUMBER 6 15 MARCH 1977

Perturbation theory at large order. II. Role of the vacuum instability

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(Received 30 November 1976)

We extend our previous study of large orders of perturbation series for nonrelativistic quantum mechanics and boson field theories to more complicated situations. It is shown that when perturbation theory is performed around an unstable vacuum and does not reveal any pathology at low orders the existence of real pseudoparticles, which are responsible for the tunneling to a more stable vacuum, also implies the divergence and the non-Borel-summability of the series. Conversely, large orders of perturbations around a stable vacuum are dominated by complex solutions to Euclidean field equations. They quantitatively characterize its behavior and indicate the Borel summability of the series. Thus the corresponding Green's functions are unambiguously determined by their perturbation series.

I. INTRODUCTION

In a recent work Lipatov[1] has shown that the nature of the perturbation series at large orders for massless renormalizable scalar field theories may be characterized by classical solutions to Euclidean field equations of the pseudoparticle type. This picture also applies to nonrelativistic quantum mechanics[2] for which it allows one to recover the results of Bender and Wu.[3] In this article we want to describe the role of pseudoparticles for large orders of perturbation series in more complex situations, in connection with the stability of the vacuum and with the possibility of spontaneous symmetry breaking.

Let us first summarize the discussion[1,2] for simple theories, such as, for instance, anharmonic oscillators with a Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}x^2+gx^{2N}, \tag{1}$$

or scalar massless field theories with (Euclidean) action

$$\mathcal{A}=\int d^dx[\tfrac{1}{2}(\partial_\mu\phi)^2+g\phi^{2N}+\text{counterterms}]. \tag{2}$$

For unphysical (i.e., negative) values of the coupling constant there exists a classical solution to the field equations with a finite value of the Euclidean action. This implies that if we consider some physical quantity $F(g)$ such as an energy level, a Green's function, etc., and if we write its perturbation expansion as

$$F(g)=\sum_K F_K g^K, \tag{3}$$

then for K large F_K behaves as

$$F_K \underset{K\ \text{large}}{\simeq} [K(N-1)]!\,a^K K^b c\left[1+O\left(\frac{1}{K}\right)\right]. \tag{4}$$

This is obtained by looking at a saddle point in combined field and coupling-constant space. The important point is that the negative value of g at the saddle point implies that the parameter a of Eq. (4) is also negative. Therefore Eq. (4) indicates that the perturbation series is Borel-summable, and in particular that the function $F(g)$ is unambiguously determined from the knowledge of the F_K's.

In some circumstances pseudoparticles may be present for real physical values of the parameters. This occurs if one expands around a minimum of the potential which does not correspond to the true vacuum of the theory. An example of such a situation is provided by any potential problem of quantum mechanics whenever one expands around a relative minimum. It is also known to occur in non-Abelian gauge theories.[4,5] It will be shown that these real pseudoparticles imply that the theory is not Borel-summable. In other words, for physical values of the coupling constant, either one is expanding right on a cut in the complex g plane or one has to add to the perturbation series terms which are exponentially small with g.

The situation is similar for spontaneous symmetry breaking. If one takes for instance the ϕ^4 theory (without internal symmetry except $\phi\to-\phi$) in Euclidean one-dimensional space, i.e., quantum mechanics with an imaginary time, there is a real pseudoparticle which leads to the non-Borel-summability of the perturbation series constructed around one of the vacuums of broken symmetry. However, above one dimension there are only complex pseudoparticles for this theory and the perturbation series becomes Borel-summable. One should of course remember that for this theory spontaneous symmetry breaking cannot exist in one dimension, but does exist above one dimension. Therefore real pseudoparticles are responsible for the fact that perturbation theory becomes meaningless if one is expanding around the wrong vacuum, and complex pseudoparticles are responsible for its Borel-summability when the vacuum is stable. These features will be illustrated below on simple models taken from potential theory or boson field theories.

