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Properties of a 20-component spin-1/2 relativistic wave equation

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A new 20-component relativistic wave equation is studied. It is shown that this manifestly Lorentz-covariant equation describes particles with unique mass m and spin 1/2. A nonsingular η matrix is also shown to exist. The free wave equation is equivalent to the Dirac equation, but when a minimally coupled external electromagnetic field is introduced the new equation behaves differently from the Dirac equation. For example, the magnetic moment of a particle described by the 20-component equation depends on a free parameter and can thus take on any value including zero. The proposed equation is irreducible and the Γ_μ matrices satisfy a fourth-order minimal algebra, and thus Γ_0 is nondiagonalizable. Previously, examples of irreducible wave equations with nondiagonalizable Γ_0, that could continue to propagate causally in an external field, were not known. The equation proposed here is an example of such a theory.

I. INTRODUCTION

Since the discovery and subsequent successes of the Dirac equation for spin-$\frac{1}{2}$ particles, equations of the same general form

$$(-i\Gamma_\mu \partial^\mu + m)\psi(x) = 0 \tag{1}$$

have been studied extensively to construct analogous theories for particles of any spin.[1] Some of these studies have revealed that there are other wave equations, or even classes of wave equations, of the form (1) that can also describe spin-$\frac{1}{2}$ particles.[2] The behavior of a physical system described by these equations may be different from the behavior of particles described by the Dirac equation when an interaction with an external field is considered.

In this paper a particular relativistic wave equation is introduced. This equation is equivalent to the Dirac equation in the free-field case, but has new properties, that the Dirac equation does not have, in an external-field interaction, and thus may be more suitable for describing spin-$\frac{1}{2}$ particles other than the electron, or physical systems whose dynamics are different from those of the electron in simple external fields. Furthermore, this equation sheds new light on the theory of general SL(2, C)-invariant arbitrary-spin wave equations.

Specifically, the following will be shown: (i) The proposed equation is equivalent to the Dirac equation in the free-field case, but in a minimally coupled external field it is equivalent to the Dirac equation with the interaction $B(x)$ where

$$B(x) = -e\not{A}(x) + i\Delta_a \sigma_{\mu\nu} F^{\mu\nu}(x) .$$

(ii) The Γ_μ satisfy

$$\sum_{\sigma(\mu\nu\lambda\rho)} (\Gamma_\mu\Gamma_\nu - g_{\mu\nu})\Gamma_\lambda\Gamma_\rho = 0 \tag{2}$$

as the minimal algebra. The $\sigma(\mu\nu\lambda\rho)$ denotes the sum over all permutations of the indices μ, ν, λ, and ρ. (iii) As a consequence of (i), the new equation will remain causal in a minimally coupled external field. The implications of this result will be discussed further in the final section.

II. GENERAL ASSUMPTIONS

Relativistic wave equations (1) are often required to satisfy several basic requirements derived from the general physical assumptions. The proposed equation will be required to satisfy the following properties in common with the Dirac equation:

(*a*) *Lorentz covariance*. The equation is of the form

$$(-i\Gamma_\mu \partial^\mu + m)\psi(x) = 0 ,$$

where $m > 0$ is a real multiple of the identity and the equation transforms covariantly under a (reducible) representation of SL(2, C), $\Lambda \to T(\Lambda)$. Consequently Γ_μ were chosen so that

$$T(\Lambda)\Gamma_\mu T(\Lambda)^{-1} = \Lambda_\mu{}^\nu \Gamma_\nu , \quad T(\Lambda)\psi(x) = \psi'(\Lambda x) . \tag{3}$$

This condition can be written in terms of the infinitesimal generators of the representation $T(\Lambda)$. If we denote the rotations for $T(\Lambda)$ by J_i and the boosts by N_i, then Eq. (3) is equivalent to choosing Γ_0 consistent with

$$[\Gamma_0, J_i] = 0 , \tag{4a}$$

$$[[N_3, \Gamma_0], N_3] = \Gamma_0 \tag{4b}$$

and defining Γ_i as

$$\Gamma_i = [\Gamma_0, iN_i] . \tag{5}$$

For the equation considered in this paper $T(\Lambda) = (1,\frac{1}{2}) \oplus (\frac{1}{2},1) \oplus (0,\frac{1}{2}) \oplus (\frac{1}{2},0) \oplus (0,\frac{1}{2}) \oplus (\frac{1}{2},0) \equiv A + \overline{A} \oplus 2(B \oplus \overline{B})$, where $\overline{A}, \overline{B}$ are the representations conjugate to A and B, respectively.

