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Noncausal propagation in spin-0 theories with external field interactions

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The two-component Sakata-Taketani (ST) spin-0 theory and the single-component Klein-Gordon theory are obtained from the five-component Duffin-Kemmer-Petiau (DKP) theory with six types of external field interactions by means of a Peirce decomposition. Whereas the DKP equation manifests the covariance, the ST equation manifests the causal properties. In particular, the presence of noncausal wave propagation when there is coupling to a second-rank tensor field is apparent from the form of the ST equation, in which the coefficients of all the space derivatives depend on the external field. Our results indicate that the causal properties of higher-spin equations should also be obvious when they are expressed in $2(2J+1)$-component Schrödinger form.

I. INTRODUCTION

Since the work of Velo and Zwanziger,[1] a great deal of attention has been given to the fact that manifestly covariant wave equations with certain types of external field interactions can possess noncausal solutions propagating with a speed exceeding that of light in a vacuum, a property that is by no means obvious from the form of the equations. In particular, when Wightman[2] studied the first-order five-component Duffin-Kemmer-Petiau (DKP) spin-0 equation,[3] he concluded that coupling to either an antisymmetric or a traceless symmetric second-rank tensor field admits noncausal wave propagation. Recently when the present authors[4] employed a Lagragian formalism to introduce interactions into the two-component Sakata-Taketani (ST) spin-0 theory,[5,6] which although relativistic is not manifestly covariant, they found that the causal or noncausal nature is apparent from the manner in which the coefficients of the space derivatives depend on the external field. In the present paper we demonstrate the equivalence of the interacting ST and DKP equations for the six types of external-field couplings that have previously been considered, obtain the corresponding one-component Klein-Gordon (KG) equation, and call attention to the evidence that the causal properties of higher-spin equations will also be manifest if they are rewritten in $2(2J+1)$-component Schrödinger form.

If an equation for a field (or wave function) Φ describing free massive particles with a definite spin yields, in addition, the Klein-Gordon relation[7]

$$(\Box+m^2)\Phi=0\,, \tag{1.1}$$

then the particles have mass m and the theory has $2(2J+1)$ degrees of freedom, corresponding to the $2J+1$ spin states and the two degrees of freedom in the "charge space."[8] The maximum number of independent components that such an equation can have is therefore $2(2J+1)$ if it is of first order in the time derivative and $2J+1$ if it is of second order in this operator. Thus, in the spin-0 ST equation, which is of the Schrödinger form

$$i\frac{\partial\Phi}{\partial t}=H\Phi\,, \tag{1.2}$$

the dimensionality accounts for the two degrees of freedom, whereas in the one-component Klein-Gordon equation[9] this freedom is accounted for by the second-order time derivative. A similar situation exists with regard to the description of spin-$\frac{1}{2}$ particles by either the four-component first-order Dirac equation or the two-component second-order Kramers equation.[10]

A manifestly covariant equation describing particles with a unique nonzero mass m and spin J can always be written in the linear form[11,12]

$$(i\alpha^\mu\partial_\mu-m+\mathcal{B})\Phi=0\,, \tag{1.3}$$

where $\mathcal{B}=0$ for free particles. Except for the Dirac equation, the dimension of (1.3) is greater than $2(2J+1)$ and the α matrices are singular. In the presence of interactions, $\mathcal{B}$ is formed by contracting the tensor indices of external fields with those of the linearly independent matrices one can form from products of the α matrices[12]; in this way one obtains all possible nonderivative couplings. However, again with the exception of the Dirac equation, the interacting-field equation may turn out to be noncausal,[1,2,13,14] and, even when it is causal, it may be unstable.[15] For $J \geq \frac{3}{2}$ no theory free of such difficulties is known.

The DKP spin-0 and spin-1 equations, both of which have the form (1.2), are known to be noncausal for particular external-field interactions of the type just discussed.[2,13] Of the six kinds of nonderivative couplings that are possible for the spin-0 equation, the two scalar and two vector ones are causal, but noncausal wave propagation occurs when the interaction is with either a symmetric or an antisymmetric second-rank tensor field. The present authors[4] considered six external field interactions of the same types when discussing a Lagrangian formalism for the spin-0 ST equation (in this case, spatial derivatives, but not time derivatives, of the ST field were required in the interaction Lagrangian), and when they applied certain standard tests[16] to their results, they found that only the two second-rank tensor couplings were noncausal. In this paper we obtain our earlier results for the interacting ST spin-0 field directly from the corresponding DKP theory by employing the same procedure followed by Taketani and Sakata[5,17] for the DKP equation with minimal electromagnetic coupling; that is to say, we perform a Peirce decomposition[18] of the DKP equation and show that three of the components can be eliminated from the field equation and from all observables, yielding a theory involving only the two physically essential components. It is also easily demonstrated that the DKP equation can be recovered from the ST equation provided that one of the two scalar fields is sufficiently well-behaved. With this provision, the two theories are equivalent for the six types of external-field interactions considered.

We remind the reader that in a manifestly covariant theory the space and time derivatives appear in a symmetrical manner, as in (1.3), whereas in the ST theory, H in (1.2) is of second order in the space derivatives. Furthermore, in a manifestly covariant theory, the relationship between the wave function $\Phi(x)$ in one reference frame and the wave function $\Phi'(x')$ in another frame, where $x' = \Lambda x$ and Λ is a proper orthochronous Lorentz transformation, has the form

$$\Phi'(x') = \mathcal{S}_\Lambda \Phi(x) , \quad (1.4)$$

where the transformation matrix $\mathcal{S}_\Lambda$ depends only on Λ; this relation is the same for both the free and interacting fields. In theories that do not manifest the covariance, even though they are relativistic, $\mathcal{S}_\Lambda$ may, in addition, depend on the spatial derivatives and upon the particular external fields with which Φ is interacting. Just how complicated $\mathcal{S}_\Lambda$ can be for an infinitesimal transformation was demonstrated in Ref. 4 for the ST spin-0 equation, which results are rederived in this paper via the Peirce decomposition procedure. We thus see that, whereas the ST theory has the advantage of working with only two components corresponding to the number of degrees of freedom, one sacrifices the simplicity of the Lorentz-transformation properties that occurs in the DKP theory with its three extra components.

If the free-particle equation satisfies the Klein-Gordon relation (1.1), then, according to one of the accepted tests,[13,16] the interacting field propagates causally if it satisfies an equation of the form

$$(\Box + \mathcal{G})\Phi = 0 , \quad (1.5)$$

where $\mathcal{G}$ contains derivatives of order one at the most; however, causal propagation also occurs if $\mathcal{G}$ contains a term $\mathcal{G}'$ with second-order derivatives satisfying $\mathcal{G}'^2 = 0$. It is therefore interesting to note that for those cases in which the spin-0 ST equation is noncausal, *all* the coefficients of the space derivatives in the operator H in (1.2) depend on the external field, so (1.5) cannot be satisfied. Of particular interest is the fact that in the noncausal case these coefficients contain a *noncovariant* denominator that can vanish for certain strengths of the external field. Thus, the ST spin-0 equation manifests the causal properties, in contrast to the corresponding DKP equation, in which the terms containing the space-time derivatives are identical to those occurring in the free-particle equation.

By expressing the four-components of the DKP spin-0 equation that transform under the $D^{(1/2,1/2)}$ representation of the proper orthochronous Lorentz group in terms of the single component that transforms under the $D^{(0,0)}$ representation, we obtain the Klein-Gordon equation. This equation, for the reasons discussed in the preceding paragraph, also manifests the causal properties.

When one eliminates the four extra components from the DKP spin-1 equation, one obtains the six-component ST spin-1 equation.[5] It is natural to ask whether the latter is changed in as noticeable a manner as the ST spin-0 equation when noncausal external-field couplings are introduced. As we shall report elsewhere,[19] the answer is positive for those interactions that we have investigated. Similar results should be expected when higher-spin equations describing particles with a unique mass and spin are likewise written in $2(2J+1)$-component Schrödinger form; free-particle equations of this type have already been studied extensively by a number of authors,[20-22] and the implications of their results are discussed in Sec. V.

II. DUFFIN-KEMMER-PETIAU THEORY

The β matrices that appear in the Duffin-Kemmer-Petiau equation satisfy the relations[3,7]

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=g_{\mu\nu}\beta_\lambda+g_{\lambda\nu}\beta_\mu . \tag{2.1}$$

The algebra generated by the four β_μ's has three irreducible representations, one of which is a trivial one-dimensional representation of all matrices by the number 0. Of the two physically interesting representations, the first consists of ten-by-ten matrices and the second of five-by-five matrices, the former case leading to a description of spin-1 particles and the latter to a theory of spin-0 particles.