The influence of Borel or non-Borel-summability may be illustrated by a simple example. If one considers the series

$$\sum_K (-1)^K K!\, g^K , \tag{5}$$

the Borel sum

$$\int_0^\infty dt\, e^{-t} \sum_K (-1)^K (tg)^K = \int_0^\infty \frac{e^{-t} dt}{1+tg} \tag{6}$$

is well defined for $g>0$, and therefore if we know *a priori* that the function which is expanded is Borel-summable,[6] this function is unambiguously determined. This feature would persist if, instead of $(-1)^K$ in Eq. (5), one had phase oscillations such as $\cos K\theta$ $(\theta \neq 0)$. The Borel transform would then have a singularity for complex t instead of real negative t.

Let us now examine the situation in which the series is

$$\sum K!\, g^K . \tag{7}$$

The Borel transform, formally $\int_0^\infty e^{-t} dt/(1-tg)$, does not exist for positive real g. Thus, either the function of g which is being expanded has indeed a cut along the positive real axis and the expansion is meaningless, or the singularity is canceled by terms which have all their derivatives vanishing at $g=0$. In the latter case one has thus to modify the previous integral representation of the function. There is a large arbitrariness; one can consider for instance the function

$$\int_0^\infty \frac{dt\, e^{-t/2}}{1-tg} \left(e^{-t/2} - \frac{P(tg)}{P(1)}\, e^{-1/2g} \right),$$

in which $P(x)$ is an arbitrary polynomial. Therefore if the theory is known to exist for real positive g, one cannot extract the function out of its expansion without further information which would fix the ambiguities.

II. PERTURBATION SERIES IN NONRELATIVISTIC QUANTUM MECHANICS

Until now the behavior at large orders of perturbation series has been characterized for polynomial potentials $V(x)$ of the form[3]

$$V(x) = \lambda (x^{2N} + a x^{2N-2} + \cdots),$$

in which λ is the expansion parameter. In this case the asymptotic orders are dominated by the highest power of x in V. However, if one performs instead an expansion in powers of $\hbar$, all the terms of the potential contribute to the leading behavior. This semiclassical expansion is of interest since it corresponds to the loop expansion in field theory and since it arises naturally when symmetries are discussed. In order to generate this expansion we shall write the potential in the form

$$\frac{1}{g^2} V(gx) , \tag{8}$$

and expand it in powers of g. It will be assumed that $x=0$ is a minimum of the potential and that around zero it is normalized according to

$$V(x) = \tfrac{1}{2} x^2 + O(x^3) . \tag{9}$$

The difference $(1/g^2)V(gx) - \frac{1}{2}x^2$ is treated as a perturbation. Furthermore, it is assumed that the potential is analytic in x in some neighborhood of the real axis. We shall focus our attention on the perturbation series for the ground-state energy generated through the limit

$$E_G - \tfrac{1}{2} = \lim_{\beta\to\infty} -\frac{1}{\beta} \ln \frac{\operatorname{Tr} e^{-\beta H}}{\operatorname{Tr} e^{-\beta H_0}} , \tag{10}$$

in which

$$H = \tfrac{1}{2} p^2 + \frac{1}{g^2} V(gx) , \tag{11a}$$

$$H_0 = \tfrac{1}{2} p^2 + \tfrac{1}{2} x^2 . \tag{11b}$$

The ratio of traces of Eq. (10) is given by the Feynman-Kac formula

$$\frac{\operatorname{Tr} e^{-\beta H}}{\operatorname{Tr} e^{-\beta H_0}} = N \int_{x(0)=x(\beta)} [dx] \exp[-\mathcal{A}\{x(\tau)\}] , \tag{12}$$

in which $\mathcal{A}$ is the Euclidean action

$$\mathcal{A}\{x\} = \int_0^\beta \left[\tfrac{1}{2} \dot{x}^2(\tau) + \frac{1}{g^2} V(gx(\tau)) \right] d\tau , \tag{13}$$

and the normalizing factor N is such that the ratio (12) reduces to one for $g=0$.

