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Perturbation theory at large order. I. The ϕ^{2N} interaction

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A new method for calculating the large orders of perturbation theory in quantum field theories has been discussed recently by Lipatov. We show that the same method applied to anharmonic oscillators in quantum mechanics allows one to rederive and generalize results previously obtained by Bender and Wu. We have also verified and generalized Lipatov's results to the case of an internal O(n) symmetry. These results show the divergence of the Wilson-Fisher ϵ expansion and indicate its Borel summability which is used for critical exponents. Similarly, the Callan-Symanzik functions for the ϕ^4 theory in three dimensions are characterized.

I. INTRODUCTION

Several years ago Bender and Wu,[1] in a series of papers, showed that perturbation theory for one-dimensional quantum mechanics with a polynomial potential is divergent, and they were able to calculate several terms which characterize the behavior of the coefficients of the perturbation series at very high order. Recently, it has been shown by Lipatov[2] that the same results hold for renormalizable scalar quantum field theories. He made the beautiful observation that the large orders of perturbation theory may be described by a classical structure with small quantum fluctuations, but around a pseudoparticle solution of the classical field equations. Such results are not only of conceptual but also of practical importance. They give a definite way of extracting better results from low-order perturbation theory. The first obvious applications concern critical phenomena in three dimensions, or the Wilson-Fisher[3] ϵ expansion, but it is clear that there are many different directions in which this could be useful.

The purpose of this article is the following: (1) We have verified that the results of Bender and Wu may also be obtained within the formalism used in field theory, and we have studied the case of a Hamiltonian with internal symmetry. (2) We have calculated the leading contributions to the Green's functions of renormalizable scalar theories and recovered Lipatov's results for the β function.[2] (3) We have generalized these calculations to the case of an O(n) internal symmetry in view of applications to critical phenomena. (4) We have applied the same techniques to the ϕ^4 theory in less than four dimensions, in which long perturbation series have been recently calculated in view of computing critical exponents.[4] (5) We have characterized the nature of the ϵ expansion.

The setup of this article is the following: We first give the basic argument which explains why the problem is tractable; we then treat completely the slightly simpler problem of the ground-state energy of the quantum-mechanical O(n)-symmetric anharmonic oscillator. We end up with field theory and its renormalization problems.

In all cases the results have the following structure: If the perturbation series for any given quantity are ordered in terms of the loop parameter ($g^{1/(N-1)}$ for a $g\phi^{2N}$ theory), the Kth order reads for large K as

$$K!\,a^K K^b c\left[1+O\left(\frac{1}{K}\right)\right]. \tag{1}$$

The parameter a, which may be simply calculated, is the same for all quantities in a given theory; it does not even depend on the possible existence of an internal symmetry. The parameter b is also simple, but it depends on which Green's function is considered and on the internal symmetry; the parameter c, which is momentum dependent in the case of Green's functions, requires detailed calculations which are given here for various theories. It is only at this level that renormalization appears, and not beyond the one-loop level. Finally, there is a systematics to calculate the inverse powers of K beyond the constant which is certainly cumbersome, but goes without conceptual difficulties.

1. Results

(*a*) *Anharmonic oscillator.*

$$H=\tfrac{1}{2}\sum_{i=1}^{n}p_i^{\,2}+\tfrac{1}{2}\sum_{i=1}^{n}x_i^{\,2}+g\left(\sum_{i=1}^{n}x_i^{\,2}\right)^N .$$

Ground-state energy:

$$E(\lambda)=\sum_0^{\infty}A_K g^K ,$$

$$A_K \underset{K\to\infty}{\approx} [K(N-1)]!\,a^K K^b c[1+O(1/K)],$$

$$a=-\frac{1}{2}\left[\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right]^{N-1},$$

$$b=n/2-1,$$

$$c=-\frac{1}{\pi}\frac{(N-1)^{n/2}}{\Gamma(n/2)}\left[\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right]^{n/2}.$$

(*b*) *Field theory.*

$$\mathcal{L}_{\text{int}}=-\frac{g}{(2N)!}\int d^dx\left(\sum_{i=1}^{n}\phi_i^{\,2}\right)^N.$$

(*i*) $d=2N/(N-1)$, $N>2$.

The $2M$-point vertex functions are given in Sec. III.

β *function*:

$$\beta(g)=\sum_2^\infty\beta_K g^K,$$

$$\beta_K \underset{K\to\infty}{\simeq}[K(N-1)]!\,a^K K^b c[1+O(1/K)],$$

$$a=-\frac{1}{N}\left(\frac{N-1}{2}\right)^{N+1}\frac{1}{\pi^N(2N-1)!}\left[\frac{\Gamma(2N/(N-1))}{\Gamma(N/(N-1))}\right]^{N-1},$$

$$b=N^2/(N-1)+n/2-1,$$

$$c=\frac{\Gamma(N+\frac{1}{2})}{\Gamma(N+n/2)}\,2^{3N-1/2-(n-1)[N/(N-1)+1/2]}\,\pi^{N-3/2}\left(\frac{N^{(n-1)}}{N-1}\right)^{1/2}\left[\left(\frac{2N-1}{N-1}\right)^n\frac{N}{3N-1}\right]^{(3N-1)/2(N-1)}\left[\frac{\Gamma(2N/(N-1))}{\Gamma^3(N/(N-1))}\right]^N$$

$$\times\exp\left\{-\tfrac{1}{2}\sum_{l=2}^{\infty}\frac{\Gamma(l+(N+1)/(N-1))[2l+(N+1)/(N-1)]}{l!\,\Gamma(2N/(N-1))}\right.$$

$$\times\left[\ln\left(1-\frac{N(2N-1)(N-1)^{-2}}{[l+1/(N-1)][l+N/(N-1)]}\right)+(n-1)\ln\left(1-\frac{N(N-1)^{-2}}{[l+1/(N-1)][l+N/(N-1]}\right)\right.$$

$$\left.\left.+\frac{N(2N-1)(N-1)^{-2}+(n-1)N(N-1)^{-2}}{[l+1/(N-1)][l+N/(N-1)]}\right]\right\}\exp\left[-\frac{(n-2+2N)N^2}{(N-1)(2N-1)}\right].$$

(*ii*) $d=4$, $N=2$.

$$\beta_K\simeq K!\,a^K K^b c[1+O(1/K)],$$

$$a=-1/16\pi^2,$$

$$b=3+n/2,$$

$$c=\frac{n+8}{\Gamma(2+n/2)}\,\pi 2^{10-2n}\,3^{3+(7/3)(n-1)}\,5^{-5/2}\exp[-8-\tfrac{16}{9}(n-1)+\tfrac{1}{3}\gamma(n+8)]$$

$$\times\exp\left(-\tfrac{1}{12}\sum_{l=2}^{\infty}(l+1)(l+2)(2l+3)\left\{\ln\left[1-\frac{6}{(l+1)(l+2)}\right]+(n-1)\ln\left[1-\frac{2}{(l+1)(l+2)}\right]+\frac{6+2(n-1)}{(l+1)(l+2)}+\frac{2(n+8)}{(l+1)^2(l+2)^2}\right\}\right)$$

$$\times\int_0^\infty dx\,x^{(n+8)/3+3}K_1^{\,4}(x).$$

γ is Euler's constant, $K_1(x)$ is the modified Bessel function

$$K_1(x)=\int_0^\infty dt\,e^{-x\cosh t}\cosh t\,dt.$$

These results are in agreement with those of Lipatov for $n=1$.[2]

This leads to the numerical values

$$c=\begin{cases}2.193, & n=1\\ 0.543, & n=2\\ 0.125, & n=3.\end{cases}$$

Comparison of the asymptotic formula with explicit perturbation calculations for $\tilde{\beta}_K = (8\pi^2)^{K-1}\beta_K$ gives, for $K=4$,

n	$\tilde{\beta}_4^{\text{exact}}$	$\tilde{\beta}_4^{\text{asymp}}$
1	4.90	5.33
2	6.08	5.28
3	7.34	4.84

A Padé-Borel analysis of the β function calculated up to order four in perturbation theory does not give any nontrivial zero, even if one modifies the analysis in order to incorporate the information beyond the $K!$ contained in $a^K K^b c$.