In this paper we shall work only with the five-dimensional irreducible representation of the algebra, for which a more convenient set of generating elements than the β matrices has been developed.[23] Following Fischbach, Nieto, and Scott,[24] we label the 25 elements of the spin-0 subalgebra, each of which is a five-by-five matrix, as P, ${}_\mu P$, P_μ, and ${}_\mu P_\nu$, and we also require that

$$P(P_\mu)=P_\mu , \tag{2.2a}$$

$$({}_\mu P)P={}_\mu P , \tag{2.2b}$$

$$(P_\mu)P=P({}_\mu P)=0 , \tag{2.2c}$$

$$({}_\mu P)(P_\nu)={}_\mu P_\nu , \tag{2.2d}$$

$$(P_\mu)({}_\nu P)=g_{\mu\nu}P . \tag{2.2e}$$

From the above, one finds that

$$P^2=P , \tag{2.3a}$$

$$(P_\mu)(P_\nu)=({}_\mu P)({}_\nu P)=0 , \tag{2.3b}$$

$$(P)({}_\mu P_\nu)=({}_\mu P_\nu)P=0 , \tag{2.3c}$$

$$(P_\mu)({}_\nu P_\lambda)=g_{\mu\nu}(P_\lambda) , \tag{2.3d}$$

$$({}_\lambda P_\nu)({}_\mu P)=g_{\mu\nu}({}_\lambda P) , \tag{2.3e}$$

$$({}_\mu P_\nu)(P_\lambda)=({}_\lambda P)({}_\mu P_\nu)=0 , \tag{2.3f}$$

$$({}_\mu P_\nu)({}_\lambda P_\rho)=g_{\nu\lambda}({}_\mu P_\rho) . \tag{2.3g}$$

We also define

$$\tilde{P}=g^{\mu\nu}({}_\mu P_\nu) , \tag{2.4}$$

and it follows that

$$(\tilde{P})^2=\tilde{P} , \tag{2.5a}$$

$$P\tilde{P}=\tilde{P}P=0 , \tag{2.5b}$$

$$(P_\mu)\tilde{P}=P_\mu , \tag{2.5c}$$

$$\tilde{P}({}_\mu P)={}_\mu P , \tag{2.5d}$$

$$({}_\mu P)\tilde{P}=\tilde{P}(P_\mu)=0 , \tag{2.5e}$$

$$({}_\mu P_\nu)\tilde{P}=\tilde{P}({}_\mu P_\nu)={}_\mu P_\nu . \tag{2.5f}$$

In order for the P_μ's and the ${}_\mu P$'s to transform as four-vectors under proper orthochronous Lorentz transformation, in which case the four-coordinates x_μ go into

$$x'_\mu=\Lambda_\mu{}^\nu x_\nu , \tag{2.6}$$

there must exist a matrix S_Λ such that

$$S_\Lambda{}^{-1}(P_\mu)S_\Lambda=\Lambda_\mu{}^\nu(P_\nu) , \tag{2.7a}$$

$$S_\Lambda{}^{-1}({}_\mu P)S_\Lambda=\Lambda_\mu{}^\nu({}_\nu P) . \tag{2.7b}$$

For an infinitesimal transformation of the type (2.6),

$$x'_\mu=x_\mu+\omega_\mu{}^\nu x_\nu , \tag{2.8}$$

where $\omega_{\mu\nu}=-\omega_{\nu\mu}$, the properties (2.7) are satisfied by

$$S_\Lambda=\tilde{I}+\omega_{\mu\nu}({}^\mu P^\nu) , \tag{2.9}$$

where $\tilde{I}$ is the five-by-five unit matrix. Furthermore, P and $\tilde{P}$ transform as scalars and ${}_\mu P_\nu$ as a second-rank tensor.

The five-by-five β matrices, which are defined by

$$\beta_\mu=P_\mu+{}_\mu P , \tag{2.10}$$

are easily seen to be consistent with (2.1). They transform as four-vectors, and one has

$$\beta_\mu\beta_\nu=g_{\mu\nu}(P)+{}_\mu P_\nu . \tag{2.11}$$

We introduce the matrix η_0 with the property

$$(\eta_0)^2=\tilde{I} , \tag{2.12}$$

where $\tilde{I}$ is the five-by-five unit matrix, by means of the usual definition

$$\eta_0=2(\beta_0)^2-\tilde{I} \tag{2.13a}$$

$$=2(P+{}_0P_0)-\tilde{I} . \tag{2.13b}$$

In order to be able to construct Hermitian observables, it is necessary to have an Hermitianizing matrix for the theory. We therefore make the usual requirement that the β matrices be Hermitian with respect to η_0; that is,[7]

$$(\beta_\mu)^\dagger=\eta_0\beta_\mu\eta_0=g_{\mu\mu}\beta_\mu , \tag{2.14}$$

where the second equality follows from (2.1). To satisfy (2.14) we impose the following property on the P matrices:

$$(P_\mu)^\dagger=\eta_0({}_\mu P)\eta_0=g_{\mu\mu}({}_\mu P) , \tag{2.15a}$$

$$({}_\mu P)^\dagger=\eta_0(P_\mu)\eta_0=g_{\mu\mu}(P_\mu) . \tag{2.15b}$$

It follows that P and $\tilde{P}$ are Hermitian with respect to η_0 and that

$$({}_\mu P_\nu)^\dagger=\eta_0({}_\nu P_\mu)\eta_0=g_{\mu\mu}g_{\nu\nu}({}_\nu P_\mu) . \tag{2.15c}$$

It is clear from (2.3a), (2.3c), and (2.3g) that the five idempotent operators P and

$${}_{\mu}\tilde{P}_{\mu} = g_{\mu\mu}({}_{\mu}P_{\mu})$$

are projection operators for orthogonal subspaces of the five-dimensional space. We may therefore write for the five-by-five unit matrix

$$I = P + g^{\mu\nu}({}_{\mu}P_{\nu}) \quad (2.16a)$$

$$= P + \tilde{P}. \quad (2.16b)$$

According to (2.3a), (2.5a), and (2.5b), P and $\tilde{P}$ are projection operators onto subspaces that are irreducible under proper orthochronous Lorentz transformations. The subspace belonging to P is one-dimensional and that belonging to $\tilde{P}$ is four-dimensional; these transform, respectively, under the $D^{(0,0)}$ and $D^{(1/2,1/2)}$ representations of the proper orthochronous Lorentz group. We note that, because of (2.11),

$$\beta^{\mu}\beta_{\mu} = 4P + \tilde{P} = 3P + \tilde{I}, \quad (2.17)$$

and with the aid of (2.16b) we find that

$$P = \tfrac{1}{3}(\beta_{\mu}\beta^{\mu} - \tilde{I}), \quad (2.18a)$$

$$\tilde{P} = \tfrac{1}{3}(4\tilde{I} - \beta_{\mu}\beta^{\mu}). \quad (2.18b)$$

Since (2.1) requires the five-dimensional Hermitian matrix β_0 to satisfy

$$(\beta_0)^3 = \beta_0,$$

it can have the eigenvalues +1, −1, or 0. In fact, we can easily find how many eigenvalues are zero by introducing

$$\mathcal{G}_S = P + {}_0P_0 = (\beta_0)^2, \quad (2.19a)$$

$$\mathcal{G}_E = \tilde{I} - \mathcal{G}_S = -\sum_j ({}_jP_j). \quad (2.19b)$$

From the properties

$$(\mathcal{G}_S)^2 = \mathcal{G}_S, \quad (2.20a)$$

$$(\mathcal{G}_E)^2 = \mathcal{G}_E, \quad (2.20b)$$

$$\mathcal{G}_S\mathcal{G}_E = \mathcal{G}_E\mathcal{G}_S = 0, \quad (2.20c)$$

it follows that $\mathcal{G}_S$ and $\mathcal{G}_E$ are projection operators for orthogonal subspaces that have two and three dimensions, respectively. From the definition (2.10) of β_0 and the relations satisfied by the P matrices, we find that

$$\mathcal{G}_S\beta_0 = \beta_0\mathcal{G}_S = \beta_0, \quad (2.21a)$$

$$\mathcal{G}_E\beta_0 = \beta_0\mathcal{G}_E = 0, \quad (2.21b)$$

which shows that three of the eigenvalues of β_0 are zero. It is easy to verify from (2.2) that the matrices P_0 and ${}_0P$ are traceless,[24] whereby β_0 also shares this property and must have one eigenvalue equal to +1 and one equal to −1.