(*b*) *Unique mass and spin.* The equation describes a unique mass m, and spin $\frac{1}{2}$ with a minimal number of independent components needed to describe the both negative- and positive-energy states on equal footing, i.e., $2(2s+1)$ which for $s=\frac{1}{2}$ is 4.

Unique mass is realized if the minimal polynomial of Γ_0 is (Harish-Chandra)

$$\Gamma_0{}^n(\Gamma_0{}^2-1)=0, \quad n\geq 0. \tag{6}$$

In the present case $n=2$.

(*c*) *Hermitizing matrix* η. One requires that there exist a Lorentz-invariant sesquilinear form on the positive-energy solutions of (1). Among other things, this allows one to derive the equation of motion (1) from an invariant Lagrangian and to construct a Lorentz-invariant Hermitian inner product on the solutions of (1). This property can be realized if there exists a nonsingular matrix η such that

$$\begin{aligned} &T^\dagger(\Lambda)\eta T(n)=\eta\,,\\ &\eta^\dagger=\eta\,,\\ &(\eta\Gamma_\mu)^\dagger=\eta\Gamma_\mu\,. \end{aligned} \tag{7}$$

For the Dirac equation, as an example, $\eta=\gamma_0$.

(*d*) *Irreducible* Γ_μ. The set of matrices Γ_μ are an irreducible set, i.e., Γ_μ considered as a set of linear transformations acting on the representation space of $T(\Lambda)$ have no invariant subspaces.[3] Not only does the proposed equation share this property with the Dirac equation, but without it the new equation in minimal coupling would not be distinguishable from the Dirac equation in minimal coupling.[4]

These properties will be explicitly verified in Appendix A.

III. THE EQUATION

(*a*) *Construction of the equation.* Let $S_i^{(1/2)}$ be the generators of $R\to D^{(1/2)}(R)$, the two-dimensional irreducible unitary representation of SU(2), and let $K_i^{(3/2)}$ denote the spin-$\frac{3}{2}$ K matrices that appear as the connectors of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ pieces in the generators of the boosts $N_i^{(3/2)}$ for the $(\frac{1}{2},1)$ or the $(1,\frac{1}{2})$ representations of SL(2, *C*). Γ_0 is chosen consistent with (4) and Γ_i are computed from (5). Figures 1 and 2 show Γ_0 and Γ_i, respectively, in their SU(2) block forms. Here f is to be chosen as a real or a pure imaginary number.

The equation (1) with the Γ_μ of Figs. 1 and 2 can be derived by varying the Lagrangian

$$L_0[\psi(x),\partial_\mu\psi(x)]=\psi^\dagger(x)\eta(-i\Gamma\cdot\partial+m)\psi(x) \tag{8}$$

Γ_0 =

	B 1/2	B 1/2	A 3/2	A 1/2	Ā 3/2	Ā 1/2	B 1/2	B̄ 1/2
1/2 B		1		f				if
1/2 B̄	1					f	-if	
3/2 A								
1/2 A	f							
3/2 Ā								
1/2 Ā		f						
1/2 B		-if						
1/2 B̄	if							

FIG. 1. The matrix Γ_0.

with respect to $\bar{\psi}(x)=\psi^\dagger(x)\eta$ (Appendix A).

The choice of coefficients of Γ_0 (and hence Γ_i) is one of many possible choices consistent with (4). However, the choice made in Fig. 1 is the simplest and the most convenient one for an equation with the desired properties.[5]

(*b*) *The equivalence of the free* Γ *equation to the free Dirac equation.* The free Γ equation with Γ_μ as in Figs. 1 and 2, is equivalent to the free Dirac equation. Consider the Γ equation in the following form:

$$\Gamma_\mu=\begin{array}{c} \begin{array}{cc} T_1(\Lambda) & T_2(\Lambda) \end{array} \\ \begin{bmatrix} \gamma_\mu & X_\mu \\ Y_\mu & 0 \end{bmatrix} \end{array} \begin{array}{c} T_1(\Lambda) \\ T_2(\Lambda) \end{array}, \tag{9}$$

$$\begin{aligned} &T_1(\Lambda)=(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\,,\\ &T_2(\Lambda)=(1,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\oplus(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\,,\\ &T(\Lambda)=T_1(\Lambda)\oplus T_2(\Lambda)\,. \end{aligned}$$