This representation may be used to generate an expansion in powers of g^2 of the ground-state energy

$$E_G = \sum_0^\infty E_K g^{2K} , \tag{14}$$

which we shall try to characterize at large orders. As explained in Ref. 2, the method consists in looking for the saddle points in the combined variables $x(\tau)$ and g of the integral giving the Kth order of the ratio

$$\left(\frac{\mathrm{Tr}e^{-\beta H}}{\mathrm{Tr}e^{-\beta H_0}}\right)_{(K)} = N\int [dx] \oint \frac{dg^2}{2i\pi (g^2)^{K+1}} \times \exp[-\mathcal{A}\{x(\tau), g\}]\,. \tag{15}$$

Let us assume that we have identified one leading saddle point (LSP), $x_c(\tau)$, g_c, corresponding to a finite action (13) in the large-β limit. This saddle point is given by the equations

$$\ddot{x}_c = \frac{1}{g_c} V'(g_c x_c)\,, \tag{16a}$$

$$\frac{2K}{g_c} = \frac{2}{g_c^3}\int_0^\beta d\tau\, V(g_c x_c) - \frac{1}{g_c^2}\int_0^\beta d\tau\, x_c V'(g_c x_c)\,, \tag{16b}$$

which in the variables $y_c(\tau) = g_c x_c(\tau)$ and g_c reads

$$\ddot{y}_c = V'(y_c)\,, \tag{17a}$$

$$K g_c^2 = \int_0^\beta d\tau[\tfrac{1}{2}\dot{y}_c^2 + V(y_c)]\,. \tag{17b}$$

This shows, as expected, that g_c^2 is of order $1/K$ and justifies the use of the steepest-descent method. The leading contribution from this saddle point comes simply from the value of the integrand at the saddle point and from the harmonic fluctuations around this classical solution. We shall first integrate over the $y(t)$ fluctuations and then evaluate the integral over g.

Let E be the energy of the $y(\tau)$ trajectory of period β, such that

$$\tfrac{1}{2}\dot{y}_c^2 = V(y_c) + E\,. \tag{18}$$

The corresponding value of the action is

$$\mathcal{A}\{y_c\} = \frac{1}{g_c^2}\left(-\beta E + 2\int_{y_-}^{y_+} dy\{2[E + V(y)]\}^{1/2}\right)\,, \tag{19}$$

in which y_- and y_+ are the turning points where the velocity vanishes. (See Fig. 1.)

For the calculation of the fluctuations around this classical trajectory we shall follow the "shifting method."[7] The contribution of these fluctuations amounts to calculating the functional integral

$$\int_{Z(0)=Z(\beta)=0} dZ(\tau) \exp\left\{-\frac{1}{2g^2}\int_0^\beta d\tau[\dot{Z}^2 + V''(\tau)Z^2]\right\}\,, \tag{20}$$

in which

$$V''(\tau) \equiv \left.\frac{\partial^2 V}{\partial y^2}\right|_{y=y_{c1}(\tau)}\,. \tag{21}$$

We now define a solution $N(\tau)$ of the differential equation

$$\ddot{N}(\tau) = V''(\tau) N(\tau)\,, \tag{22}$$

which we can always find since, from translational invariance, we know that $\dot{y}_{c1}(\tau)$ satisfies (22), and perform the change of variables

$$Z(\tau) = \xi(\tau) + N(\tau)\int_0^\tau d\theta\, \xi(\theta)\left[\frac{\dot{N}(\theta)}{N^2(\theta)} - \frac{1}{N(\theta)}\right]\,. \tag{23}$$

This reduces the problem to that of a harmonic oscillator with a time-independent frequency. Thus, one is led to

$$\left(\frac{\mathrm{Tr}e^{-\beta H}}{\mathrm{Tr}e^{-\beta H_0}}\right)_{(K)} \underset{K\ \text{large}}{=} \frac{\beta}{2\pi}\left(\frac{2}{\pi}\right)^{1/2} \sinh\frac{\beta}{2}\left(\frac{\partial E}{\partial \beta}\right)^{1/2} \times \int \frac{dg^2}{(g^2)^{K+3/2}} \exp\left(-\frac{1}{g^2}\mathcal{A}\{y_c\}\right)\,. \tag{24}$$