(iii) ϵ expansion, $\epsilon \equiv 4-d$.

The series are of the form

$$\sum_{K\ \text{large}} \epsilon^K K!\, a^K K^b c.$$

For the infrared-stable fixed point $g^*(\epsilon)$

$$a = -3/(n+8),\quad b = 4+n/2.$$

Critical exponents:

$$a = -\frac{3}{n+8},\quad b = \begin{cases} 3+n/2 \text{ for } \eta \\ 4+n/2 \text{ for } 1/\nu \\ 5+n/2 \text{ for } \omega. \end{cases}$$

A Padé-Borel analysis of the bad ω series gives, for $n=1,2,3$, $\omega = 0.79 \pm 0.01$.

If we include the additional information provided by a and b in the Padé-Borel technique, the results are essentially unchanged.

(iv) Callan-Symanzik equation in three dimensions.

$$\beta(g) = \sum_{K\ \text{large}} g^K K!\, a^K K^b c,$$

$$a = (-\pi\ 36.091)^{-1},$$

$$b = n/2 + 3.$$

2. Outline of the derivation

Consider the scalar massless quantum field theory[5] $g\phi^{2N}$ in dimension $d = 2N/(N-1)$, in which it is renormalizable. The Euclidean action $\mathcal{A}(\phi)$ reads

$$\mathcal{A}(\phi) = \int d^d x \left[\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{g}{(2N)!}\phi^{2N}\right] + \delta\mathcal{A}, \qquad (2)$$

in which $\delta\mathcal{A}$ includes all the counterterms of the theory. The Kth order of perturbation theory may be extracted by considering the contour integral

$$\frac{1}{2i\pi}\oint \frac{dg\, e^{-\mathcal{A}(\phi)}}{g^{K+1}}, \qquad (3)$$

inserted in a functional integral over ϕ. For large K we look at the saddle point in the $\{g, \phi\}$ variables,

$$\frac{K}{g_c} = -\frac{1}{(2N)!}\int d^d x\, \phi_c^{2N} - \frac{\partial}{\partial g}(\delta\mathcal{A}), \qquad (4)$$

$$\partial^2\phi_c = \frac{g_c}{(2N-1)!}\phi_c^{2N-1} + \text{counterterms}, \qquad (5)$$

and through the rescaling

$$\phi \to (-g_c)^{-1/(2N-2)}\psi,$$

we discover that g_c, which is negative, goes to zero as $(1/K)^{1/(N-1)}$ provided there exists a solution to the equation

$$\partial^2\psi = -\psi^{2N-1},$$

for which $\int d^d x\, \psi^{2N} < \infty$. Such solutions do exist and will be given later. Since g_c is infinitesimal for large K, the counterterms may be treated as small perturbations, and are to be omitted from Eqs. (4), (5). They do not play any role at leading order and may be included in an expansion around the saddle point. The leading contribution is given by

$$(-)^K \exp\{-[\mathcal{A}(\phi_c) + K\ln(-g_c)]\},$$

and since $\mathcal{A}(\phi_c)$ is proportional to K this gives already the $K!$ and the a^K of Eq. (1). Variations around the saddle point are necessary to obtain b and c. This is slightly simpler for quantum mechanics, which we treat first as an illustration.

II. ANHARMONIC OSCILLATORS

Consider the Hamiltonian

$$H = \tfrac{1}{2}\sum_{i=1}^{n} P_i^2 + \tfrac{1}{2}\sum_{i=1}^{n} X_i^2 + g\left[\sum_{i=1}^{n}(X_i^2)\right]^N \qquad (6)$$

The ground-state energy may be obtained by taking the limit

$$E - E_0 = \lim_{\beta\to\infty} -\frac{1}{\beta}\ln\frac{\mathrm{Tr}\,e^{-\beta H}}{\mathrm{Tr}\,e^{-\beta H_0}}, \qquad (7)$$

and these traces are expressed through the path integral

$$\frac{\mathrm{Tr}\,e^{-\beta H}}{\mathrm{Tr}\,e^{-\beta H_0}} = \mathfrak{N}\int \mathcal{D}\vec{x}(t) \times \exp\left\{-\int_0^\beta d\tau\left[\tfrac{1}{2}(\dot{\vec{x}}^2 + \vec{x}^2) + g(\vec{x}^2)^N\right]\right\}, \qquad (8)$$

in which $\mathfrak{N}$ is such that this ratio reduces to one for $g=0$. The Kth order in g is projected out by the Cauchy formula as in Eq. (3) and, in principle, the integration over g is to be performed first. In this particular problem this of course may be done easily, and it leads to the following calculations:

$$\left(\frac{\mathrm{Tr}e^{-\beta H}}{\mathrm{Tr}e^{-\beta H_0}}\right)^{(K)}=\frac{(-)^K}{K!}\mathfrak{N}\int \mathfrak{D}\vec{x}(t)\exp\left(-\int_0^\beta d\tau\left\{\tfrac{1}{2}(\dot{\vec{x}}^2+\vec{x}^2)-K\ln\int_0^\beta d\tau[\vec{x}^2(\tau)]^N\right\}\right). \tag{9}$$

The saddle points in $\vec{x}$ are given by the equations

$$\ddot{\vec{x}}=\vec{x}-\frac{2NK}{I\{x\}}\vec{x}(\vec{x}^2)^{N-1}, \tag{10}$$

in which we have denoted by $I\{x\}$

$$I\{\vec{x}\}=\int_0^\beta dt[\vec{x}^2(t)]^N. \tag{11}$$

Through a rescaling of $x(t)$ this equation reduces to

$$\ddot{\vec{x}}=\vec{x}-\vec{x}(t)(\vec{x}^2)^{N-1}. \tag{12}$$

The leading contribution to the action in the large-β limit is a trajectory labeled by an arbitrary time origin τ and an arbitrary direction characterized by a unit vector $\vec{u}$:

$$\vec{x}_c(t)=\vec{u}x_0(t-\tau), \tag{13}$$

$$x_0^{2N-2}(t)=N/\cosh^2(N-1)t. \tag{14}$$

The trace operation of Eq. (8) involves solutions such that $x(t_0)=x(t_0+\beta)$. In the large-β limit the solution (14) fulfills this requirement provided one takes a large interval symmetric around $t=0$. It is convenient to define

$$J=\int_{-\infty}^{+\infty}dt\,x_0^{2N}(t)$$
$$=\frac{N^{N/(N-1)}2^{(N+1)/(N-1)}}{N-1}\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}. \tag{15}$$

The corresponding argument of the exponential in Eq. (9) has a limit when the interval $[0,\beta]$ goes to infinity

$$\mathfrak{A}_c=\lim_{\beta\to\infty}\int_{-\beta/2}^{\beta/2}d\tau\tfrac{1}{2}(\dot{\vec{x}}_c^2+\vec{x}_c^2)-K\ln\int_{-\beta/2}^{\beta/2}d\tau[\vec{x}_c^2(\tau)]^N$$
$$=NK-NK\ln 2NK+K(N-1)\ln J. \tag{16}$$