Now,

$$(-i\Gamma\cdot\partial+m)\psi(x)=0$$

is equivalent to

$$\begin{bmatrix} -i\partial\!\!\!/+m & -iX\cdot\partial \\ -iY\cdot\partial & +m \end{bmatrix}\begin{bmatrix} \phi(x) \\ \omega(x) \end{bmatrix}=0\,,$$

Γ_i =

	$-2S_i^{(\frac{1}{2})}$	$\frac{2}{3}fK_i^{(\frac{3}{2})}$	$\frac{2}{3}fS_i^{(\frac{1}{2})}$				$-2ifS_i^{(\frac{1}{2})}$
$2S_i^{(\frac{1}{2})}$				$-\frac{2}{3}fK_i^{(\frac{3}{2})}$	$-\frac{2}{3}fS_i^{(\frac{1}{2})}$	$-2ifS_i^{(\frac{1}{2})}$	
$-\frac{2}{3}fK_i^{+(\frac{3}{2})}$							
$-\frac{2}{3}fS_i^{(\frac{1}{2})}$							
	$\frac{2}{3}fK_i^{+(\frac{3}{2})}$						
	$\frac{2}{3}fS_i^{(\frac{1}{2})}$						
	$2ifS_i^{(\frac{1}{2})}$						
$2ifS_i^{(\frac{1}{2})}$							

FIG. 2. The matrices Γ_i.

which yields

$$(-i\partial\!\!\!/ + m)\phi(x) - iX\cdot\partial\omega(x) = 0\,,$$
$$-iY\cdot\partial\phi(x) + m\omega(x) = 0\,, \tag{10}$$

or on rearranging

$$(-i\partial\!\!\!/ + m)\phi(x) + \frac{1}{m}X\cdot\partial\, Y\cdot\partial\phi(x) = 0\,,$$
$$\omega(x) = \frac{1}{m}\, iY\cdot\partial\phi(x)\,; \tag{11}$$

but X_0, Y_0 are chosen so that

$$X\cdot\partial\, Y\cdot\partial = 0\,, \tag{12}$$

and (11) becomes

$$(-i\partial\!\!\!/ + m)\phi(x) = 0\,,$$
$$\omega(x) = \frac{1}{m}\, iY\cdot\partial\phi(x)\,.$$

Hence, the Γ equation is equivalent to the Dirac equation and a set of dependent components $\omega(x)$ are completely determined by the independent components $\phi(x)$ (which satisfy the Dirac equation).

(*c*) *External field interaction by minimal coupling and magnetic moments*. The structure of the Γ equation in an interaction $B(x) = -e\Gamma_\mu A^\mu(x)$ can be considered as follows:

$$X_0 Y_0 = 0 \Rightarrow X\cdot\partial\, Y\cdot\partial = 0$$

by boosting to arbitrary Lorentz frames.

$$X\cdot\partial\, Y\cdot\partial = 0 \Rightarrow X_\mu Y_\nu + X_\nu Y_\mu = 0\,,$$

since

$$\partial_\mu \partial_\nu = \partial_\nu \partial_\mu\,.$$

The equation

$$(-i\Gamma\cdot D + m)\Psi(x) = 0\,,$$

where

$$D_\mu = \partial_\mu - ieA_\mu(x)$$

can again be written as

$$(-iD\!\!\!\!/ + m)\Phi(x) + \frac{1}{m}X^\mu Y^\nu D_\mu D_\nu \Phi(x) = 0\,, \tag{13a}$$

$$\Omega(x) = \frac{i}{m}\, Y\cdot D\Phi(x)\,. \tag{13b}$$

$$X_\mu Y_\nu = -X_\nu Y_\mu$$

$$\Rightarrow X_\mu Y_\nu D^\mu D^\nu \Phi(x) = \tfrac{1}{2}(X_\mu Y_\nu D^\mu D^\nu + X_\nu Y_\mu D^\nu D^\mu)\Phi(x)$$
$$= \tfrac{1}{2} X_\mu Y_\nu (D^\mu D^\nu - D^\nu D^\mu)\Phi(x)\,,$$

since

$$[D^\mu, D^\nu]\Phi(x) = ieF_{\mu\nu}(x)\Phi(x)\,,$$
$$X_\mu Y_\nu D^\mu D^\nu \Phi(x) = \tfrac{1}{2} ie X_\mu Y_\nu F^{\mu\nu}(x)\Phi(x)\,.$$