For large β the energy goes exponentially to zero as

$$E \underset{\beta\to\infty}{\sim} -2e^{-\beta} y_+^2 \exp\left[2\int_0^{y_+} dy\left(\frac{1}{[2V(y)]^{1/2}} - \frac{1}{y}\right)\right]\,, \tag{25}$$

and the resulting ground-state energy is

$$E_G = \sum E_K (g^2)^K\,,$$

where for K large, collecting the contribution of all leading saddle points, we obtain

$$E_K = \sum_{\mathrm{LSP}} -\frac{1}{2\pi^{3/2}} \frac{\Gamma(K+\frac{1}{2})}{[\mathcal{A}(y_c)]^{K+1/2}} y_+ \times \exp\left[\int_0^{y_+} dy\left(\frac{1}{[2V(y)]^{1/2}} - \frac{1}{y}\right)\right]\,. \tag{26}$$

This formula and the determination of the leading saddle points will be illustrated by the discussion of two simple examples.

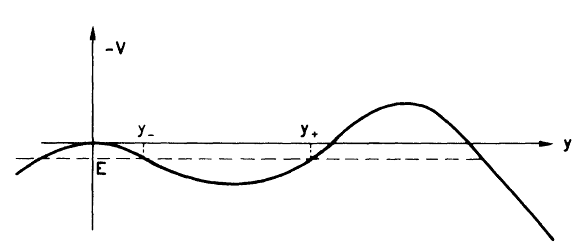

FIG. 1. The (real) turning points for an (unstable) minimum.

(a) As the first example, we have

$$V(y)=\tfrac{1}{2}y^2-\gamma y^3+\tfrac{1}{2}y^4\,. \qquad (27)$$

(i) For $|\gamma|>1$ we are not expanding around the absolute minimum of V and there is a real trajectory leaving the origin, reflecting at y_+ and coming back to the origin:

$$y_+=\gamma-(\gamma^2-1)^{1/2}\,,$$

$$\int_0^{y_+}dy\left(\frac{1}{[2V(y)]^{1/2}}-\frac{1}{y}\right)=-\ln\{\tfrac{1}{2}(\gamma^2-1)^{1/2}[\gamma-(\gamma^2-1)^{1/2}]\}\,, \qquad (28)$$

$$\mathcal{A}(y_c)=-\tfrac{2}{3}+\gamma^2-\tfrac{1}{2}\gamma(\gamma^2-1)\ln\frac{\gamma+1}{\gamma-1}\,.$$

Therefore, applying Eq. (26) we obtain

$$E_K \underset{K\ \text{large}}{=} -\frac{1}{\pi^{3/2}}\frac{\Gamma(K+\frac{1}{2})}{\mathcal{A}^{K+1/2}}(\gamma^2-1)^{-1/2}\,, \qquad (29)$$

which exhibits, since from (28) $\mathcal{A}$ is positive for $|\gamma|>1$, the non-Borel-summability of the series.

(ii) For $|\gamma|<1$ the origin is the absolute minimum of V and the previous pseudoparticle becomes complex. Actually there are now two complex-conjugate saddle points which both contribute to the leading order.

The result for each saddle point is obtained by analytic continuation of the previous one and gives

$$E_K \underset{K\ \text{large}}{=} \frac{2}{\pi^{3/2}}\Gamma(K+\tfrac{1}{2})\,\mathrm{Im}\,\mathcal{A}^{-(K+1/2)}(1-\gamma^2)^{-1/2}\,, \qquad (30)$$

in which $\mathcal{A}$ has been defined by (28).