The last part consists of quantizing the vibrations over one classical solution (13) and obtaining their leading effect. We thus differentiate the exponent (9) once more and obtain the stability matrix of the second derivatives at the saddle point,

$$M^{\alpha\beta}(t_1,t_2)=\frac{\delta^2}{\delta x^\alpha(t_1)\delta x^\beta(t_2)}\times\left[\int_0^\beta\frac{d\tau}{2}(\dot{\vec{x}}^2+\vec{x}^2)-K\ln\int_0^\beta d\tau(\vec{x}^2)^N\right], \tag{17}$$

$$M^{\alpha\beta}(t_1,t_2)=\left\{\left[-\frac{d^2}{dt_1^2}+1-\frac{N}{\cosh^2(N-1)t_1}\right]\delta^{\alpha\beta}-\frac{2(N-1)Nu^\alpha u^\beta}{\cosh^2(N-1)t_1}\right\}\delta(t_1-t_2)+2\frac{N}{J}u^\alpha u^\beta x_0^{2N-1}(t_1)x_0^{2N-1}(t_2)$$
$$(\alpha,\beta=1,\ldots,n),$$

which can be decomposed into a longitudinal and a transverse part,

$$M^{\alpha\beta}=M_Lu^\alpha u^\beta+M_T(\delta^{\alpha\beta}-u^\alpha u^\beta),$$

$$M_L(t_1,t_2)=\left[-\frac{d^2}{dt_1^2}+1-\frac{(2N-1)N}{\cosh^2(N-1)t_1}\right]\delta(t_1-t_2)+\frac{2N}{J}x_0^{2N-1}(t_1)x_0^{2N-1}(t_2), \tag{18}$$

$$M_T(t_1,t_2)=\left[-\frac{d^2}{dt_1^2}+1-\frac{N}{\cosh^2(N-1)t_1}\right]\delta(t_1-t_2). \tag{19}$$

The Gaussian integrals over the small fluctuations yield the inverse square roots of the determinants of these operators, but one has to extract first the collective motion of the classical solution (14), which should be properly quantized.

We first consider the transverse part. Let $\mathcal{H}$ be the one-dimensional Hamiltonian

$$\mathcal{H}=p^2-\frac{\lambda(\lambda+1)}{\cosh^2 x}. \tag{20}$$

The bound and scattering states of $\mathcal{H}$ are easily determined in terms of Jacobi functions (for integer values of λ this is a Bargmann potential[6]), and it is easy to show that

$$\frac{\det(\mathcal{H}-z)}{\det(\mathcal{H}_0-z)}=\frac{\Gamma(1+\sqrt{-z})\Gamma(\sqrt{-z})}{\Gamma(1+\lambda+\sqrt{-z})\Gamma(\sqrt{-z}-\lambda)}. \tag{21}$$

For the transverse problem we have to apply this formula for $\lambda = 1/(N-1)$, $-z = 1/(N-1)^2$. However, we must separate first the zero eigenvalues of M_T corresponding to a global rotation of the saddle-point solution (13); $x_0(t)$ is thus the corresponding eigenfunction. We then obtain, taking into account the 2π of the Gaussian integral which goes with every eigenvalue,

$$\lim_{z\to -1/(N-1)^2} \frac{2\pi}{-1/(N-1)^2 - z} \frac{\det(\mathcal{H}-z)}{\det(\mathcal{H}_0 - z)} \frac{1}{(N-1)^2} = 2\pi \frac{N+1}{2(N-1)} \frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}.$$

Since there are $n-1$ transverse directions, the total contribution of these modes is equal to

$$\left[2\pi \frac{N+1}{2(N-1)} \frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]^{-(n-1)/2}. \qquad (22)$$

There is an additional factor generated by the separation of this zero eigenmode. Indeed, the proper quantization around such solutions is by now standard.[7] If one chooses for instance the collective-coordinates method[7] we will separate the rotation mode by the change of variables $\vec{x}(t) \to (\vec{u}, \vec{c}_n)$,

$$\vec{x} = \vec{u} x_0(t) + \sum \vec{c}_n \psi_n(t), \qquad (23)$$

in which the $\vec{c}_n$ are coefficients on the orthonormal basis ψ_n corresponding to the degrees of freedom orthogonal to the rotations, whose contribution has been taken into account in Eq. (22). The Jacobian of this transformation is equal to the $(n-1)$th power of the norm of x_0,

$$\|x_0\| = \left[\int_{-\infty}^{+\infty} x_0^2(t)\,dt\right]^{1/2} = [K(N+1)]^{1/2}. \qquad (24)$$

The total contribution of the transverse modes is thus

$$\frac{2\pi^{n/2}}{\Gamma(n/2)} [K(N+1)]^{(n-1)/2} \left[\frac{\pi(N+1)}{N-1} \frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]^{-(n-1)/2}, \qquad (25)$$

in which the integration over the directions of $\vec{u}$ has given the factor $2\pi^{n/2}/\Gamma(n/2)$.

The longitudinal problem is similar; the operator M_L is a sum of the local operator

$$\bar{M}_L = \left[-\frac{d^2}{dt_1^2} + 1 - \frac{(2N-1)N}{\cosh^2(N-1)t_1}\right]\delta(t_1 - t_2), \qquad (26)$$

and of a one-dimensional projector. Again M_L has a zero eigenmode corresponding to time translations which should be properly quantized; the corresponding eigenfunction is thus dx_0/dt, which is orthogonal to the projector on x_0^{2N-1}. We can thus apply the formula

$$\det(\bar{M} + |u\rangle\langle u|) = (\det\bar{M})(1 + \langle u|\bar{M}^{-1}|u\rangle), \qquad (27)$$

since $|u\rangle$ is the subspace orthogonal to the zero eigenvalue of $\bar{M}$. Noting that

$$x_0^{2N-1}(t) = \frac{1}{2(1-N)} \bar{M}_L x_0,$$

we obtain

$$\det M_L = -\frac{1}{N-1} \det \bar{M}_L. \qquad (28)$$

We are thus led to the same soluble potential problem. However, $\bar{M}_L$ has now one negative eigenvalue corresponding to the ground state of the Hamiltonian problem. Thus, as expected (once we have extracted the zero mode) we are left with a positive $\det M_L$. We apply again Eq. (21), with now

$$\lambda = \frac{N}{N-1}$$

and obtain

$$\lim_{z\to -1/(N-1)^2} \frac{2\pi}{-z - 1/(N-1)^2} \frac{\det(\mathcal{H}-z)}{\det(\mathcal{H}_0 - z)} \frac{1}{(N-1)^2} = -2\pi \frac{1}{2} \frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}. \qquad (29)$$

The proper extraction of the time-translation mode involves here again a Jacobian proportional to the norm of dx_0/dt.

A simple calculation yields

$$\left\|\frac{dx_0}{dt}\right\| = \left[\int_{-\infty}^{+\infty} \left(\frac{dx_0}{dt}\right)^2 dt\right]^{1/2} = [K(N-1)]^{1/2}. \qquad (30)$$

Therefore the total contribution of the longitudinal modes is

$$\beta\left\{\left(-\frac{1}{N-1}\right)\left[-2\pi \frac{1}{2} \frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]\right\}^{-1/2} [K(N-1)]^{1/2}, \qquad (31)$$

in which integration over time translation yields the factor β.

This ends the calculation of the Kth order of $\mathrm{Tr}\,e^{-\beta H}/\mathrm{Tr}\,e^{-\beta H_0}$. The logarithm of this series is now to be taken, but it is easy to verify that the leading contribution of the coefficient of g^K in the series $\ln(1+\sum_1^\infty c_p g^p)$ is $c_K[1+O(1/K)]$ when the c_K's grow faster than $K!$ for large K. Dividing out by the factor $-\beta$ which appeared above, we collect the formulas (16), (25), (31) and end up with

$$(E-E_0)^{(K)}=(-)^{K+1}g^K\left(\frac{2}{\pi}\right)^{1/2}\frac{(N-1)^{(n+1)/2}}{\Gamma(n/2)}K^{(n-1)/2}2^{-K}\times\left[\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right]^{K(N-1)+n/2}\times\exp\left[K(N-1)\ln\frac{K(N-1)}{e}\right]\left[1+O\left(\frac{1}{K}\right)\right], \quad (32)$$

and if we set n equal to 1 in this result we recover the result of Ref. 1.