A second four-vector that is Hermitian with respect to η_0 and linearly independent of β_μ is

$$\tilde{\beta}_{\mu} = i(P_{\mu} - {}_{\mu}P) \quad (2.22a)$$

$$= i[P, \beta_{\mu}]_{-} \quad (2.22b)$$

$$= \frac{i}{3}[(\beta_{\nu}\beta^{\nu} - \tilde{I}), \beta_{\mu}]_{-}. \quad (2.22c)$$

The remaining linearly independent matrices that are Hermitian with respect to η_0 are the second-rank tensors

$$\sigma_{\mu\nu} = \frac{i}{2}({}_{\mu}P_{\nu} - {}_{\nu}P_{\mu}) \quad (2.23a)$$

$$= \frac{i}{2}[\beta_{\mu}, \beta_{\nu}]_{-}, \quad (2.23b)$$

and

$$\beta_{\mu\nu} = \tfrac{1}{2}({}_{\mu}P_{\nu} + {}_{\nu}P_{\mu}) - \tfrac{1}{4}g_{\mu\nu}\tilde{P} \quad (2.24a)$$

$$= \tfrac{1}{2}[\beta_{\mu}, \beta_{\nu}]_{+} - \tfrac{1}{4}g_{\mu\nu}\beta_{\lambda}\beta^{\lambda}. \quad (2.24b)$$

Note that $\sigma_{\mu\nu}$ is antisymmetric in the tensor indices μ and ν, whereas $\beta_{\mu\nu}$ is symmetric and traceless in these indices. We have thus found six subsets of linearly independent and Hermitian (with respect to η_0) matrices that are irreducible under proper orthochronous Lorentz transformations and that serve as a basis in terms of which to express any five-by-five matrix: two scalars, $\tilde{I}$ and P (or alternatively, $\tilde{I}$ and $\tilde{P}$), two vectors, β_μ and $\tilde{\beta}_\mu$, an antisymmetric second-rank tensor, $\sigma_{\mu\nu}$, and a traceless symmetric second-rank tensor, $\beta_{\mu\nu}$. Our results agree with those obtained by Glass,[12] but our use of the P matrices rather than the β matrices greatly simplifies the calculations in the next two sections of this paper.

The free DKP spin-0 equation may be written

$$(i\beta_{\mu}\partial^{\mu} - m)\Psi = 0. \quad (2.25)$$

As is customary, we define

$$\bar{\Psi} = \Psi^{\dagger}\eta_0, \quad (2.26)$$

which, in the free-particle theory, satisfies

$$i\partial_{\mu}\bar{\Psi}\beta^{\mu} + m\bar{\Psi} = 0. \quad (2.27)$$

The invariance of (2.25) and of (2.27) under proper orthochronous Lorentz transformations requires, because of (2.7), that

$$\Psi'(x') = S_{\Lambda}\Psi(x), \quad (2.28a)$$

$$\bar{\Psi}'(x') = \bar{\Psi}(x)S_{\Lambda}^{-1}. \quad (2.28b)$$

In particular, for an infinitesimal Lorentz transformation the above reads, because of (2.9),

$$\Psi'(x') = [\tilde{I} + \omega_{\mu\nu}({}^{\mu}P^{\nu})]\Psi(x), \quad (2.29a)$$

$$\bar{\Psi}'(x') = \bar{\Psi}(x)[\tilde{I} - \omega_{\mu\nu}({}^{\mu}P^{\nu})]. \quad (2.29b)$$

It follows that

$$\mathcal{J}_{\mu} = \bar{\Psi}\beta_{\mu}\Psi \quad (2.30)$$

is a conserved four-current and that

$$Q=\int d^3x\bar{\Psi}\beta_0\Psi=\int d^3x\Psi^\dagger\beta_0\Psi \quad (2.31)$$

is independent of the time and may be interpreted as the charge of the particles described by the field Ψ [in the single-particle (first quantized) theory, (2.31) gives the normalization of the wave function Ψ]. As is well known, (2.31) is positive for positive-energy states and negative for negative-energy ones.

When interactions are introduced, the Lorentz-transformation relations (2.28) remain the same, and (2.25) becomes

$$\mathcal{K}\Psi=0\,, \quad (2.32a)$$

where

$$\mathcal{K}=i\beta_\mu\partial^\mu-m+B\,, \quad (2.32b)$$

and where the most general form B can have is[2,12,13]

$$B=PS^{(1)}+\tilde{I}S^{(2)}+\beta^\mu J_\mu^{(1)}+\tilde{\beta}^\mu J_\mu^{(2)}+\sigma^{\mu\nu}\mathfrak{F}_{\mu\nu}+\beta^{\mu\nu}T_{\mu\nu} \quad (2.33a)$$

$$=PS^{(1)}+\tilde{I}S^{(2)}+P^\mu[J_\mu^{(1)}+iJ_\mu^{(2)}]+({}^\mu P)[J_\mu^{(1)}-iJ_\mu^{(2)}]+({}^\mu P^\nu)\{i\mathfrak{F}_{\mu\nu}+T_{\mu\nu}\}. \quad (2.33b)$$

Here $S^{(1)}$, $S^{(2)}$, $J_\mu^{(1)}$, $J_\mu^{(2)}$, $\mathfrak{F}_{\mu\nu}$, and $T_{\mu\nu}$ are all real external fields, the first two being four-scalars, the second two four-vectors, and the last two second-rank four-tensors that are, respectively, antisymmetric and traceless symmetric in the indices μ and ν. The coupling constants have been included in the external fields; e.g., one has minimal electromagnetic coupling when one puts $J_\mu^{(1)}=-eA_\mu$, where e is the charge of the particles described by the field and A_μ is the electromagnetic four-potential, while allowing the other fields to vanish identically. It is easily verified that $\mathcal{J}_\mu$, defined in (2.30), is still a conserved four-current and that Q in (2.31) retains its earlier significance.

When only the second type of scalar coupling in (2.33) is present, one in effect has a mass $m'=m(1-S^{(2)}/m)$. We shall assume that there is no open set of space-time points at which $S^{(2)}=m$, since otherwise one would have a region in which (2.32) becomes

$$i\beta_\mu\partial^\mu\Psi=0\,,$$

and does not describe massive spin-0 particles. Wightman's proof of causality for the second type of scalar coupling would certaintly fail in such a region,[25] and it is possible that even more stringent requirements concerning the behavior of $S^{(2)}$ in small neighborhoods of space-time points where $S^{(2)}=m$ are also required to avoid problems with causality. As pointed out in the Introduction, Wightman showed that (2.32) also admits noncausal propagation unless the two second-rank tensor fields in (2.33) vanish identically.[2]

III. SAKATA-TAKETANI THEORY

The DKP spin-0 equation involves five components, whereas, as discussed in the Introduction, only two of the components are physically independent. In this section we shall eliminate the three components that are not needed to describe the physics in the DKP theory, and arrive at the two-component Sakata-Taketani equation in the Schrödinger form (1.2). We first perform a Peirce decomposition[18] of the operator $\mathcal{K}$ in (2.32) into the four pieces projected by the operators $\mathcal{I}_s$ and $\mathcal{I}_E$ in (2.19), the former of which belongs to the subspace of nonzero eigenvalues of β_0. This is the same procedure that was originally employed to obtain the ST equation from the DKP equation with minimal electromagnetic coupling.[5,17]

Suppose one is given two projection operators $\mathcal{I}_1$ and $\mathcal{I}_2$ with the properties

$$\mathcal{I}_a\mathcal{I}_b=\delta_{ab}\mathcal{I}_a\,, \quad (3.1a)$$

$$\tilde{I}=\mathcal{I}_1+\mathcal{I}_2\,. \quad (3.1b)$$

Then a Peirce decomposition of an equation of the type

$$\mathcal{O}\Psi=0\,, \quad (3.2)$$

where $\mathcal{O}$ is any five-by-five operator, involves a decomposition into the two sets of equations

$$\sum_b \mathcal{O}_{ab}\Psi_b=0\,, \quad (3.3)$$

where

$${}_a\mathcal{O}_b=\mathcal{I}_a\mathcal{O}\mathcal{I}_b\,, \quad (3.4a)$$

$$\Psi_a=\mathcal{I}_a\Psi\,. \quad (3.4b)$$

From (2.19) and the relations satisfied by the P matrices, it is easily verified that

$${}_S\mathcal{K}_S=i\beta_0\frac{\partial}{\partial t}-m+{}_SB_S \quad (3.5a)$$

$${}_S\mathcal{K}_E=i\vec{\mathrm{P}}\cdot\vec{\nabla}+{}_SB_E\,, \quad (3.5b)$$

$${}_E\mathcal{K}_E=-m+{}_EB_E\,, \quad (3.5c)$$

$${}_E\mathcal{K}_S=i\,\overleftarrow{\mathrm{P}}\cdot\vec{\nabla}+{}_EB_S\,, \quad (3.5d)$$

where the notation

$$\vec{\mathrm{P}}\cdot\vec{a}=\sum_j(P_j)a_j\,, \quad (3.6a)$$

$$\overleftarrow{\mathrm{P}}\cdot\vec{a}=\sum_j({}_jP)a_j\,, \quad (3.6b)$$

is adopted for any three-vector $\vec{a}$. To find the various projections of B that occur in (3.5), it is convenient first to relabel the time components of the external four-vector fields as