In the limit $\gamma=0$ this reduces to the result previously derived,[3,2]

$$E_K \underset{K\ \text{large}}{=} (-)^{K+1}\frac{6^{1/2}}{\pi^{3/2}}3^K\,\Gamma(K+\tfrac{1}{2})\,.$$

Thus one sees that, apart from the exceptional values $|\gamma|=1$ for which two saddle points coincide, this solves the problem. At the exception values $\gamma=\pm1$, one has a triple saddle point for one degree of freedom, and in order to integrate over the corresponding fluctuations one has to take into account the third-order terms in the expansion around the saddle point. However, this integration will not yield any K-dependent phase factor; thus the series will not be Borel-summable for $|\gamma|=1$, which is to be related to the impossibility of spontaneously breaking the symmetry $x\to1-x$ in one dimension.

(b) As the second example, we have

$$V(y)=\tfrac{1}{2}y^2-\gamma y^4+\tfrac{1}{2}y^6\,,$$

$$y_+^{\,2}=\gamma-(\gamma^2-1)^{1/2}\,,$$

$$y_+\exp\left[\int_0^{y_+}dy\left(\frac{1}{[2V(y)]^{1/2}}-\frac{1}{y}\right)\right]=2^{1/2}(\gamma^2-1)^{1/4}\,,$$

$$\mathcal{A}(y_c)=\tfrac{1}{2}\gamma-\tfrac{1}{4}(\gamma^2-1)\ln\frac{\gamma+1}{\gamma-1}\,.$$

(i) $\gamma>1$. The pseudoparticle is real and

$$E_K=-\frac{1}{\pi^{3/2}}\Gamma(K+\tfrac{1}{2})2^{1/2}(\gamma^2-1)^{-1/4}\mathcal{A}^{-K-1/2}\,,$$

which is not Borel-summable.

(ii) $-1<\gamma<1$. The pseudoparticle is complex and

$$E_K=-2^{1/2}\pi^{-3/2}(1-\gamma^2)^{-1/4}\Gamma(K+\tfrac{1}{2})\times\mathrm{Re}\left\{\left[\frac{\pi}{4}(1-\gamma^2)+i\left(\tfrac{1}{2}\gamma+(1-\gamma^2)\ln\frac{1+\gamma}{1-\gamma}\right)\right]^{-1/2}\left[\tfrac{1}{2}\gamma+\tfrac{1}{4}(1-\gamma^2)\left(\ln\frac{1+\gamma}{1-\gamma}-i\pi\right)\right]^{-K}\right\},$$

which reduces in the known case[3,2] $\gamma=0$ to the correct result, and, thanks to the phase factor, is indeed Borel-summable with complex singularities of the Borel transform.

(iii) $\gamma<-1$. The pseudoparticle is still complex, but the action at the saddle point is real and negative. Then

$$E_K=(-)^{K+1}2^{1/2}\pi^{-3/2}\left(1-\frac{1}{\gamma^2}\right)^{-1/4}\left[-\tfrac{1}{2}\gamma+\tfrac{1}{4}(\gamma^2-1)\ln\frac{\gamma+1}{\gamma-1}\right]^{-K}\left[\tfrac{1}{2}\gamma^2-\tfrac{1}{4}\gamma(\gamma^2-1)\ln\frac{\gamma+1}{\gamma-1}\right]^{-1/2}.$$

For large negative values of γ this formula coincides with the result[3] of a pure X^4 anharmonicity. For arbitrary $\gamma<-1$ the leading singularity of the Borel transform lies on the negative real axis. For $\gamma=\pm1$ we are again in a situation of a triple saddle point as before. However, for $\gamma=+1$, a situation in which we have degenerate classical minima, the series will remain non-Borel-summable. For $\gamma=-1$ the action at the saddle point remains real negative; the series is Borel-summable, but the result is not given by the same formula. In particular, the correct result for E_K involves a different power of K.

(c) Quantum mechanics with more than one degree of freedom. For a Hamiltonian

$$H=\sum_{i=1}^{n}\tfrac{1}{2}p_i^2+\frac{1}{g^2}V(g\vec{x}),$$

the same analysis may be repeated. The calculation of the fluctuations around the saddle point can be done following the lines of Dashen, Hasslacher, and Neveu.[7] We shall give here simply the explicit result for an O(n) – invariant potential normalized to

$$V(|\vec{x}|)=\tfrac{1}{2}(\vec{x})^2+O(|\vec{x}|^3)$$

for $|\vec{x}|$ small. The only trajectories which go back to the origin at large time are of the form

$$x_\alpha(t)=u_\alpha r(t),$$

in which $r(t)$ is the solution of the $n=1$ problem.