Let us now briefly describe an alternative method, which generalizes more easily to field theory, where the Kth order of perturbation theory is a mixture of interaction and counterterms. We have not done it directly since its significance is better understood through the previous calculations. Assume we do not integrate over g the contour integral (3) and consider the saddle point in the $\{g,\vec{x}(t)\}$ space. They are determined by the equations

$$\ddot{\vec{x}}_c=\vec{x}_c+2Ng_c\vec{x}_c(\vec{x}_c^{\,2})^{N-1}, \quad (33a)$$

$$\frac{K}{g_c}=-\int dt(\vec{x}_c^{\,2})^N. \quad (33b)$$

Note that g_c is negative, and that through the rescaling

$$\vec{x}_c\to(-2Ng_c)^{-1/(2N-2)}\vec{x}$$

Eq. (33a) becomes identical to (12), which thus explains the sign of g_c. The rest of the calculation is very similar. The only differences are at the following stages:

We have to integrate over fluctuations around g_c. If we rescale x by $(g_c/g)^{1/(2N-2)}$ there is no coupling between the $\vec{x}$-$\vec{x}_c$ and g-g_c fluctuations.

The transverse stability matrix is unchanged, but the longitudinal one is $\overline{M}_L$ and not M_L. This matrix has a negative eigenvalue. If we integrate formally over these fluctuations and take

$$(\det\overline{M}_L)^{1/2}=-i(-\det\overline{M}_L)^{1/2}, \quad (34)$$

we find exactly the results that were established by the previous method.

III. FIELD THEORY

Now that we understand the origin of the various factors which enter in the result, we are in a position to go to field theory. Let us first discuss what counterterms will be relevant at the order for which we are doing the present calculation. We take n massless scalar fields $\vec{\phi}$ coupled by a $g(\vec{\phi}^2)^N/(2N)!$ interaction. An arbitrary $2M$-point function will be evaluated by a functional integral

$$G^{(2M)}(x_1\cdots x_{2M})=\int\mathfrak{D}\phi\,\phi(x_1)\cdots\phi(x_{2M})e^{-\mathfrak{a}(\phi)}.$$

If we write

$$\frac{1}{2i\pi}\oint\frac{dg}{g^{K+1}}=\frac{(-)^K}{2i\pi}\oint\frac{dg}{g}e^{-K\ln(-g)},$$

the saddle points, as shown in the Introduction, are given by

$$\frac{K}{g_c}=-\frac{1}{(2N)!}\int d^dx(\vec{\phi}_c^{\,2})^N, \quad (35)$$

$$\partial^2\vec{\phi}_c=\frac{g_c}{(2N-1)!}\vec{\phi}_c(\vec{\phi}_c^{\,2})^{N-1}, \quad (36)$$

and if we perform the rescaling

$$\vec{\phi}_c=\left[\frac{(2N-1)!}{-g_c}\right]^{1/(2N-2)}\vec{\psi}_c \quad (37)$$

so that

$$\partial^2\vec{\psi}_c=-\vec{\psi}_c(\vec{\psi}_c^{\,2})^{N-1}, \quad (38)$$

we see that $-g_c$ is of order $(1/K)^{N-1}$ and $\vec{\phi}_c$ of order $K^{1/2}$.

Therefore the only counterterms relevant, as long as we do not go beyond the order of Eq. (1), are $g(\vec{\phi}^2)^{N-1}$ for $N>2$, and in addition $g^2(\vec{\phi}^2)^2$ for $N=2$. Any higher counterterm would yield negative powers of K. We thus treat first the interactions with $N>2$ and discuss afterwards the ϕ^4 theory.

In dimension $d=2N/(N-1)$, in which the theory is renormalizable, the massless Eq. (38) admits conformally covariant, spherically symmetric solutions of finite action which are[2]

$$\vec{\psi}_c(\vec{x})=\vec{u}\lambda^{(d-2)/2}\psi_0(\lambda(\vec{x}-\vec{a})), \quad (39)$$

$$\psi_0(\vec{x})=\left[\frac{4N}{(N-1)^2}\right]^{1/2(N-1)}(1+x^2)^{-1/(N-1)}. \quad (40)$$

The possible existence of solutions of finite action in lower dimension will be discussed below. The solutions (39) are labeled by an arbitrary translation a, dilatation λ, and direction in internal space $\vec{u}$. Let us note that for these solutions

$$J\equiv\int(\partial_\mu\vec{\psi}_c)^2d^dx=\int(\psi_c^{\,2})^Nd^dx=\left[\frac{4N\pi}{(N-1)^2}\right]^{N/(N-1)}\frac{\Gamma(N/(N-1))}{\Gamma(2N/(N-1))}. \quad (41)$$

If we consider an arbitrary $2M$-point Green's function, the first contribution to the result is thus equal to the exponential of the action at the saddle point (without the counterterm) which, noting that $-g_c = (2N-1)!\,(J/2NK)^{N-1}$, is equal to

$$C_1 \equiv \exp\left[-K(N-1) + K(N-1)\ln\frac{2NK}{J} - K\ln(2N-1)!\right]. \qquad (42)$$

1. Small oscillations around the saddle point

We ignore at present the possible ultraviolet divergences and postpone their discussion to the stage where the question will arise.

If we make the change of variables

$$\phi = \left(\frac{-g_c}{-g}\right)^{1/(2N-2)}\psi,$$

the oscillations of g and ψ decouple. The oscillations around g_c, together with the $1/g_c$ left over from the g integration, give a contribution equal to

$$C_2 \equiv \left[\frac{2\pi}{K}(N-1)\right]^{1/2}. \qquad (43)$$

Small oscillations around $\vec{\phi}_c$ lead again to transverse and longitudinal modes corresponding to the operators

$$M_T = \left[-\partial^2 - \frac{4N}{(N-1)^2}\frac{1}{(1+r^2)^2}\right]\delta^d(r-r'), \qquad (44)$$

$$M_L = \left[-\partial^2 - \frac{4N(2N-1)}{(N-1)^2}\frac{1}{(1+r^2)^2}\right]\delta^d(r-r'), \qquad (45)$$

in which the differential operator ∂^2 is in spherical coordinates

$$\partial^2 = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{L^2}{r^2}. \qquad (46)$$

The Casimir operator of the d-dimensional rotation group has the eigenvalues

$$L^2 = l(l+d-2), \quad l = 0, 1, \ldots, \qquad (47)$$

with a degeneracy

$$\delta_l = \frac{\Gamma(l+d-2)(d+2l-2)}{\Gamma(d-1)\Gamma(l+1)}. \qquad (48)$$

In order to determine the eigenvalues of the radial problems it is convenient to make the change of variables and of function

$$\psi(r) = r^{-(d-2)/2}\chi(e^t), \qquad (49)$$

in which dilatations are represented by t translations.

The corresponding local operators in t are for a given l state

$$P_T^l = -\frac{d^2}{dt^2} + \left(l + \frac{1}{N-1}\right)^2 - \frac{N}{(N-1)^2}\frac{1}{\cosh^2 t}, \qquad (50)$$

$$P_L^l = -\frac{d^2}{dt^2} + \left(l + \frac{1}{N-1}\right)^2 - \frac{N(2N-1)}{(N-1)^2}\frac{1}{\cosh^2 t}, \qquad (51)$$

in which we have defined the P operators by

$$P = r^{(d-2)/2-2} M r^{-(d-2)/2}. \qquad (52)$$

We are led again to the same one-dimensional potential problems as in Sec. II and we have to use Eq. (21) in order to obtain the determinants of these operators.