$$V^{(a)}=J_0^{(a)}, \quad a=1,2 \tag{3.7}$$

and the components of the antisymmetric second-rank tensor field as

$$e_j=-\mathfrak{F}_{0j}=\mathfrak{F}_{j0}, \tag{3.8a}$$

$$b_j=-\tfrac{1}{2}\sum_{k,l}\epsilon_{jkl}\mathfrak{F}_{kl}. \tag{3.8b}$$

Furthermore, we relabel the components of the traceless symmetric tensor field as follows:

$$u=T_{00}, \tag{3.9a}$$

$$s_j=T_{0j}=T_{j0}. \tag{3.9b}$$

$$t_{jk}=t_{kj}=T_{jk}-\tfrac{1}{3}\delta_{jk}T_{00}, \tag{3.9c}$$

where t_{jk} has been defined so as to be traceless in the indices j and k. From (2.33) we now find that

$$_SB_S=({}_0P)V+(P_0)V^*+PS+({}_0P_0)\mathfrak{U}, \tag{3.10a}$$

$$_SB_E=-\vec{\mathrm{P}}\cdot\vec{\mathrm{J}}^*-\sum_j({}_0P_j)c_j^*, \tag{3.10b}$$

$$_EB_S=-\vec{\mathrm{P}}\cdot\vec{\mathrm{J}}-\sum_j({}_jP_0)c_j, \tag{3.10c}$$

$$_EB_E=\sum_{j,k}({}_jP_k)d_{jk}, \tag{3.10d}$$

where

$$V=V^{(1)}-iV^{(2)}, \tag{3.11a}$$

$$S=S^{(1)}+S^{(2)}, \tag{3.11b}$$

$$\mathfrak{U}=S^{(2)}-u, \tag{3.11c}$$

$$\vec{\mathrm{J}}=\vec{\mathrm{J}}_1-i\vec{\mathrm{J}}_2, \tag{3.11d}$$

$$\vec{\mathrm{c}}=\vec{\mathrm{s}}+i\vec{\mathrm{e}}, \tag{3.11e}$$

$$d_{jk}=(\tfrac{1}{3}u-S^{(2)})\delta_{jk}-i\sum_l\epsilon_{jkl}b_l+t_{jk}. \tag{3.11f}$$

Because of (3.4), we may now rewrite (2.32) as

$$\left(i\frac{\partial}{\partial t}-m\beta_0+\beta_0({}_SB_S)\right)\Psi_S+\left(i\sum_j({}_0P_j)\nabla_j+\beta_0({}_SB_E)\right)\Psi_E=0, \tag{3.12a}$$

$$-[m-({}_EB_E)]\Psi_E+[i\vec{\mathrm{P}}\cdot\vec{\nabla}+({}_EB_S)]\Psi_S=0. \tag{3.12b}$$

A similar procedure enables one to decompose the relation (2.29a) governing the behavior of the DKP field under infinitesimal Lorentz transformations. We introduce

$$\theta_j=\tfrac{1}{2}\sum_{k,l}\epsilon_{jkl}\omega_{kl} \tag{3.13a}$$

to describe space rotations and

$$\lambda_j=-\omega_{0j}=\omega_{j0} \tag{3.13b}$$

to describe boosts, so that (2.8) can be rewritten

$$\vec{\mathrm{x}}'=\vec{\mathrm{x}}+\vec{\theta}\times\vec{\mathrm{x}}+\vec{\lambda}t, \tag{3.14a}$$

$$t'=t+\vec{\lambda}\cdot\vec{\mathrm{x}}. \tag{3.14b}$$

Upon performing a Peirce decomposition of (2.29a) we find that

$$\Psi_S'(\vec{\mathrm{x}}',t')=\Psi_S(\vec{\mathrm{x}},t)+\sum_j\lambda_j({}_0P_j)\Psi_E(\vec{\mathrm{x}},t), \tag{3.15a}$$

$$\Psi_E'(\vec{\mathrm{x}}',t')=\left(I+\sum_{j,k,l}\epsilon_{jkl}\theta_j({}_kP_l)\right)\Psi_E(\vec{\mathrm{x}},t)-\sum_j\lambda_j({}_jP_0)\Psi_S(\vec{\mathrm{x}},t), \tag{3.15b}$$

a result which emphasizes the fact that the decomposition (3.12) is not manifestly covariant, since Ψ_S and Ψ_E mix under Lorentz boosts. It is easy to verify that Q in (2.31) may be written

$$Q=\int d^3x\,\Psi_S^\dagger\beta_0\Psi_S, \tag{3.16}$$

so Ψ_E does not contribute to the charge of the field (to the scalar product of the wave function in the first quantized theory), but it may appear in the expressions for other observables.

In the free-particle theory B vanishes identically, and one may use (3.12b) to express Ψ_E in terms of Ψ_S and its space derivatives. When the second type of scalar coupling or either of the two second-rank tensor couplings is present, it follows from (3.10d) and (3.11f) that the matrix $[m-({}_EB_E)]$ may be singular for particular strengths of the external field. Let us write

$$_E(m-B)_E=-m\sum_{j,k}({}_jP_k)W_{jk}, \tag{3.17a}$$

where

$$W_{jk}=\delta_{jk}+m^{-1}d_{jk}. \tag{3.17b}$$

We furthermore let

$$\Delta=\det W, \tag{3.18a}$$

and assume Y to be the three-by-three matrix such that

$$\sum_l Y_{jl}W_{lk}=\sum_l W_{jl}Y_{lk}=\Delta\delta_{jk}. \tag{3.18b}$$

Then

$$W_{jk}^{-1}=\Delta^{-1}Y_{jk}, \tag{3.18c}$$

$$m^{-1}\sum_{j,k}({}_jP_k)Y_{jk}=-\Delta({}_E[(m-B)^{-1}]_E), \tag{3.18d}$$

and the inverse of $[m-({}_EB_E)]$ exists unless $\Delta=0$.

If only one of the six types of coupling in (2.33) is present we obtain, with the aid of (3.11f), the following results for Δ and Y.

(a) First type of scalar coupling or either type of vector coupling:

$$\Delta=1\,, \tag{3.19a}$$

$$Y_{jk}=\delta_{jk}\,. \tag{3.19b}$$

(b) Second type of scalar coupling:

$$\Delta=\Delta_S=1-S^{(2)}/m\,, \tag{3.20a}$$

$$Y_{jk}=\delta_{jk}\,. \tag{3.20b}$$

(c) Antisymmetric second-rank tensor coupling:

$$\Delta=\Delta_A=1-m^{-2}\vec{b}^2\,, \tag{3.21a}$$

$$Y_{jk}=Y^A_{jk}=\delta_{jk}-m^{-2}b_jb_k+im^{-1}\sum_l \epsilon_{jkl}b_l\,. \tag{3.21b}$$

(d) Traceless symmetric second-rank tensor coupling:

$$\Delta=\Delta_T =\left(1+\frac{u}{3m}\right)^3-\frac{1}{2}\left(1+\frac{u}{3m}\right)\left(\frac{u}{m}\right)^2 m^{-2}\sum_{j,k}(t_{jk})^2 +\frac{1}{3}m^{-3}\sum_{j,k,l} t_{jk}t_{kl}t_{lj}\,, \tag{3.22a}$$

$$Y_{jk}=Y^T_{jk}=\left[\left(1+\frac{u}{3m}\right)^2-\frac{1}{2}m^{-2}\sum_{l,r}(t_{lr})^2\right]\delta_{jk} -\frac{1}{m}\left[\left(1+\frac{u}{3m}\right)t_{jk}-m^{-1}\sum_l t_{jl}t_{lk}\right]. \tag{3.22b}$$

We see that Δ cannot vanish in case (a) above, but it can vanish for the others.