Therefore, Eq. (13a) is

$$(-iD\!\!\!\!/ + m)\Phi(x) - \frac{2}{3}\frac{ie}{m} f^2 \sigma_{\mu\nu} F^{\mu\nu}(x)\Phi(x) = 0\,.$$

It will be shown that

$$X_\mu Y_\nu = -\tfrac{4}{3} f^2 \sigma_{\mu\nu} = -\tfrac{2}{3} f^2 [\gamma_\mu, \gamma_\nu]$$

in Appendix B.

The magnetic moment for a particle described by the Γ equation is

$$M = \frac{e}{2m}(1 + \tfrac{4}{3} f^2)\,,$$

or the gyromagnetic ratio g is

$$g = (1 + \tfrac{4}{3} f^2)\,. \tag{14}$$

Originally, f was an arbitrary complex number. It is chosen to be either real or pure imaginary so that M is real. With this arrangement g can have any real value one desires, negative or positive, including zero.

(*d*) *Causality*. Velo and Zwanziger[6] have shown that equations of the form (1) are sometimes "unstable" with respect to external-field perturbations in the sense that they develop modes of propagation that exceed the velocity of light in regions with sufficiently strong external fields. This is the phenomenon of noncausal propagation. There are classes of equations that do not suffer from this problem, including the Dirac equation for spin $\frac{1}{2}$ and the equation proposed here.

In the case at hand, a detailed analysis is not carried out since the causality problem for the Γ equation can be reduced to the same problem for the Dirac equation as follows:

The original equation for the wave function $\Psi(x)$ can be reduced to the equation for the independent components $\Phi(x)$,

$$[-iD\!\!\!\!/ + m + \alpha\sigma^{\mu\nu}F_{\mu\nu}(x)]\Phi(x) = 0\,,$$

and a "constraint" equation

$$\Omega(x) = \frac{1}{m}\, iY\cdot D\Phi(x)\,;$$

or, alternatively, $\Omega(x)$ can be given in terms of only the spatial derivatives of $\Phi(x)$:

$$\Omega(x) = \frac{i}{m}[Y_0\gamma_0(\vec{\gamma}\cdot\vec{\mathrm{D}} + m + \alpha\sigma^{\mu\nu}F_{\mu\nu}) - \vec{\mathrm{Y}}\cdot\vec{\mathrm{D}}]\Phi(x)\,.$$

Since the independent components satisfy the Dirac equation in the external field $B(x)$, which is known to be causal, so the Γ equation is also causal.

The same arguments apply for other more complicated interactions of the Γ equation as long as the dependent component $\Omega(x)$ can be solved in terms of the components $\Phi(x)$.

A more detailed proof can be constructed from the techniques discussed in the references.[7]

IV. DISCUSSION AND SUMMARY

First, it has been shown that an anomalous-magnetic-moment term can be obtained from a first-order Lorentz-invariant, Dirac-type equation interacting with an external field by minimal coupling. On the other hand, in the free-field case, the new equation is completely equivalent to the Dirac equation.[8]

The g factor can have any value for this new equation in a minimally coupled external field except $g=1$. The case $g=1$ makes the Γ equation reducible by forcing f to be zero. Otherwise the assumptions of Sec. II are not strong enough to specify the value of g any further. It is a matter of choice just as the mass m of the particle is. However, one may pick a value by fixing f as, for example, $g=0$ where $f=\frac{1}{2}i\sqrt{3}$. In such a case the equation describes a spin-$\frac{1}{2}$ system with no intrinsic magnetic moment, which has also been discussed by Chang.[9]

It should also be pointed out that similar equations exist for any spin, and thus a first-order wave equation for a given spin does not always have a fixed magnetic moment in minimally coupled external electromagnetic fields.