The transverse operator has one zero eigenvalue for $l=0$ corresponding to the rotational collective motion, as in Sec. II, which has to be eliminated. Its eigenfunction is proportional to the transform by (49) of ψ_0 of Eq. (40) and it has a norm equal to $[K(N^2-1)(N-1)]^{1/2}$. Thus we obtain from the $l=0$ transverse contribution

$$C_3 \equiv \frac{2\pi^{n/2}}{\Gamma(n/2)}\left\{[K(N^2-1)(N-1)]^{1/2} \times\left[2\pi\,\frac{(N^2-1)}{2}\,\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]^{-1/2}\right\}^{(n-1)}. \qquad (53)$$

For $l \geq 1$ there is no spurious eigenvalue; we thus obtain a contribution which is

$$\prod_{l=1}^{\infty}\left(\frac{\Gamma^2(l+N/(N-1))}{\Gamma(l)\Gamma(l+2N/(N-1))}\,\frac{l+(N+1)/(N-1)}{l+1/(N-1)}\right)^{-\delta_l(n-1)/2}, \qquad (54)$$

which results of application of Eq. (21) with $-z = [l+1/(N-1)]^2$ and $\lambda = 1/(N-1)$; δ_l is the degeneracy given by Eq. (48).

For the longitudinal operators we have to treat separately the first two l's:

(i) $l=0$. The corresponding P_L^0 operator has one negative eigenvalue as in Sec. II, one spurious zero eigenvalue corresponding to the dilatation mode, with an eigenfunction proportional to $\sinh t(\cosh t)^{-d/2}$. Once this mode has been extracted, we obtain by application of (21) a contribution equal to

$$iC_4 \equiv [K(N-1)]^{1/2} i\left[\frac{2\pi}{2}(N-1)^2\,\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]^{-1/2}. \qquad (55)$$

(ii) $l=1$. There is now a collective translation mode. The eigenfunction is proportional to $(\cosh t)^{-N/(N-1)}$, and gives a Jacobian in the collective-coordinates change of variables equal to $\lambda[K(N-1)/N]^{1/2}$. We obtain then the contribution

$$\lambda^d C_5 \equiv \left\{\lambda[K(N-1)/N]^{1/2} \times\left[2\pi\,\frac{N-1}{4N}\,\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]^{-1/2}\right\}^{d} \qquad (56)$$

in which λ is the arbitrary scale of the solution (39).

(iii) $l \geq 2$. There is no zero eigenvalue, and the result is

$$\prod_{l=2}^{\infty}\left\{\frac{\Gamma^2(l+N/(N-1))}{[l+1/(N-1)]\Gamma(l-1)\Gamma(l+2N/(N-1))}\right\}^{-\delta_l/2}. \tag{57}$$

Before we go on let us summarize what has been obtained up to now: We have explicitly calculated the contribution coming from the saddle point and from the harmonic vibrations around it. The spurious zero eigenvalues of rotation, translation, and dilatation has been extracted. It remains though to evaluate the integrals over the collective coordinates of translation and dilatation. The integral over the translation may be evaluated without difficulty, and this restores translation invariance; however, before we integrate over dilatation we have to consider the renormalization problems.

2. Renormalization

The infinite products (54) and (57) are in fact divergent for large l; this calculation has been up to now meaningless since we have neither regularized nor renormalized the theory. Any regularization would cut off the large-l contribution to (54), (57). Assume that this has been done in some way, for instance, by adding the regulator $(1/2\Lambda^2)\vec{\phi}\cdot(\partial^2)^2\vec{\phi}$ to the action. The cutoff dependence will be removed by the addition of the one-loop counterterm proportional to $g(\vec{\phi}^2)^{N-1}$ which makes the $(2N-2)$-point function vanish at zero momentum. We are going to evaluate this term at the saddle point in the l basis, which we have used above, and multiply it by the one-loop Feynman diagram which goes with it. This may be conveniently done by the following procedure. This one-loop counterterm may be found by expanding to first order in g the operator given by the second derivative of the action

$$-\tfrac{1}{2}\mathrm{Tr}\ln\left\{\left[-\partial^2+g\frac{(\phi_c{}^2)^{N-1}}{(2N-1)!}\right]\delta_{\alpha\beta}+g\frac{2(N-1)}{(2N-1)!}\phi_{c\alpha}\phi_{c\beta}(\phi_c{}^2)^{N-2}\right\}$$
$$=-\tfrac{1}{2}\mathrm{Tr}\ln\left[-\partial^2+g\frac{(\phi_c{}^2)^{N-1}}{(2N-2)!}\right]-\left(\frac{n-1}{2}\right)\mathrm{Tr}\ln\left[-\partial^2+\frac{g(\phi_c{}^2)^{N-1}}{(2N-1)!}\right] \tag{58}$$

in which ∂^2 stands, in fact, for $\partial^2-(1/\Lambda^2)(\partial^2)^2$.

In the l representation this calculation may be done by expanding in powers of X the operator

$$\sum_{l=0}^{\infty}\delta_l\left\{-\frac{n-1}{2}\mathrm{Tr}\ln\left[-\frac{d^2}{dt^2}+\left(l+\frac{1}{N-1}\right)^2-X\frac{N}{(N-1)^2}\frac{1}{\cosh^2 t}\right]-\tfrac{1}{2}\mathrm{Tr}\ln\left[-\frac{d^2}{dt^2}+\left(l+\frac{1}{N-1}\right)^2-X\frac{N(2N-1)}{(N-1)^2\cosh^2 t}\right]\right\} \tag{59}$$

to first order in X and setting X equal to 1 (regularization is meant here again). This calculation may be done easily again with Eq. (21), which gives that expression (59), to first order in X, reduces to

$$\delta A=\sum_{l=0}\delta_l\frac{n-2+2N}{2[l+1/(N-1)]}\frac{N}{(N-1)^2}, \tag{60}$$

which has to be added to the classical action since this is the one-loop counterterm at the saddle point. If we take into account (54), (57) together with the exponential of $-\delta A$ given by (60), we find the sum

$$\exp\left(\sum_{l=2}^{\infty}-\tfrac{1}{2}\delta_l\left\{(n-1)\ln\left[\frac{\Gamma^2(l+N/(N-1))}{\Gamma(l)\Gamma(l+2N/(N-1))}\frac{l+(N+1)/(N-1)}{l+1/(N-1)}\right]+\ln\left[\frac{\Gamma^2(l+N/(N-1))}{\Gamma(l-1)\Gamma(l+2N/(N-1))}\frac{1}{l+1/(N-1)}\right]\right.\right.$$
$$\left.\left.+\frac{n-2+2N}{l+1/(N-1)}\frac{N}{(N-1)^2}\right\}\right)\exp\left(-\left\{\frac{(n-1)N}{N-1}\ln\left[\frac{N}{N-1}\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\right]+\frac{(n-2+2N)N(N+1)}{2(N-1)^2}\right\}\right). \tag{61}$$

In going to expression (61) we have let the cutoff go to infinity before summing over l, and it is elementary to verify that for $N>2$ the sum over l is now, as expected, convergent. This may be seen more easily if we simplify the expression (61). Using the identity

$$\delta_l(d+1)-\delta_{l-1}(d+1)=\delta_l(d), \tag{62}$$

the expression (61) takes the simpler form

$$C_6 \equiv \exp\Bigg(-\tfrac{1}{2}\sum_{l=2}^{\infty}\frac{\Gamma(l+(N+1)/(N-1))[2l+(N+1)/(N-1)]}{l!\,\Gamma(2N/(N-1))}\times\Bigg\{(n-1)\ln\frac{l[l+(N+1)/(N-1)]}{[l+1/(N-1)][l+N/(N-1)]}+\ln\frac{(l-1)[l+2N/(N-1)]}{[l+1/(N-1)][l+N/(N-1)]}+(n-2+2N)\frac{N}{(N-1)^2}\frac{1}{[l+1/(N-1)][l+N/(N-1)]}\Bigg\}\Bigg)$$
$$\times\exp\Bigg\{\frac{3N-1}{2(N-1)}\left[\ln\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\frac{N(2N-1)}{2(N-1)(3N-1)}+(n-1)\ln\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}\frac{N(2N-1)}{2(N-1)^2}\right]-\left[\frac{(n-1)N}{N-1}\ln\frac{N}{N-1}\frac{\Gamma^2(N/(N-1))}{\Gamma(2N/(N-1))}+\frac{(n-2+2N)N^2}{(N-1)(2N-1)}\right]\Bigg\}. \tag{63}$$

3. Integration over translations and dilatation

The regularization breaks the scale invariance of the classical action, and will cutoff the integration over dilatations.