When we multiply (3.12b) from the left by (3.18d), we obtain the relation

$$\Delta\Psi_E=-m^{-1}\sum_{j,k}({}_jP_k)Y_{jk}[i\vec{P}\cdot\vec{\nabla}+({}_EB_S)]\Psi_S\,. \tag{3.23}$$

Provided $\Delta\neq 0$, one therefore has

$$\Psi_E=-(m\Delta)^{-1}\sum_{j,k}({}_jP_k)Y_{jk}[i\vec{P}\cdot\vec{\nabla}+({}_EB_S)]\Psi_S \tag{3.24a}$$

$$=\{{}_E[(m-B)^{-1}]_E\}[i\vec{P}\cdot\vec{\nabla}+({}_EB_S)]\Psi_S\,. \tag{3.24b}$$

When either of the two second-rank tensor couplings is present Δ is not a Lorentz scalar, so even if $[m-({}_EB_E)]$ is singular at all space-time points in one reference frame, it is not singular in other frames reached by means of a Lorentz boost; with the aid of (3.15) one can easily see that the limit of (3.24) will exist as one approaches the original frame through a set of frames in which $\Delta\neq 0$ by letting the boost parameter $\vec{\lambda}\to 0$. If the second type of scalar coupling is present, the inverse does not exist when $S^{(2)}=m$, but, according to an assumption in the last paragraph of Sec. II, there is no open set of space-time points at which this equality holds; any such point is therefore a limit point of points at which $S^{(2)}\neq m$, and by taking this limit in (3.24) one should be able to define Ψ_E at such a point. Of course, the existence of this limit may require certain assumptions concerning the behavior of $S^{(2)}$ in small neighborhoods of space-time points at which $S^{(2)}=m$; it would be interesting to see if these conditions are precisely those needed to guarantee the causality of the original DKP equation, as mentioned at the end of Sec. II.

When (3.24b) is substituted into (3.12a), one obtains an equation for Ψ_S alone[26] in the Schrödinger form (1.2):

$$i\frac{\partial\Psi_S}{\partial t}=\left[-\left(i\sum_j({}_0P_j)\nabla_j+\beta_0({}_SB_E)\right)\{{}_E[(m-B)^{-1}]_E\}[i\vec{P}\cdot\vec{\nabla}+({}_EB_S)]+m\beta_0-\beta_0({}_SB_S)\right]\Psi_S\,. \tag{3.25a}$$

The same substitution in (3.15a) tell us that under an infinitesimal Lorentz transformation Ψ_S goes into

$$\Psi_S'(\vec{x}',t')=\left(\tilde{I}+\sum_j\lambda_j({}_0P_j)\{{}_E[(m-B)^{-1}]_E\}[i\vec{P}\cdot\vec{\nabla}+({}_EB_S)]\right)\Psi_S(x,t)\,. \tag{3.25b}$$

Similarly, one can eliminate Ψ_E from all observables formed from the DKP field and its adjoint to obtain expressions involving only Ψ_S and its adjoint. We refer to (3.25a) as the Sakata-Taketani equation and to the resulting theory, which involves only the field Ψ_S [or the two-component field ψ to be introduced in (3.32)] and which is covariant (but not manifestly covariant) under the transformation (3.25b), as the Sakata-Taketani theory.

To proceed, we use (3.10) and (3.18) to rewrite (3.25) in the form

$$i\frac{\partial\Psi_S}{\partial t}=\left[({}_0P)\left(\sum_{j,k}(i\nabla_j-J_j^*)(W_{jk}^{-1})(i\nabla_k-J_t)-S\right)-P\left(\sum_{j,k}c_j^*(W_{jk}^{-1})(i\nabla_k-J_k)+V\right) -({}_0P_0)\left(\sum_{j,k}(i\nabla_j-J_j^*)(W_{jk}^{-1})c_k-V^*\right)+(P_0)\left(\sum_{j,k}c_j^*(W_{jk}^{-1})c_k-\mathfrak{U}\right)+m\beta_0\right]\Psi_S\,, \tag{3.26a}$$

$$\Psi_S'(\vec{x}',t')=\left[\tilde{I}-\sum_j \lambda_j\left((_0P)\sum_k(W_{jk}^{-1})(i\nabla_k-J_k)+(_0P_0)\sum_k(W_{jk}^{-1})c_k\right)\right]\Psi_S(\vec{x},t). \quad (3.26b)$$

We now note that β_0 and $\tilde{\beta}_0$, defined by (2.10) and (2.22a), respectively, and

$$\mathcal{P}=-P+{}_0P_0 \quad (3.27)$$

satisfy the algebraic relations

$$\tilde{\beta}_0\mathcal{P}=-\mathcal{P}\tilde{\beta}_0=i\beta_0, \quad (3.28a)$$

$$\mathcal{P}\beta_0=-\beta_0\mathcal{P}=i\tilde{\beta}_0, \quad (3.28b)$$

$$\beta_0\tilde{\beta}_0=-\tilde{\beta}_0\beta_0=i\mathcal{P}, \quad (3.28c)$$

as well as

$$(\tilde{\beta}_0)^2=\mathcal{P}^2=(\beta_0)^2=\mathcal{I}_S. \quad (3.29)$$

Note that the relations (3.28) and (3.29) are, within the two-dimensional subspace belonging to $\mathcal{I}_S$, the same as the relations satisfied by the three Pauli matrices. There exists a two- (row) by-five (column) matrix $\mathcal{K}_S$ with the properties

$$\mathcal{K}_S^\dagger\mathcal{K}_S=\mathcal{I}_S, \quad (3.30a)$$

$$\mathcal{K}_S\mathcal{K}_S^\dagger=I, \quad (3.30b)$$

where I represents the two-by-two unit matrix, such that

$$\mathcal{K}_S\beta_0\mathcal{K}_S^\dagger=\rho_1, \quad (3.31a)$$

$$\mathcal{K}_S\mathcal{P}\mathcal{K}_S^\dagger=\rho_2, \quad (3.31b)$$

$$\mathcal{K}_S\beta_0\mathcal{K}_S^\dagger=\rho_3. \quad (3.31c)$$

Here the two-by-two ρ_j's are equal to the Pauli matrices, but are associated with the charge-space freedom rather than with spin.[8,27] We may thus define the (two-component) Sakata-Taketani field

$$\psi=\mathcal{K}_S\Psi_S. \quad (3.32)$$

As a result, (3.26) yields the two-component ST equation for ψ and its Lorentz-transformation properties:

$$i\frac{\partial\psi}{\partial t}=\Big[\tfrac{1}{2}(\rho_3+i\rho_1)\Big(\sum_{j,k}(i\nabla_j-J_j^*)(W_{jk}^{-1})(i\nabla_k-J_k)-S\Big)-\tfrac{1}{2}(I-\rho_2)\Big(\sum_{j,k}c_j^*(W_{jk}^{-1})(i\nabla_k-J_k)+V\Big)$$
$$-\tfrac{1}{2}(I+\rho_2)\Big(\sum_{j,k}(i\nabla_j-J_j^*)(W_{jk}^{-1})c_k-V^*\Big)+\tfrac{1}{2}(\rho_3-i\rho_1)\Big(\sum_{j,k}c_j^*(W_{jk}^{-1})c_k-\mathfrak{U}\Big)+\rho_3 m\Big]\psi(\vec{x},t), \quad (3.33a)$$

$$\psi'(\vec{x}',t')=\Big[I-\sum_j\lambda_j\Big(\tfrac{1}{2}(\rho_3+i\rho_1)\sum_k(W_{jk}^{-1})(i\nabla_k-J_k)+\tfrac{1}{2}(I+\rho_2)\sum_k(W_{jk}^{-1})c_k\Big)\Big]\psi(\vec{x},t). \quad (3.33b)$$

It is easy to verify that Q in (3.16) may be written

$$Q=\int d^3x\,\psi^\dagger\rho_3\psi. \quad (3.34)$$

We now evaluate (3.33) explicitly, using (3.11) and (3.18)–(3.22) for those cases in which only one of the six external fields is present. The results are given below.

(i) First type of scalar coupling in (2.33):

$$i\frac{\partial\psi}{\partial t}=\Big[-\tfrac{1}{2}(\rho_3+i\rho_1)\Big(\frac{1}{m}\vec{\nabla}^2+S^{(1)}\Big)+\rho_3 m\Big]\psi, \quad (3.35a)$$

$$\psi'(\vec{x}',t')=\Big(I-\frac{i}{2m}(\rho_3+i\rho_1)\vec{\lambda}\cdot\vec{\nabla}\Big)\psi(\vec{x},t), \quad (3.35b)$$

in agreement with (4.10) of Ref. 4.

(ii) Second type of scalar coupling in (2.33):

$$i\frac{\partial\psi}{\partial t}=\Big(-\frac{1}{2m}(\rho_3+i\rho_1)\vec{\nabla}\cdot(\Delta_S)^{-1}\vec{\nabla}+\rho_3 m\Delta_S\Big)\psi, \quad (3.36a)$$

$$\psi'(\vec{x}',t')=\Big(I-\frac{i}{2m}(\rho_3+i\rho_1)(\Delta_S)^{-1}\vec{\lambda}\cdot\vec{\nabla}\Big)\psi(\vec{x},t), \quad (3.36b)$$

which is the same result given in (4.11) of Ref. 4.