The second general point is related to the causal propagation property of the new equation. Equations of the form (1) do not suffer from the Velo-Zwanziger pathology as long as Γ_0 is diagonalizable and the external-field interaction is minimal coupling: $B(x)=-e\Gamma^{\mu}A_{\mu}(x)$.[10] The equation given here is the first example of an irreducible wave equation with nondiagonalizable Γ_0 that remains causal in an external field. If the irreducibility requirement is suspended, then one can easily construct causal wave equations for arbitrary spins with $B(x)=-e\Gamma_{\mu}A^{\mu}(x)$ and

$$\Gamma_0{}^n(\Gamma_0{}^2-1)=0$$

for any $n>1$; but for irreducible wave equations the same statement is not known to be true. In fact, examples of irreducible causal equations for $B(x)=-e\Gamma_{\mu}A^{\mu}(x)$, with $n>1$, were not known. In the case of multimass equations, analogous statements apply where the degree of nilpotency of the null-eigenvalue submatrix of Γ_0 exceeds 1.

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APPENDIX A

In Sec. II, four conditions were stated, that the equation proposed in this paper was required to satisfy. In this appendix these conditions will be verified more explicitly. The first three conditions are easy to check and fairly standard, so these will not be considered in detail.

(a) The equation is manifestly Lorentz covariant under the SL(2, C) representation $\Lambda\to T(\Lambda)$, where $T(\Lambda)=(\frac{1}{2},1)\oplus(1,\frac{1}{2})\oplus 2[(0,\frac{1}{2})\oplus(\frac{1}{2},0)]$ by construction.

(b) The fact that the equation describes unique spin is clear from the observation that Γ_0 connects only spin-$\frac{1}{2}$ pieces. Alternatively, the parts of the wave function $\psi(x)$ that transform under $R\to D^{3/2}(R)$, the representation of SU(2) occurring in the SU(2) restriction of $T(\Lambda)$, are zero in the rest frame. The independent components, which are the nonzero components in the rest frame, transform under the spin-$\frac{1}{2}$, $R\to D^{1/2}(R)$ representations of SU(2).

If one considers the eigenvalue equation for Γ_0, it is found that

$$|\Gamma_0-\lambda I|=\lambda^2(\lambda^2-1)=0;$$

so by the Hamilton-Cayley theorem

$$\Gamma_0{}^2(\Gamma_0{}^2-1)=0 \tag{A1}$$

is the characteristic polynomial. It is also the minimal polynomial since

$$\Gamma_0(\Gamma_0{}^2-1)=\begin{pmatrix}0 & 0\\ 0 & Y_0\gamma_0X_0\end{pmatrix}\neq 0.$$

Owing to the results of Harish-Chandra, (A1) implies that the equation describes a unique mass m. Lastly, the independent components arise from γ_0 in Γ_0, and thus there are $2(2s+1)$ independent components.

All these qualities are obvious from the Jordan canonical form (JCF) of Γ_0:

$$\Gamma_0(\text{JCF})=\Gamma_0^{(1/2)}(\text{JCF})\oplus\Gamma_0^{(3/2)}(\text{JCF}),$$

$$\Gamma_0^{(3/2)}=0.$$

$\Gamma_0^{(1/2)}$(JCF) in block form, with each block being a 2×2 matrix, is given in Fig. 3.

$$\Gamma_0^{(1/2)}(\text{JCF})=\begin{pmatrix}1&&&&&\\&-1&&&&\\&&&1&&\\&&&&&\\&&&&&1\\&&&&&\end{pmatrix},\quad \Gamma_0^{(3/2)}(\text{JCF})=0$$

FIG. 3. The Jordan canonical form of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ submatrices of Γ_0.

(c) When f is real, choose η as in Fig. 4, and when f is pure imaginary, choose η as in Fig. 5. It can be easily checked that $\eta^\dagger = \eta$, $T^\dagger(\Lambda)\eta T(\Lambda)$, and $(\eta\Gamma_\mu)^\dagger = \eta\Gamma_\mu$.