Consider first the $2N$-point function

$$\tilde G_{\alpha_1\cdots\alpha_{2N}}(p_1\cdots p_{2N})(2\pi)^d\delta(\textstyle\sum p_i)=\mathfrak{N}\displaystyle\int\prod_1^{2N}dx_i e^{i\Sigma p_i x_i}\int\mathfrak{D}\phi\,\phi_{\alpha_1}(x_1)\cdots\phi_{\alpha_{2N}}(x_{2N})e^{-\mathfrak{A}(\phi)}. \tag{64}$$

In addition to the previous factors, for a given scale λ and origin a we obtain a contribution proportional to

$$\prod_1^{2N}\lambda^{(d-2)/2}\phi^c_{\alpha_i}(\lambda(\vec{x}_i-\vec{a})). \tag{65}$$

It remains to integrate over $\vec{a}$, and over λ with the measure $d\lambda/\lambda$ [corresponding to t-translation invariance of the P operators of Eqs. (50),(51)]. We collect from (56) and from the regularization a factor $\lambda^d\exp(-\rho^2\lambda^2/\Lambda^2)$, with

$$\rho^2\equiv\tfrac{1}{2}\int(\partial^2)^2\vec{\phi}_c^{\,2}(x)d^dx. \tag{66}$$

This yields, collecting Eqs. (42),(43),(53),(55),(56),(63),

$$(2\pi)^d\delta(\textstyle\sum p_i)\tilde G_{(K)}=\dfrac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\displaystyle\int d\vec{a}\frac{d\lambda}{\lambda}\lambda^d\exp\left(-\rho^2\frac{\lambda^2}{\Lambda^2}\right)\times\int\prod_1^{2N}dx_i e^{i\Sigma\vec{p}_i\cdot\vec{x}_i}\prod_1^{2N}\lambda^{(d-2)/2}\phi^c(\lambda(\vec{x}_i-\vec{a}))\langle u_{\alpha_1}\cdots u_{\alpha_{2N}}\rangle, \tag{67}$$

in which $\langle u_{\alpha_1}\cdots u_{\alpha_{2N}}\rangle$ stands for the O(n) rotation-group tensor

$$\langle u_{\alpha_1}\cdots u_{\alpha_{2N}}\rangle=\frac{\Gamma(n/2)}{2\pi^{n/2}}\int d\hat{u}\,u_{\alpha_1}\cdots u_{\alpha_{2N}}=\frac{1}{\sqrt{\pi}}\frac{2^N N!}{2N!}\frac{\Gamma(N+\frac{1}{2})\Gamma(n/2)}{\Gamma(N+n/2)}\sum_{\text{all possible pairings}}\delta_{\alpha_1\alpha_i}\cdots\delta_{\alpha_j\alpha_k}. \tag{68}$$

From (67), extracting the obvious O(n) tensor structure (68), we get

$$\tilde G_{(K)}(p_1\cdots p_{2N})=\frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^{\infty}\frac{d\lambda}{\lambda}e^{-\rho^2\lambda^2/\Lambda^2}\prod_1^{2N}\left[\tilde\phi_c\left(\frac{p_i}{\lambda}\right)\lambda^{-2}\right], \tag{69}$$

in which

$$\tilde\phi_c(\vec{p})=\pi^{-N/2(N-1)}\left[\frac{K(N-1)^2}{2}\frac{\Gamma(2N/(N-1))}{\Gamma(N/(N-1))}\right]^{1/2}\int d^dx\,\frac{e^{i\vec{p}\cdot\vec{x}}}{(1+x^2)^{1/(N-1)}}. \tag{70}$$

Let us note that

$$\int d^d x \frac{e^{i\vec{p}\cdot\vec{x}}}{(1+x^2)^{1/(N-1)}} = \frac{\pi^{N/(N-1)}}{\Gamma(1/(N-1))}\int_0^\infty dt\, e^{-tp^2/4-1/t} \equiv \frac{\pi^{N/(N-1)}}{\Gamma(1/(N-1))}\frac{4}{|\vec{p}|}K_1(|\vec{p}|), \tag{71}$$

in which the modified Bessel function K_1 decreases exponentially for large $|\vec{p}|$, but for small p behaves as

$$K_1(p) = \frac{1}{p}\left[1 + \frac{p^2}{2}(\ln p + \gamma - \tfrac{1}{2}) + O(p^4 \ln p)\right]. \tag{72}$$

From this formula we see that the λ integral (69) diverges logarithmically when Λ goes to infinity as

$$\tilde{G}_{(K)}(p_1\cdots p_{2N}) = \frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\pi^{N^2/(N-1)}\left[8K\frac{\Gamma(2N/(N-1))}{\Gamma^3(N/(N-1))}\right]^N\left(\prod_1^{2N}\frac{1}{p_i^2}\right)(\ln\Lambda + \text{constant}). \tag{73}$$

A few comments concerning this result should be made. First, at leading order in K there is no contribution from the disconnected diagrams. Secondly, the Green's function

$$\Gamma_{(K)}^{(2N)} = \tilde{G}_{(K)}(p_1\cdots p_{2N})\prod_1^{2N} p_i^2 \tag{74}$$

in which the external legs have been removed is automatically one-particle irreducible, again for K large. Finally, the leading diagrams at order K give a single power of $\ln\Lambda$, they are those which do not involve any divergent subgraph; i.e., they are the completely irreducible diagrams.

This logarithmic divergence is removed by a coupling renormalization. If we define the renormalized coupling by some prescription, for instance, as the value of $\Gamma^{(2N)}$ for all internal indices equal, at the symmetry point

$$p_i p_j = \frac{\mu^2}{2N-1}(2N\delta_{ij} - 1), \tag{75}$$

we obtain

$$\Gamma_{(K)}^{(2N)}(p_1\cdots p_{2N}) = \left(\frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^\infty \frac{d\lambda}{\lambda}\left\{\left[\prod_1^{2N}\left(\frac{p_i}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{p_i}{\lambda}\right)\right] - \left[\left(\frac{\mu}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{\mu}{\lambda}\right)\right]^{2N}\right\}\right)\left[1 + O\left(\frac{1}{K}\right)\right]. \tag{76}$$

Let us consider now a function with less than $2N$ points; for instance, the propagator, through similar calculations, takes the form

$$\tilde{G}_{(K)}^{(2)}(p) = \frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^\infty \frac{d\lambda}{\lambda}e^{-p^2\lambda^2/\Lambda^2}\lambda^2\left[\tilde{\phi}_c\left(\frac{p}{\lambda}\right)\lambda^{-2}\right]^2. \tag{77}$$

Again it behaves as $1/p^4$ for p small, which shows that there is no contribution from one-particle-reducible subgraphs, and thus the inverse propagator $\Gamma^{(2)}$ is

$$\Gamma_{(K)}^{(2)}(p) = -\frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^\infty d\lambda\, e^{-p^2\lambda^2/\Lambda^2}\lambda\left[\tilde{\phi}_c\left(\frac{p}{\lambda}\right)\frac{p^2}{\lambda^2}\right]^2. \tag{78}$$

Mass renormalization gives

$$\Gamma_{(K)}^{(2)}(p) = -\frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^\infty d\lambda\, e^{-p^2\lambda^2/\Lambda^2}\lambda\left\{\left[\frac{p^2}{\lambda^2}\tilde{\phi}_c\left(\frac{p}{\lambda}\right)\right]^2 - \lim_{x\to 0}[x^2\tilde{\phi}_c(x)]^2\right\}, \tag{79}$$

which is proportional to $(\ln\Lambda)^2$ for large Λ, as one can see from (72). This means that the leading contribution at large K to wave-function renormalization is proportional to $\ln^2\Lambda$. We shall come back to this point below.