The ground-state energy

$$E_G=\sum g^{2K}E_K$$

behaves asymptotically for large K as

$$E_K \underset{K\text{ large}}{=} \sum_{\text{LSP}}\frac{-1}{\pi\Gamma(n/2)}\Gamma(K+\tfrac{1}{2}n)[\mathcal{A}(y_c)]^{-K-n/2}\,y_+^{\,n}\exp\left[n\int_0^{y_+}dy\left(\frac{1}{[2V(y)]^{1/2}}-\frac{1}{y}\right)\right],$$

in which the notations follow those of Eq. (26).

This formula allows one to recover the result previously derived for the potential $V(x)=(\vec{x}^2)^N$, if one notes that there are $N-1$ saddle points giving the same contribution.

III. NATURE OF THE PERTURBATION SERIES IN FIELD THEORY

The discussion of the preceding section may be repeated here. The existence of real pseudoparticles would lead necessarily to serious difficulties for the perturbation series, whereas complex pseudoparticles would indicate Borel-summability. We shall restrict the discussion to boson theories.

A. Scalar fields

The Euclidean action is

$$\mathcal{A}(\vec{\phi})=\int d^dx\left[\tfrac{1}{2}\partial_\mu\phi_\alpha M_{\alpha\beta}(\phi)\partial_\mu\phi_\beta+\frac{1}{g^2}V(g\phi)\right], \tag{31}$$

in which M is a positive-definite matrix. Let us assume that one has found a real classical solution $\phi_\alpha^c(x)$. We now calculate[8] $\mathcal{A}\{\vec{\phi}^c(\lambda x)\}$,

$$\mathcal{A}\{\vec{\phi}^c(\lambda x)\}=\lambda^{2-d}A+\lambda^{-d}B. \tag{32}$$

The coefficient A is positive, and B is also positive whenever one is looking for a stable minimum. The stationarity of $\mathcal{A}\{\vec{\phi}^c(\lambda x)\}$ for $\lambda=1$ thus eliminates any real solution for $d>2$ since $(d-2)A+dB$ remains positive.

In two dimensions a minimum of $\mathcal{A}$ should correspond to a vanishing B. In the stable case this implies $V(\vec{\phi}^c(x))=0$ for any x. This may only occur whenever $V(\vec{\phi})$ has a set of degenerate vacuums continuously connected; an example of such a situation is provided by a potential invariant under a continuous symmetry group. This picture is consistent with the known result of the absence of a phase transition with ordering in two dimensions for a continuous symmetry.[9]

Let us study now the quantitative consequences of this analysis and outline the calculation of the large-order behavior of perturbation series. For an N-point function, at order $2K$ if N is even (the same would apply at order $2K+1$ for odd N),

$$G_N^{(2K)}(x_1,\ldots,x_N)=\int[d\phi]\phi(x_1)\cdots\phi(x_N)\times\oint\frac{dg}{2i\pi}\frac{1}{g^{2K+1}}\exp(-\mathcal{A}\{\phi,g\}). \tag{33}$$

A saddle point has to be of the form

$$\phi_c(x)=\frac{1}{g}F_c(x,\xi_i), \tag{34}$$

in which the ξ_i $(i=1,2,\ldots,q)$ are arbitrary constants reflecting the possible invariances of the field equations (translations, dilatation, internal symmetries, etc.). We have to integrate over the fluctuations of ϕ around ϕ_c. The collective motions corresponding to ξ changes have to be quantized,[10] and this gives a Jacobian

$$\left\{\det\left[\frac{1}{g^2}\left\langle\frac{\partial F_c}{\partial\xi_i},\frac{\partial F_c}{\partial\xi_j}\right\rangle\right]\right\}^{1/2}.$$

The remaining integrations over the ϕ fluctuations give a g-independent contribution and the integral over g may now be evaluated. This yields for each saddle point

$$G_N^{(2K)} = \sum_{\mathrm{LSP}} C_N(x_1, \ldots, x_N)\Gamma(K + \tfrac{1}{2}N + \tfrac{1}{2}q) \times (\mathfrak{A}\{F_c(x), g=1\})^{-(K+N/2+q/2)}, \quad (35)$$

in which the C_N are K independent.