(d) For a wave equation of the form (1) irreducibility[11] means that Γ_μ, when considered as linear transformations on the representation space $R(T)$ of $T(\Lambda)$, leave no proper subspace invariant. First, if Γ_μ leaves any subspace invariant, it also leaves a maximal subspace invariant which is an SL(2, C) subspace of $R(T)$; i.e., the maximal-invariant subspace is a representation space of some combination of representation in $T(\Lambda)$. Second, since

$$R(T) = \bigoplus_{j=1}^{k} \alpha_j R(T_j)$$

when

$$T(\Lambda) = \bigoplus_{j=1}^{k} \alpha_j T_j ,$$

there exist projection operators that are Hermitian and idempotent, which can project any SL(2, C) subspace in $R(T)$. All the possible projectors commute with the representation matrices $T(\Lambda)$, and some of the projectors have the following property with respect to Γ_0:

$$(\Gamma_0 P - P\Gamma_0 P) = 0 \Longleftrightarrow [\Gamma_0, P]P = 0 \Longleftrightarrow (1-P)\Gamma_0 P = 0 . \tag{A2}$$

Now, for every vector ϕ in $R(T)$, (A2) is satisfied if and only if $P\phi$, an SL(2, C) subspace of $R(T)$, is an invariant subspace of Γ_0 (and hence of Γ_μ); i.e., Γ_0 (and Γ_i) leave this subspace invariant when considered as a set of linear transformations on $R(T)$. If the only projectors P that satisfy (A2) for a given wave equation are I and 0, then Γ_0 leaves no proper subspace invariant and hence the wave equation is irreducible. Otherwise it is reducible.

The most general projector of this kind for $\Gamma(\Lambda)$, considered in this paper, is of the form given in Fig. 6.

Now, $(1-P)\Gamma_0 P = 0$ gives the following relations for the coefficients of P:

(1) $(1-a)b - ihfb + (1-a)qif = 0$,

(2) $(1-a)fc = 0$,

(3) $(1-a)j - ihfj + (1-a)ifg = 0$,

(4) $(1-b)a + ijfa - (1-b)ifp = 0$,

(5) $(1-b)fd = 0$,

(6) $(1-b)h + ijfh - (1-b)ife = 0$,

(7) $(1-c)fa = 0$,

(8) $(1-c)fh = 0$,

(9) $(1-d)fb = 0$,

(10) $(1-d)fj = 0$,

(11) $pb - (1-e)ifb + pqif = 0$,

(12) $pfc = 0$,

(13) $pj - (1-e)ifj + pifg = 0$,

(14) $qa + (1-g)ifa - ifqp = 0$,

(15) $qfd = 0$,

(16) $qh + (1-g)ifh - ifqe = 0$.

We can now consider all the possible solutions of these equations for $f \neq 0$. Conditions (12) and (15)

η =

B	$\bar{B}$	A		$\bar{A}$		B	$\bar{B}$
	1						
1							
				1			
					1		
		1					
			1				
							1
						1	

FIG. 4. The η matrix for real f.

η =

B	$\bar{B}$	A		A		B	$\bar{B}$
	1						
1							
				-1			
					-1		
		-1					
			-1				
							-1
						-1	

FIG. 5. The η matrix for imaginary f.

P = , a, b, c, d, e, g ∈ R

1/2	1/2	3/2	1/2	3/2	1/2	1/2	1/2	
a						h		1/2 B
	b						j	1/2 $\bar{B}$
		c						3/2 A
			c					1/2 A
				d				3/2 $\bar{A}$
					d			1/2 $\bar{A}$
p						e		1/2 B
	q						g	1/2 $\bar{B}$

FIG. 6. The general projector.

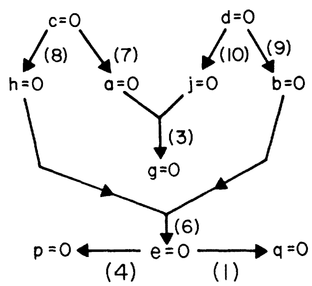

FIG. 7. Case (a): assuming $c=0$, $d=0$ leads to $P=0$.

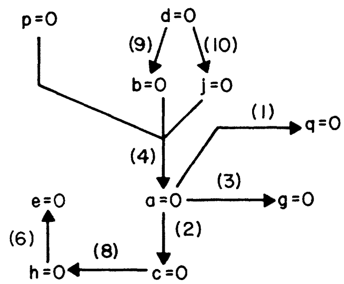

FIG. 9. Case (c): $p=0$, $d=0$.

yield four cases: case (a): $c=0$, $d=0$; case (b): $c=0$, $q=0$; case (c): $p=0$, $d=0$; and case (d): $p=0$, $q=0$. The only solution for case (a) is $P=0$ as shown in Fig. 7. Furthermore, for cases (b) and (c), once again P has to be zero to satisfy (A2) as shown in Figs. 8 and 9, respectively.