There are no divergences in the λ integration for a $2M$-point function with $M > N$, and the result is

$$\Gamma_{(K)}^{(2M)}(p_1\cdots p_{2M}) = -\frac{(-1)^{K+1}}{2\pi}C_1\cdots C_6\int_0^\infty \frac{d\lambda}{\lambda^{2(M-N)/(N-1)+1}}\prod_1^{2M}\left[\frac{p_i^2}{\lambda^2}\tilde{\phi}_c\left(\frac{p_i}{\lambda}\right)\right], \quad M > N. \tag{80}$$

4. Renormalization-group equations

The previous results will allow us to calculate the coefficients of g^K, for large K, of the functions which appear in the renormalization-group equations

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - M\gamma(g)\right]\Gamma^{(2M)} = 0. \tag{81}$$

We first notice that, since $\vec{\phi}_c$ is proportional to $K^{1/2}$, the wave-function renormalization will not modify at this order the result (76) for the renormalized vertex function. Indeed, this means that the coefficients for the perturbation series of $\Gamma^{(2N)}$ are not modified at leading order in K by division by $[\Gamma^{(2)}(p)/p^2]^{2N}$. It is then easy to derive that

$$\beta(g)=\sum_2^\infty \beta_K g^K, \tag{82}$$

has the behavior[8]

$$\beta_K \underset{K\to\infty}{\simeq} (-1)^K \frac{1}{2\pi} C_1 \cdots C \lim_{6\, x\to 0}[x^2\vec{\phi}_c(x)]^{2N}. \tag{83}$$

This may be seen, for instance, by demanding that $\Gamma^{(2N)}$, given by (76), fulfills (81) with no leading contribution due to $\gamma(g)$. In fact, Eq. (76) involves the bare coupling constant. However, when we substitute the renormalized it modifies Eq. (81) only at order $1/K$.

The calculation of the large-K behavior of the expansion of $\gamma(g)$ requires going beyond the order considered in this work. The leading order vanishes as implied by the $\ln^2\Lambda$ behavior of the field-strength renormalization constant. Indeed,

$$\gamma(g)=\mu\frac{\partial}{\partial\mu}\bigg|_{g_0\Lambda}\ln Z,$$

and

$$\ln Z = c_2 g_0^2 \ln\frac{\Lambda}{\mu}+\cdots+\sum_K c_K g_0^K\left(\ln\frac{\Lambda}{\mu}\right)^2,$$

$$\gamma(g)=c_2 g_0^2+\cdots+\sum_K 2c_K g_0^K\ln\frac{\Lambda}{\mu}.$$

If we substitute the renormalized coupling constant

$$g_0=g+\cdots+\sum_K d_K g^K\ln\frac{\Lambda}{\mu},$$

the leading contribution to $\gamma(g)$ is

$$\gamma(g)=c_2g^2+\cdots+\sum_K g^K\ln\frac{\Lambda}{\mu}(2c_K+2c_2d_{K-1}).$$

The finiteness of $\gamma(g)$ when Λ goes to infinity implies that the coefficient of $\ln(\Lambda/\mu)$ vanishes, as may be checked from the explicit expressions given above.

IV. ϕ^4 THEORY

There are two slight additional complications for the four-dimensional field theory. First, we need to take into account the one-loop counterterm of coupling renormalization of the form $g^2\int(\vec{\phi}^2)^2d^4x$. We shall see that this term subtracts the remaining infinity which appears in the sums (63) when N goes to 2. Second, the mass counterterm $g\int\vec{\phi}^2d^4x$ gives a fictitious infrared divergence at the saddle point. We should, in principle, repeat the calculations with an infrared regulator. However, it will be shown that this divergence disappears by going to the representation (l,t) used above. In fact, ϕ^{2N} with $N>2$ may be regarded as an infrared regularization of ϕ^4.

The form of the coupling-constant counterterm depends on the renormalization prescriptions. We will make a particular choice which simplifies the present calculation. For any other choice, the coefficient c of Eq. (1) would be modified by a finite calculable amount. The easiest one-loop counterterm results from expansion of (58) up to second order in g, i.e., it is

$$\frac{\mu_0^{(d-4)}}{4}\left[1+\frac{n-1}{(2N-1)^2}\right] \times \mathrm{Tr}\left[\frac{1}{(-\partial^2)}g\frac{(\vec{\phi}_c^{\,2})^{N-1}}{(2N-2)!}\frac{1}{(-\partial^2)}g\frac{(\vec{\phi}_c^{\,2})^{N-1}}{(2N-2)!}\right], \tag{84}$$

in which μ_0 is some arbitrary scale. This will yield a one-loop counterterm for the $(4N-4)$-point function which will subtract the infinity of the 4-point function when N goes to 2. The function $\vec{\phi}_c$ in Eq. (84) depends of the dilatation parameter λ, as in Eq. (39). Thus after rescaling, from the expansion of (59) up to second order and through the identity (62), we obtain now

$$\delta\mathfrak{a}=-\frac{1}{4}\left(\frac{\lambda}{\mu_0}\right)^{4-d}\left[1+\frac{n-1}{(2N-1)^2}\right]\sum_0^\infty\frac{\Gamma(l+(N+1)/(N-1))[2l+(N+1)/(N-1)]}{l!\,\Gamma(2N/(N-1))}\left[\frac{N^2(2N-1)^2}{(N-1)^4}\right] \times\frac{1}{[l+1/(N-1)]^2[l+N/(N-1)]^2}. \tag{85}$$

If we separate $\delta\mathfrak{a}$ into two parts through

$$\left(\frac{\lambda}{\mu_0}\right)^{4-d}=1+\left[\left(\frac{\lambda}{\mu_0}\right)^{4-d}-1\right],$$

the first term subtracts the infinity of the sum (63) when N goes to 2, and yields a modified C_6 factor

$$\tilde{C}_6 = \exp\left\{+2(n-1)\ln 3 - \tfrac{4}{3}(n+2) - \tfrac{5}{2}\ln 5 - \tfrac{5}{2}n\ln 2 - \tfrac{7}{18}(n+8) - \tfrac{1}{12}\sum_2^\infty (l+1)(l+2)(2l+3)\left[\ln\frac{(l-1)(l+4)}{(l+1)(l+2)} + (n-1)\ln\frac{l(l+3)}{(l+1)(l+2)} + \frac{2(n+2)}{(l+1)(l+2)} + \frac{2(n+8)}{(l+1)^2(l+2)^2}\right]\right\}. \tag{86}$$

The second term of $\delta\alpha$ gives a finite limit when d goes to 4 equal to

$$\left(\frac{n+8}{3}\right)\ln\frac{\lambda}{\mu_0}. \tag{87}$$

The resulting integral over dilatations is thus modified and gives instead of (76) the expression

$$\Gamma^{(4)}_{(K)}(p_1\cdots p_4) = \frac{(-1)^{K+1}}{2\pi}C_1\cdots\tilde{C}_6\int_0^\infty \frac{d\lambda}{\lambda}\exp\left(-\frac{n+8}{3}\ln\frac{\lambda}{\mu_0}\right)\left\{\prod_1^4\left[\left(\frac{p_i}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{p_i}{\lambda}\right)\right] - \left[\left(\frac{\mu}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{\mu}{\lambda}\right)\right]^4\right\}. \tag{88}$$

The coupling constant g, which multiplies (88), is partially renormalized since we have only performed one-loop subtractions and it depends on the arbitrary scale μ_0.