(iii) First type of vector coupling in (2.33):

$$i\frac{\partial\psi}{\partial t}=\Big(\frac{1}{2m}(\rho_3+i\rho_1)(-i\vec{\nabla}+\vec{J}^{(1)})^2+\rho_3 m-V^{(1)}\Big)\psi, \quad (3.37a)$$

$$\psi'(\vec{x}',t')=\Big(I+\frac{1}{2m}(\rho_3+i\rho_1)\vec{\lambda}\cdot(-i\vec{\nabla}+\vec{J}^{(1)})\Big)\psi(\vec{x},t), \quad (3.37b)$$

as given in (4.13) of Ref. 4.

(iv) Second type of vector coupling in (2.33):

$$i\frac{\partial\psi}{\partial t}=\Big(-\frac{1}{2m}(\rho_3+i\rho_1)[\vec{\nabla}^2+(\vec{\nabla}\cdot\vec{J}^{(2)})-\vec{J}^{(2)2}]$$
$$-i\rho_2V^{(2)}+\rho_3 m\Big)\psi, \quad (3.38a)$$

$$\psi'(\vec{x}',t')=\Big(I-\frac{i}{2m}(\rho_3+i\rho_1)\vec{\lambda}\cdot(\vec{\nabla}+\vec{J}^{(2)})\Big)\psi(\vec{x},t), \quad (3.38b)$$

as given in (4.14) of Ref. 4.

(v) Antisymmetric second-rank tensor coupling in (2.33)[26]:

$$i\frac{\partial\psi}{\partial t}=\Big(-\frac{1}{2m}\sum_{j,k}[(\rho_3+i\rho_1)\nabla_j(\Delta_A)^{-1}Y^A_{jk}\nabla_k+(I-\rho_2)(\Delta_A)^{-1}e_jY^A_{jk}\nabla_k$$
$$-(I+\rho_2)\nabla_j(\Delta_A)^{-1}Y^A_{jk}e_k-(\rho_3-i\rho_1)(\Delta_A)^{-1}e_jY^A_{jk}e_k]+\rho_3 m\Big)\psi\,, \tag{3.39a}$$

$$\psi'(\vec{x}',t')=\Big(I-\frac{i}{2m}(\rho_3+i\rho_1)(\Delta_A)^{-1}\sum_{j,k}\lambda_j Y^A_{jk}(\nabla_k-\rho_3 e_k)\Big)\psi(\vec{x},t)\,, \tag{3.39b}$$

the result found in (4.17) of Ref. 4.

(vi) Traceless symmetric second-rank tensor coupling in (2.33):

$$i\frac{\partial\psi}{\partial t}=\Big(-\frac{1}{2m}\sum_{j,k}[(\rho_3+i\rho_1)\nabla_j(\Delta_T)^{-1}Y^T_{jk}\nabla_k+i(I-\rho_2)(\Delta_T)^{-1}s_jY^T_{jk}\nabla_k$$
$$+i(I+\rho_2)\nabla_j(\Delta_T)^{-1}Y^T_{jk}s_k-(\rho_3-i\rho_1)(\Delta_T)^{-1}s_jY^T_{jk}s_k]-\tfrac{1}{2}(\rho_3-i\rho_1)u+\rho_3 m\Big)\psi\,, \tag{3.40a}$$

$$\psi'(\vec{x}',t')=\Big(I-\frac{i}{2m}(\rho_3+i\rho_1)(\Delta_T)^{-1}\sum_{j,k}\lambda_j Y^T_{jk}(\nabla_k+i\rho_3 s_k)\Big)\psi(\vec{x},t)\,, \tag{3.40b}$$

in agreement with (4.20b) of Ref. 4.

It is easy to specialize (3.37) and (3.39) to account for the simultaneous presence of minimal electromagnetic coupling and direct coupling to the electromagnetic four-tensor $F_{\mu\nu}$. Since these results have been stated elsewhere[4,26] we shall not give them here.

We have shown that there exists a mapping from the DKP spin-0 theory to the ST theory developed in Ref. 4. To verify that the two theories are physically equivalent, it is necessary to show that the inverse mapping from the ST to the DKP theory exists, but this is simple to do. Given the theory involving only the ST field, one may define Ψ_E using (3.24), keeping in mind the remarks following that equation concerning the possible need for certain additional assumptions regarding the behavior of $S^{(2)}$, and then make the appropriate substitutions into the right-hand sides of (3.25) to arrive at (3.12a) and (3.15a). The relations (3.12b) and (3.15b) are easily seen to follow from (3.15a) and (3.24). From the known causality of the DKP theory, except when either of the two second-rank tensor couplings is present,[2] and from the equivalence of the two theories, it follows that the ST theory has the same causal properties.

In our earlier paper[28] we discussed in considerable detail the significance of the fact that Δ_S in (3.20a) is a Lorentz scalar, whereas Δ_A and Δ_T in (3.21a) and (3.22a), respectively, are not. At the same time we verified that the two scalar and two vector couplings are causal in the ST theory and that the two second-rank tensor couplings admit noncausal propagation. As we pointed out in the Introduction, one advantage of the ST theory is that the causal properties are apparent from the manner in which the coefficients of the space derivatives depend on the external field.

IV. KLEIN-GORDON THEORY

In the Introduction we called attention to the different ways in which the single-component KG equation and the two-component ST equation exhibit the two degrees of freedom in the "charge space," the former doing so by "doubling" the time derivative and the latter by "doubling" the single component corresponding to a spin-0 particle[8]; thus either theory can be said to possess the minimum number of components required to describe spin-0 particles and their antiparticles. It is interesting to see how the KG theory may be obtained from the DKP theory with external fields by means of a Peirce decomposition,[26] as we shall do in this section.

Subsequent to Eqs. (2.16) we called attention to the fact that

$$\mathcal{G}_K=P \tag{4.1a}$$

and

$$\mathcal{G}_V=\tilde{P}\,, \tag{4.1b}$$

are, respectively, the projection operators for one-dimensional and four-dimensional subspaces of Ψ, the DKP field, and that these subspaces are invariant under proper orthochronous Lorentz transformations. Thus,

$$\Psi_K=P\Psi=\mathcal{G}_K\Psi \tag{4.2a}$$

transforms under the $D^{(0,0)}$ representation of the proper orthochronous Lorentz group and

$$\Psi_V=\tilde{P}\Psi=\mathcal{G}_V\Psi \tag{4.2b}$$

transforms under the $D^{(1/2,1/2)}$ representation. In fact, as is easily verified using (2.29) and the algebraic relations in Sec. II,

$$\Psi'_K(x')=\Psi_K(x)\,, \tag{4.3a}$$

$$\Psi'_V(x') = [\bar{I} + \omega_{\mu\nu}({}^\mu P^\nu)]\Psi_V(x) , \tag{4.3b}$$

so Ψ_K transforms like the Klein-Gordon field.

To obtain the Klein-Gordon equation from (2.32) for the interacting DKP spin-0 field, we once more perform a Peirce decomposition of the operator $\mathcal{H}$, but now consider the four pieces projected by $\mathcal{I}_K$ and $\mathcal{I}_V$. The various steps follow closely those employed in the preceding section. From (3.4a) and the algebra of the P matrices we find that

$$ {}_K\mathcal{H}_K = -m + {}_K B_K , \tag{4.4a}$$

$$ {}_K\mathcal{H}_V = i(P_\mu)\partial^\mu + {}_K B_V , \tag{4.4b}$$

$$ {}_V\mathcal{H}_V = -m + {}_V B_V , \tag{4.4c}$$

$$ {}_V\mathcal{H}_K = i({}_\mu P)\partial^\mu + {}_V B_K . \tag{4.4d}$$

In the above

$$ {}_K B_K = PS , \tag{4.5a}$$

$$ {}_K B_V = (P^\mu) J^*_\mu , \tag{4.5b}$$

$$ {}_V B_K = ({}^\mu P) J_\mu , \tag{4.5c}$$

$$ {}_V B_V = ({}^\mu P^\nu) D_{\mu\nu} , \tag{4.5d}$$

where

$$ S = S^{(1)} + S^{(2)} , \tag{4.6a}$$

$$ J_\mu = J_\mu^{(1)} - iJ_\mu^{(2)} , \tag{4.6b}$$

$$ D_{\mu\nu} = g_{\mu\nu}S^{(2)} + i\mathcal{F}_{\mu\nu} + T_{\mu\nu} . \tag{4.6c}$$

As a result of (3.3), and (4.4), we may rewrite (2.32) in the form

$$ [i(P_\mu)\partial^\mu + ({}_K B_V)]\Psi_V - [m - ({}_K B_K)]\Psi_K = 0 , \tag{4.7a}$$

$$ -[m - ({}_V B_V)]\Psi_V + [i({}_\mu P)\partial^\mu + ({}_V B_K)]\Psi_K = 0 . \tag{4.7b}$$

The charge Q in (2.31) may be written

$$ Q = \int d^3x [\bar{\Psi}_K(P_0)\Psi_V - \bar{\Psi}_V({}_0 P)\Psi_K] . \tag{4.8}$$