In case (d) equations (1)–(16) simplify to

(1′) $(1-a)b - ifhb = 0$,

(2′) $(1-a)c = 0$,

(3′) $(1-a)j - ihfj + (1-a)ifg = 0$,

(4′) $(1-b)a + ifja = 0$,

(5′) $(1-b)d = 0$,

(6′) $(1-b)h + ijfh - (1-b)ife = 0$,

(7′) $(1-c)a = 0$,

(8′) $(1-c)h = 0$,

(9′) $(1-d)b = 0$,

(10′) $(1-d)j = 0$,

(11′) $(1-e)b = 0$,

(12′) $(1-e)j = 0$,

(13′) $(1-g)a = 0$,

(14′) $(1-g)h = 0$.

Since P is idempotent, e is either 0 or 1 because

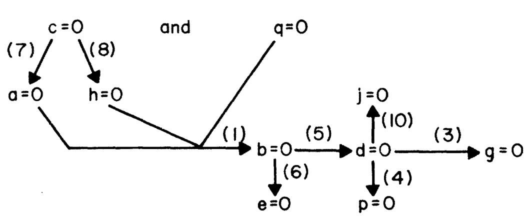

FIG. 8. Case (b): $c=0$ and $q=0$.

$p=0$. If $e=0$, Fig. 10 applies to show $P=0$. If $e=1$, Figs. 11, 12, and 13 show that $P=I$. It can be concluded that for the equation given here, when $f \neq 0$, there does not exist any projector onto a proper subspace of $R(T)$ that satisfies (A2). If Γ_μ was reducible, such a projector would exist; since it does not exist, Γ_μ is irreducible.

APPENDIX B

Considering the possible Lorentz-invariant interactions of the Dirac equation, $X_\mu Y_\nu$ must be a multiple of $\sigma^{\mu\nu}$ since it couples to $F^{\mu\nu}(x)$. To deduce the proportionality constant, the following explicit calculation is carried out.

We want to show that $(e/2m)X^\mu Y^\nu = \Delta_a \sigma^{\mu\nu}$ with $\Delta_a = -(e/2m)\frac{4}{3}f^2$.

To check the relationship we can consider the four cases (i) X^0Y^0, (ii) X^iY^0, (iii) X^0Y^i, (iv) X^iY^i.

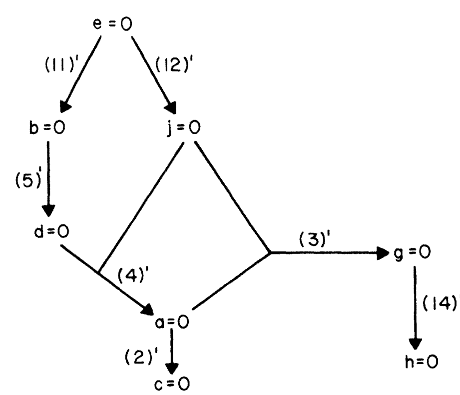

FIG. 10. Case (d1): $e=0$.

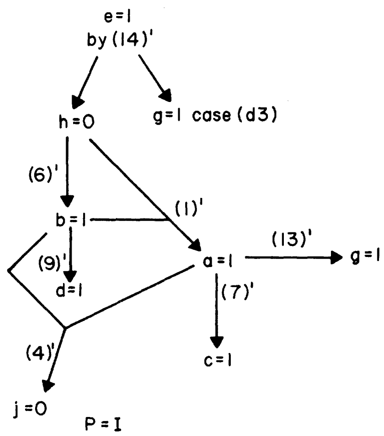

FIG. 11. Case (d2): $e=1$.

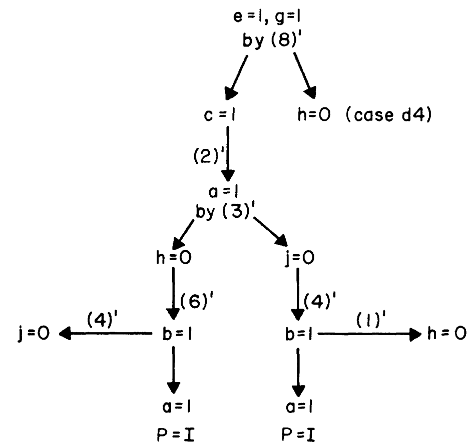

FIG. 12. Case (d3).