B. Scalar and gauge fields

For the case of pure gauge field, whenever the gauge group contains SU(2) as a subgroup, it has been shown[4] that there are real pseudoparticles in four dimensions. Therefore a perturbation series constructed around one given vacuum $A_\mu^\alpha = 0$ is not Borel-summable. This is consistent with the fact that pseudoparticles are responsible for the tunneling between different vacuums.[5] This feature persists if the gauge field is coupled to scalar ϕ^a if $\phi^a = 0$ is a stable nondegenerate minimum of the potential $V(\vec{\phi})$.

Indeed let us consider the d-dimensional Euclidean action

$$\mathfrak{A}\{\phi, A\} = \int d^dx[\tfrac{1}{4}F^\alpha_{\mu\nu}F^\alpha_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\phi_a - gA^\alpha_\mu T^\alpha_{ab}\phi_b)^2 + V(\vec{\phi})],$$

and assume that one has found a pseudoparticle $A^c(x)$, $\phi^c(x)$. We then consider[8]

$$\mathfrak{A}\{\phi^c(\lambda x), \lambda A^c(\lambda x)\} = \lambda^{4-d}\int d^dx(\tfrac{1}{4}F^2) + \lambda^{2-d}\int d^dx\tfrac{1}{2}(D_\mu\phi)^2 + \lambda^{-d}\int d^dx\, V(\phi).$$

The stationarity at $\lambda = 1$ implies

$$(4-d)\int d^dx(\tfrac{1}{4}F^2) + (2-d)\int d^dx\tfrac{1}{2}(D_\mu\phi)^2 - d\int d^dx\, V(\phi) = 0,$$

and this eliminates all real solutions for $d>4$ whenever $\phi = 0$ is a stable minimum of V. In four dimensions the solution would have to fulfill the constraints

$$D_\mu\phi^c = 0,$$
$$V(\phi^c) = 0,$$

in addition to the field equations for A_μ^c. This is only possible whenever $A_\mu^c(x)$ is not a pure gauge (otherwise the total action would vanish), and shows in particular that for QED of scalar particles no real pseudoparticles may exist and presumably the perturbation series is Borel-summable. In the non-Abelian case if $\phi = 0$ is the only stable minimum of V, then the solution is $\phi^c = 0$, with the pure gauge field solution of Ref. 4. Below four dimensions this argument does not prevent the existence of real pseudoparticles.

C. Higgs-Kibble Lagrangian[11]

If $V(\vec{\phi})$ has a stable minimum away from the origin, a constant ϕ is no longer allowed. If we assume for simplicity that the gauge group is SU(2) and that $\vec{\phi}$ is a vector, its length should be fixed at the minimum of V. The integrability conditions[12] of the system

$$D_\mu\vec{\phi}^c = 0$$

imply that for any x,

$$F^{c\alpha}_{\mu\nu}(x)\phi^c_\alpha(x) = 0,$$

in which $F^{c\alpha}_{\mu\nu}$ is the gauge field solution of Ref. 4. If we make explicit the corresponding conditions on ϕ^c, it follows immediately that there is no solution. Therefore the vacuum is, as expected, stabilized by the addition of Higgs scalars and the perturbation series becomes presumably Borel-summable.

ACKNOWLEDGMENTS

It is a pleasure to thank C. Itzykson, G. Parisi, and J.-B. Zuber for stimulating discussions.

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[12]We thank Dr. C. Itzykson for helpful discussions on this point.