We want to use (4.7b) to express Ψ_V in terms of Ψ_K, substituting the result into (4.7a) to obtain an equation for Ψ_K alone. We see from (4.5d) and (4.6c) that the matrix $[m - ({}_V B_V)]$ in (4.7b) may be singular when the second type of scalar coupling or either of the two second-rank tensor couplings is present. In analogy with (3.17) we write

$$ {}_V(m - B)_V = m({}^\mu P_\nu)\mathcal{W}_\mu{}^\nu , \tag{4.9a}$$

where

$$ \mathcal{W}_\mu{}^\nu = \delta_\mu{}^\nu - \frac{1}{m} D_\mu{}^\nu . \tag{4.9b}$$

We also let

$$ \tilde{\Delta} = \det \mathcal{W} , \tag{4.10a}$$

and assume $\mathcal{Y}$ to be the four-by-four matrix such that

$$ (\mathcal{Y}_\mu{}^\lambda)(\mathcal{W}_\lambda{}^\nu) = (\mathcal{W}_\mu{}^\lambda)(\mathcal{Y}_\lambda{}^\nu) = \tilde{\Delta}\delta_\mu{}^\nu . \tag{4.10b}$$

Consequently,

$$ \mathcal{W}^{-1}{}_\mu{}^\nu = \tilde{\Delta}^{-1}\mathcal{Y}_\mu{}^\nu , \tag{4.10c}$$

$$ m^{-1}({}_\mu P^\nu)\mathcal{Y}_\nu{}^\mu = \tilde{\Delta}({}_V[(m - B)^{-1}]_V) . \tag{4.10d}$$

When only one of the six types of coupling in (2.33) is present we obtain the following results with the aid of (4.6).

(a) First type of scalar coupling or either type of vector couplings:

$$ \tilde{\Delta} = 1 , \tag{4.11a}$$

$$ \mathcal{Y}_\mu{}^\nu = \delta_\mu{}^\nu . \tag{4.11b}$$

(b) Second type of scalar coupling:

$$ \tilde{\Delta} = \tilde{\Delta}_S = 1 - S^{(2)}/m , \tag{4.12a}$$

$$ \mathcal{Y}_\mu{}^\nu = \delta_\mu{}^\nu . \tag{4.12b}$$

(c) Antisymmetric second-rank tensor coupling:

$$ \tilde{\Delta} = \tilde{\Delta}_A = 1 - \tfrac{1}{2}m^{-2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - m^{-4}(\mathcal{F}^{\mu\nu}\mathcal{F}^D_{\mu\nu})^2 , \tag{4.13a}$$

$$ \begin{aligned} \mathcal{Y}_\mu{}^\nu = \mathcal{Y}^{A\nu}_\mu = \delta_\mu{}^\nu(1 - \tfrac{1}{2}m^{-2}\mathcal{F}_{\lambda\rho}\mathcal{F}^{\lambda\rho}) + im^{-1}\mathcal{F}_\mu{}^\nu \\ - m^{-2}\mathcal{F}_{\mu\lambda}\mathcal{F}^{\lambda\nu} + \tfrac{1}{4}im^{-3}(\mathcal{F}^{\lambda\rho}\mathcal{F}^D_{\lambda\rho})\mathcal{F}^{D\nu}_\mu , \end{aligned} \tag{4.13b}$$

with

$$ \mathcal{F}^D{}_\mu{}^\nu = \tfrac{1}{2}\epsilon_\mu{}^{\nu\lambda\rho}\mathcal{F}_{\lambda\rho} . \tag{4.13c}$$

(d) Traceless symmetric second-rank tensor coupling:

$$ \begin{aligned} \tilde{\Delta} = \tilde{\Delta}_T = 1 - \tfrac{1}{2}T_\mu{}^\nu T_\nu{}^\mu - \tfrac{1}{3}T_\mu{}^\nu T_\nu{}^\lambda T_\lambda{}^\mu \\ - \tfrac{1}{4}T_\mu{}^\nu T_\nu{}^\lambda T_\lambda{}^\sigma T_\sigma{}^\mu + \tfrac{1}{8}(T_\mu{}^\nu T_\nu{}^\mu)^2 , \end{aligned} \tag{4.14a}$$

$$ \begin{aligned} \mathcal{Y}_\mu{}^\nu = \mathcal{Y}^T{}_\mu{}^\nu = \delta_\mu{}^\nu(1 - \tfrac{1}{2}T_\lambda{}^\rho T_\rho{}^\lambda + \tfrac{1}{3}T_\lambda{}^\rho T_\rho{}^\sigma T_\sigma{}^\lambda) + T_\mu{}^\lambda T_\lambda{}^\nu \\ + T_\mu{}^\nu(1 + \tfrac{1}{2}T_\lambda{}^\rho T_\rho{}^\lambda) - T_\mu{}^\lambda T_\lambda{}^\rho T_\rho{}^\nu . \end{aligned} \tag{4.14b}$$

After operating on (4.7b) from the left with (4.10d) we have

$$ \tilde{\Delta}\Psi_V = m^{-1}({}_\mu P^\nu)\mathcal{Y}_\nu{}^\mu[i({}_\lambda P)\partial^\lambda + {}_V B_K]\Psi_K , \tag{4.15}$$

and when $\tilde{\Delta} \neq 0$ it follows that

$$ \Psi_V = (m\tilde{\Delta})^{-1}({}_\mu P^\nu)\mathcal{Y}_\nu{}^\mu[i({}_\lambda P)\partial^\lambda + {}_V B_K]\Psi_K \tag{4.16a}$$

$$ = ({}_V[(m - B)^{-1}]_V)[i({}_\lambda P)\partial^\lambda + {}_V B_K]\Psi_K . \tag{4.16b}$$

For the reasons mentioned following (3.24), there are no problems presented by the vanishing of $\tilde{\Delta}$ in the presence of the second type of scalar coupling, which is causal. However, the arguments used there for the two second-rank tensor couplings are not valid now because $\tilde{\Delta}$ is a four-scalar, as (4.13a) and (4.14a) demonstrate. Thus, if the external second-rank tensor field is such that $\tilde{\Delta}$ vanishes over some space-time region, $[m-({}_E B_E)]$ does not possess an inverse, and it is not possible to eliminate Ψ_V in this region to obtain an equation for Ψ_K alone; we must then confine our derivation of the KG equation to space-time regions where there is no open set of points at which $\tilde{\Delta}=0$.

When (4.16b) is substituted into (4.7a) and (4.8) we obtain, respectively,

$$\{[i(P_\mu)\partial^\mu + {}_K B_V](_V[(m-B)^{-1}]_V)[i({}_\mu P)\partial^\mu + {}_V B_K] - (m - {}_K B_K)\}\Psi_K = 0\,, \tag{4.17a}$$

$$Q=\int d^3x\{\overline{\Psi}_K(P_0)(_V[(m-B)^{-1}]_V)[i({}_\mu P)\partial^\mu + {}_V B_K]\Psi_K - [i(\partial^\mu\overline{\Psi}_K)P_\mu - \overline{\Psi}_K({}_K B_V)](_V[(m-B)^{-1}]_V)({}_0 P)\Psi_K\}\,. \tag{4.17b}$$

In analogy with the procedure followed commencing with (3.30), we can introduce a one-(row) by-five (column) matrix $\mathcal{K}_K$ with the properties

$$\mathcal{K}_K^\dagger \mathcal{K}_K = P\,, \tag{4.18a}$$

$$\mathcal{K}_K \mathcal{K}_K^\dagger = 1\,, \tag{4.18b}$$

and we may define the (one-component) Klein-Gordon wave function

$$\phi = m^{-1/2}\mathcal{K}_K \Psi_K\,. \tag{4.19}$$

From (4.5), (4.10), and (4.17) we obtain the Klein-Gordon equation

$$[(\partial^\mu - iJ^{\mu *})(\mathcal{W}^{-1}{}_\mu{}^\nu)(\partial_\nu - iJ_\nu) - mS + m^2]\phi = 0 \tag{4.20a}$$

and the charge operator

$$Q=\int d^3x\{\phi^*(\mathcal{W}^{-1}{}_0{}^\mu)[(i\partial_\mu + J_\mu)\phi] - [(i\partial^\mu - J^{\mu *})\phi^*](\mathcal{W}^{-1}{}_\mu{}^0)\phi\}\,. \tag{4.20b}$$

Finally, because of (4.3a), we have under Lorentz transformations

$$\phi'(x') = \phi(x)\,. \tag{4.21}$$

We now evaluate (4.20), using (4.6) and (4.10)–(4.14) for all cases in which only one of the four external fields is present, as was done for the Sakata-Taketani theory in (3.35) through (3.40). Note, however, that whereas we evaluated the Lorentz-transformation properties for the ST field, we are instead going to express the charge Q in terms of the KG field. The reason, of course, is that Q always has the form (3.34) when expressed in terms of the ST field and the KG field always transforms as in (4.21).