(i) $X^0Y^0=0$,

(ii) $X^iY^0=\frac{8}{3}f^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes S_i^{1/2}$,

(iii) $X^0Y^i=-\frac{8}{3}f^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes S_i^{1/2}$,

(iv) $X^iY^i=-\frac{4}{9}f^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes (K_i^{(3/2)}K_j^{(3/2)} - 8S_i^{(1/2)}S_j^{(1/2)})$.

In the case of $S_i^{(1/2)}S_j^{(1/2)}$ it can be checked that

$$S_i^{(1/2)}S_j^{(1/2)}=\tfrac{1}{2}i\epsilon_{ijk}S_k^{(1/2)}+\tfrac{1}{4}\delta_{ij}\,;$$

then since $[N_i,N_j]=-i\epsilon_{ijk}J_k$ this implies

$$K_i^{(3/2)}K_j^{(3/2)\dagger}+S_i^{(1/2)}S_j^{(1/2)}=-\tfrac{3}{2}i\epsilon_{ijk}S_k^{(1/2)}+\tfrac{9}{4}\delta_{ij}\,,$$

$$K_iK_j^\dagger=-2i\epsilon_{ijk}S_k^{(1/2)}+2\delta_{ij}\,.$$

Therefore

$$(K_i^{(3/2)}K_j^{(3/2)\dagger}-8S_i^{(1/2)}S_j^{(1/2)})=-6i\epsilon_{ijk}S_k^{(1/2)}$$

and

$$X^iY^j=\tfrac{8}{3}f^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes i\epsilon_{ijk}S_k^{(1/2)}\,.$$

Clearly

$$X^\mu Y^\nu=-H^\nu J^\mu\,.$$

Now,

(i) $\sigma^{00}=0$, $\sigma^{\mu\nu}=\frac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$,

(ii) $\sigma^{0i}=2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes S_i^{(1/2)}$,

(iii) $\sigma^{i0}=-2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes S_i^{(1/2)}$,

(iv) $\sigma^{ij}=-2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes i\epsilon_{ijk}S_i^{(1/2)}$.

So $X^\mu Y^\nu=-\frac{4}{3}f^2\sigma^{\mu\nu}$. In the basis chosen for this equation γ_μ are given as follows:

$$\gamma_0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\gamma_i=\begin{pmatrix}0 & -2S_i^{(1/2)}\\ 2S_i^{(1/2)} & 0\end{pmatrix}$$

$$=2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\otimes S_i^{(1/2)}\,.$$

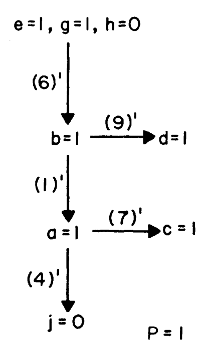

FIG. 13. Case (d4).

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[2]A. Z. Capri, Phys. Rev. 187, 1811 (1969); 178, 2427 (1969).

[3]The question of irreducibility is considered in Ref. 2, however, these definitions do not take into account several types of reducible equations which are indecomposable. A more general definition is used in this study. See also, A. S. Wightman, in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami*, 1968, edited by A. Perlmutter, C. Angas Hurst, and B. Kurşunoğlu (Benjamin, New York, 1968), pp. 291–314.

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[5]It should be pointed out that there are, as usual, other ways of constructing this free equation. One may start with $\phi(x)$, a wave function satisfying the Dirac equations, and then construct ψ from ϕ and ω, where ω are constructed as linear combinations of $\partial_\mu \phi$; so that the larger ψ will satisfy the equations considered here. However, it may not be possible to carry out an analogous procedure for this equation when general external field interactions are introduced.

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[8]An attempt to obtain results similar to those reported here was made by Capri using the equation of the first paper of Ref. 2, and reported in A. Z. Capri, Prog. Theor. Phys. 48, 1364 (1972). The author does not agree with the computations in that paper. The 16-component spin-$\frac{1}{2}$ system of Capri is reducible, has no nonsingular η matrix, and has the same magnetic moment as the Dirac spin-$\frac{1}{2}$ theory in a minimally coupled external electromagnetic field.

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