The renormalized coupling constant is defined in terms of $\Gamma^{(4)}$ at the symmetry point μ; thus

$$g_R = g + \alpha g^2 + \cdots + \frac{g^K(-1)^{K+1}}{2\pi}C_1\cdots\tilde{C}_6\int_0^\infty\frac{d\lambda}{\lambda}\exp\left(-\frac{n+8}{3}\ln\frac{\lambda}{\mu_0} - \frac{\rho^2\lambda^2}{\Lambda^2}\right)\left[\left(\frac{\mu}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{\mu}{\lambda}\right)\right]^4, \tag{89}$$

in which $\alpha(\mu_0/\mu)$ is a finite number which we need now to calculate. This may be done by computing the $(4N-4)$-point function at the one-loop order, taking into account the counterterm (84), in the limit d goes to 4. The result is

$$\alpha = -\frac{n+8}{6}\frac{1}{(2\pi)^4}\int d^4p\left[\frac{1}{p^2(p+q)^2} - \frac{1}{p^2}\frac{\int d^4k u^2(k)/(p+k)^2}{\int d^4k\, u^2(k)}\right] \tag{90}$$

taken for $q^2 = \frac{4}{3}\mu^2$, in which $u(k)$ is the Fourier transform of $\psi_c^2(x)$:

$$u(k) = \frac{1}{(2\pi)^2}\int d^4x\, e^{ikx}\frac{8\mu_0^2}{(1+\mu_0^2x^2)^2}. \tag{91}$$

This gives

$$\alpha = -\frac{n+8}{\pi^2}2^{-9}\int_0^\infty dk\, k^3u^2(k)\ln\frac{3k^2}{4\mu^2}, \tag{92}$$

and it is simple to derive

$$\alpha = -\frac{n+8}{96\pi^2}\left(\tfrac{1}{3} - \ln\frac{\mu^2}{3\mu_0^2} - 2\gamma\right), \tag{93}$$

in which $\gamma = 0.577\ldots$ is Euler's constant.

We can now proceed to the calculation of the β function. The first step is to calculate it in terms of the intermediate constant g:

$$\beta(g_R) = \beta_2 g^2 + \cdots + \sum_K (-1)^{K+1}\frac{g^K}{2\pi}C_1\cdots\tilde{C}_6\int_0^\infty\frac{d\lambda}{\lambda}\exp\left(-\frac{n+8}{3}\ln\frac{\lambda}{\mu_0}\right)\mu\frac{d}{d\mu}\left[\left(\frac{\mu}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{\mu}{\lambda}\right)\right]^4, \tag{94}$$

and to replace g by g_R. This will generate an additional factor. Indeed, let us write

$$\beta(g) = \sum_2^\infty \beta_K g^K,$$

in which

$$\beta_K \underset{K\to\infty}{\sim} K!\,a^K K^b c,$$

$$g = g_R - \alpha g_R^2 + \cdots.$$

It is easy to verify that

$$\beta(g_R) = \sum_2^\infty \tilde{\beta}_K g_R^K,$$

in which, for K large,

$$\tilde{\beta}_K = \beta_K e^{-\alpha/a}.$$

Thus, taking α from (93) and $a = -1/16\pi^2$ from (42), we obtain

$$\beta(g_R)=\beta_2 g_R^2+\cdots+\sum_K(-1)^{K+1}\frac{g^K}{2\pi}C_1\cdots\tilde{C}_6\exp\left[-\frac{n+8}{6}(\tfrac{1}{3}+\ln 3-2\gamma)\right]\int_0^\infty\frac{d\lambda}{\lambda}\mu\frac{d}{d\mu}\left[\left(\frac{\mu}{\lambda}\right)^2\tilde{\phi}_c\left(\frac{\mu}{\lambda}\right)\right]^4\exp\left(-\frac{n+8}{3}\ln\frac{\lambda}{\mu}\right). \tag{95}$$

If we use the explicit formula (71) for $\tilde{\phi}$ we end up with

$$\beta(g_R)=\beta_2 g_R^2+\cdots+\sum_K(-1)^K g_R{}^K\frac{1}{2\pi}C_1\cdots\tilde{C}_6\exp\left[-\frac{n+8}{6}(\tfrac{1}{3}+\ln 3-2\gamma)\right](4\pi)^4(3K)^2\frac{n+8}{3}$$
$$\times\int_0^\infty dx\,x^{(n+8)/3+3}K_1{}^4(x). \tag{96}$$

A few words about the dependence of β on the renormalization conditions is in order. Assume we have chosen two different renormalization schemes with coupling constants g_1 and g_2:

$$\beta_1(g_1)=\beta g_1{}^2+\cdots+\beta_{1,K}g_1{}^K+\cdots,$$
$$\beta_2(g_2)=\beta g_2{}^2+\cdots+\beta_{2,K}g_2{}^K+\cdots, \tag{97}$$

and

$$g_1=g_2+\delta_2 g_2{}^2+\cdots+\delta_K g_2{}^K+\cdots. \tag{98}$$

From the previous calculations we know that

$$\beta_{1,K}\underset{K\to\infty}{\simeq}K!\,a^K K^b C_1,$$
$$\beta_{2,K}\simeq K!\,a^K K^b C_2, \tag{99}$$
$$\delta_K\simeq K!\,a^K K^b\delta.$$

Expressing the covariance law

$$\beta_1(g_1)=\beta_2(g_2)\frac{\partial g_1}{\partial g_2} \tag{100}$$

we obtain

$$\beta_{1,K}e^{\delta_2/a}\simeq\left(\beta_{2,K}+\beta\frac{\delta_K}{a}\right)[1+O(1/K)]. \tag{101}$$

It is therefore easy to go from scheme (2) to scheme (1) provided we know how to relate g_1 to g_2.

V. IMPLICATIONS FOR CRITICAL PHENOMENA

Critical behavior for ferromagnets is related to the infrared-stable fixed point of $(\vec{\phi}^2)^2$ below four dimensions. It is possible to calculate various physical quantities like critical exponents, in powers of $\epsilon=4-d$, as first shown by Wilson and Fisher.[3] From numerical grounds it was expected that this expansion was only asymptotic.[9] The previous results give a proof of this fact. Indeed, let us consider first the expansions of the infrared-stable fixed point $g^*(\epsilon)$. If we use dimensional regularization[10] to calculate the β function (which we called W in previous work connected with statistical mechanics) it has the structure

$$\beta(g,\epsilon)=-\epsilon g+\tilde{\beta}(g), \tag{102}$$

in which $\tilde{\beta}$ is the four-dimensional function. In the preceding section it was shown that, for any definition of β, one has

$$\tilde{\beta}(g)=\cdots+\sum_{K\text{ large}}g^K K!\left(-\frac{1}{16\pi^2}\right)^K K^{3+n/2}C\left[1+O\left(\frac{1}{K}\right)\right]. \tag{103}$$

Therefore solving $\beta(g^*,\epsilon)=0$ one finds asymptotically

$$g^*(\epsilon)=\cdots+\sum_{K\text{ large}}\epsilon^K K!\left(-\frac{3}{n+8}\right)^K K^{4+n/2}C'. \tag{104}$$

As a consequence the critical exponents $1/\nu$ and η have the same character, governed in fact by the lowest-order contribution to the functions $1/\nu(g)$, $\eta(g)$, for $g=g^*$.[11]

These results show the divergence of the ϵ expansion; however, it indicates its Borel summability and one should try to calculate critical quantities through Padé-Borel transforms or more sophisticated transforms which incorporate the K^b factor. Preliminary results for a "bad" ϵ series like $\omega(\epsilon)$ indicate that an accuracy of a few percent can now be expected from three terms of the ϵ expansion.

Similarly, three-dimensional calculations have been performed by computing the coefficients of the three-dimensional Callan-Symanzik equations.[12,4] In this three-dimensional case the solutions to the classical field equations can only be obtained numerically. From the leading solution one finds

$$\beta(g)=\cdots+\sum K!\,g^K(-\pi\,36.091\ldots)^{-K}K^{3+n/2}C. \tag{105}$$

If we use the normalization of Ref. 4 this means

$$\beta(v)=-v+v^2+\cdots+\sum K!\,v^K(-0.148\ldots)^K K^{3+n/2}C. \tag{106}$$

The other two functions from which critical exponents are calculated have exactly the same structure as Eq. (106). This justifies the use of the Padé-Borel transformation of Ref. 4.

The application of these methods to a general situation in nonrelativistic quantum mechanics and other field theories will be discussed in a subsequent publication.[13]

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