(i) First type of scalar coupling in (2.33):

$$[\Box - mS^{(1)} + m^2]\phi = 0\,, \tag{4.22a}$$

$$Q = i\int d^3x[\phi^*(\partial_0\phi) - (\partial_0\phi^*)\phi]\,. \tag{4.22b}$$

(ii) Second type of scalar coupling in (2.33):

$$[\partial^\mu(\Delta_S)^{-1}\partial_\mu + m^2\Delta_S]\phi = 0\,, \tag{4.23a}$$

$$Q = i\int d^3x(\Delta_S)^{-1}[\phi^*(\partial_0\phi) - (\partial_0\phi^*)\phi]\,. \tag{4.23b}$$

(iii) First type of vector coupling in (2.33):

$$[(\partial^\mu - iJ^{(1)\mu})(\partial_\mu - iJ_\mu^{(1)}) + m^2]\phi = 0\,, \tag{4.24a}$$

$$Q = i\int d^3x\{\phi^*[(\partial_0 - iJ_0^{(1)})\phi] - [(\partial_0 + iJ_0^{(1)})\phi^*]\phi\}\,. \tag{4.24b}$$

(iv) Second type of vector coupling in (2.33):

$$[(\partial^\mu + J^{(2)\mu})(\partial_\mu - J_\mu^{(2)}) + m^2]\phi = 0\,, \tag{4.25a}$$

$$Q = i\int d^3x[\phi^*(\partial_0\phi) - (\partial_0\phi^*)\phi]\,. \tag{4.25b}$$

(v) Antisymmetric second-rank tensor coupling in (2.33)[26]:

$$[\partial^\mu(\tilde{\Delta}_A)^{-1}\mathcal{Y}^A{}_\mu{}^\nu\partial_\nu + m^2]\phi = 0\,, \tag{4.26a}$$

$$Q = i\int d^3x(\tilde{\Delta}_A)^{-1}[\phi^*(\mathcal{Y}^A{}_0{}^\mu)(\partial_\mu\phi) - (\partial^\mu\phi^*)(\mathcal{Y}^A{}_\mu{}^0)\phi]\,. \tag{4.26b}$$

(vi) Symmetric second-rank tensor coupling in (2.33):

$$[\partial^\mu(\tilde{\Delta}_T)^{-1}\mathcal{Y}^T{}_\mu{}^\nu\partial_\nu + m^2]\phi = 0\,, \tag{4.27a}$$

$$Q = i\int d^3x(\tilde{\Delta}_T)^{-1}[\phi^*(\mathcal{Y}^T{}_0{}^\mu)(\partial_\mu\phi) - (\partial^\mu\phi^*)(\mathcal{Y}^T{}_\mu{}^0)\phi]\,. \tag{4.27b}$$

The two scalar and two vector couplings in (4.22) through (4.25) are equivalent to the corresponding DKP results provided $S^{(2)}$ is sufficiently well-behaved, since it is easy to see, just as we did for the ST equation, that we can always find a mapping from the KG theory to the DKP theory; owing to the causality of the DKP theory in these cases we know that the KG theory is also causal. But if the Lorentz scalars $\tilde{\Delta}_A$ and $\tilde{\Delta}_T$, which are defined in (4.13a) and (4.14a), respectively, and which appear in the equations (4.26) and (4.27) for the two second-rank tensor couplings, vanish on some open set of space-time points, the KG equation is not defined at these points even though the corresponding DKP equation exists; thus the DKP and KG theories are not necessarily equivalent when noncausal effects are present. On the other hand, as we saw in the preceding section, the ST and DKP theories should always be equivalent for the six types of external-field couplings we have considered. It is interesting to note that Fischbach, Nieto, and Scott[17] have previously commented, in discussing symmetry-breaking interactions, that it should always be possible to associate the ST equation with the DKP equation, but not necessarily with the KG equation.

The causality of (4.22a), (4.24a), and (4.25a) can also be verified directly from the test mentioned in connection with (1.5). By multiplying (4.23a) from the left by Δ_S, it is evident that this case also passes the test, except when $S^{(2)}=m$. Both (4.26a) and (4.27a) fail the test, and the existence of noncausal effects can be proved by examining the nature of the normals to their characteristic surfaces.[1,16]

V. SUMMARY AND DISCUSSION

Starting from the (five-component) Duffin-Kemmer-Petiau spin-0 equation with six types of external field interactions, two scalar, two vector, an antisymmetric second-rank tensor, and a traceless symmetric second-rank tensor, we have obtained the corresponding (two-component) Sakata-Taketani equation by means of a Peirce decomposition and have also found its Lorentz-transformation properties. The results agree with those found elsewhere by means of a Lagrangian formalism.[4] We have also obtained the (one-component) Klein-Gordon equation for each type of coupling.

The DKP field Ψ was written as the sum of the ST field Ψ_S and the field Ψ_E. To arrive at the ST theory one completely eliminates Ψ_E from all equations and all expressions for physical observables, obtaining expressions involving Ψ_S alone, but this is done at the expense of manifest covariance. If one insists on maintaining manifest covariance, it is necessary to retain Ψ_E along with Ψ_S; following Krajcik and Nieto,[29] one may refer to Ψ_S as the "particle components" and to Ψ_E as the "subsidiary components." Since, however, the former are all that is needed to represent the physics involved, we are also justified in following Heitler[5] by referring to the latter components of the DKP field as being "redundant."

An advantage of the ST theory is that it enables one to express the field equation in the Schrödinger form (1.2) without the need for any subsidiary conditions on the field.[30] When an equation has the Schrödinger form, H does not involve any time derivatives acting on the field and plays the role of the Hamiltonian in the first quantized theory. A knowledge of the Hamiltonian permits the use of the Heisenberg picture, which, as has been emphasized by Dirac,[31] is important in obtaining a physical understanding of any quantum theory. If one seeks an equation of the form (1.2) for Ψ_E alone, as Krajcik and Nieto[29] did for the case of minimal electromagnetic coupling, it is found that H involves time derivatives operating on Ψ_E in a nonlocal manner and is not a Hamiltonian in the first-quantized theory; furthermore, a one-to-one relation between the number of components and the number of physical degrees of freedom does not exist.

An important result of this paper is the fact that the noncausal nature of the two second-rank tensor couplings is at once evident from inspection of the coefficients of the derivatives in the ST or KG equation, but not from the form of the DKP equation. A similar conclusion can be reached when one investigates various types of coupling for the DKP and ST spin-1 equations.[19] The reason that the ST equations manifest the causal properties is that both have $2(2J+1)$ components, the minimum number required to describe a massive spin-J particle by an equation in the Schrödinger form (1.2).

It is evident that, in order to manifest the causal properties of higher-spin theories, one should express them in $2(2J+1)$-component Schrödinger form. If the metric is required to commute with $\vec{x}$ and if the discrete symmetry operators are required to be such that $\vec{x}$ transforms as a three-vector, conditions whose importance has been stressed elsewhere,[22] then it follows[21] that H in (1.2) is a nonlocal operator in the free-particle theory for $J \geq \frac{3}{2}$; that is to say, it cannot be expressed as a polynomial in the space derivatives. Reference 22 stated the significance of this result: When a free manifestly covariant theory of the form (1.3) possesses a constant Hermitianizing matrix[11] it should not be possible to eliminate the extra components in a local manner, in contrast to the situation that exists for the DKP spin-0 and spin-1 equations, and we thus obtain an explanation of why even minimal electromagnetic coupling is noncausal for $J \geq \frac{3}{2}$. [By avoiding the use of such an Hermitianizing matrix one can obtain causal equations of the form (1.3), but they are unstable.[15]] This conclusion is consistent with the results of Moldauer and Case,[32] who obtained nonlocal operators H when they reduced the manifestly covariant free Rarita-Schwinger spin-$\frac{3}{2}$ and spin-$\frac{5}{2}$ equations[33] to the form (1.2) with $2(2J+1)$ components. A similar conclusion has recently been reached independently by Capri and

Shamaly,[34] who performed a Peirce decomposition of the manifestly covariant Gupta form[35] of the Rarita-Schwinger spin-$\frac{3}{2}$ equation with minimal electromagnetic coupling.

The simplest form[22] that a $2(2J+1)$-component Schrödinger-like equation can have appears to be the linear form proposed by Guth.[27] The resulting equations are either the arbitrary-spin generalizations of the Dirac equation found earlier by several authors[20,21] or the arbitrary-spin generalizations of the Sakata-Taketani spin-0 and spin-1 equations.[21,36]

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