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Feynman propagator in curved space-time

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The Wick rotation is generalized in a covariant manner so as to apply to curved manifolds in a way that is independent of the analytic properties of the manifold. This enables us to show that various methods for defining a Feynman propagator to be found in the literature are equivalent where they are applicable. We are also able to discuss the relation between certain regularization methods that have been employed.

I. INTRODUCTION

A central question in the construction of a quantum field theory in curved space-time is the definition of the Feynman propagator. Once this is known, much of the information required from such a theory becomes readily available. The propagator is a Green's function for the hyperbolic equation

$$LG(x,x')=-g^{-1/2}\delta(x,x'), \qquad (1)$$

but unlike the advanced or retarded Green's functions, which can be defined locally, the Feynman propagator is a solution defined globally through certain boundary or positive-frequency conditions. In the case of, say, a Dirac field interacting with a classical external electromagnetic field, the positive-frequency conditions imposed on the Feynman function $G(x,x')$ are the same as those for a free Dirac field; thus, although the electromagnetic potentials enter the operator L explicitly, and may make (1) difficult to solve, they do not enter into the positive-frequency conditions. However, for an external gravitational field, the background metric enters the problem twice: once through the operator L, which contains the metric explicitly, and again through the global structure of the manifold, and, consequently, the boundary conditions.

Suppose $G(x,x')$ to be the propagator for a massive Klein-Gordon field:

$$(\Box-m^2)G(x,x')=-g^{-1/2}\delta(x,x'). \qquad (2)$$

In *Minkowski space* $G(x,x')$ exhibits a number of properties which one might attempt to generalize to curved space-time. In particular, (i) G is the (unique) analytic continuation to Minkowski space of the preferred fundamental solution to the elliptic equation obtained from (2) by the replacement $t\to it$ of the Minkowski coordinate time t; (ii) G is the unique solution to (2) which is analytic in the upper half $\sigma=\frac{1}{2}(x-x')^2$ plane, and such that $G\to 0$ as $\sigma\to\infty$ along any ray in the upper half-plane (i.e., as $\mathrm{Im}\sigma$ and $\mathrm{Re}\sigma$ tend to infinity together); (iii) G is the unique solution to (2) which is analytic in the lower half m^2 plane and such that $G\to 0$ as $m^2\to\infty$ along any ray in the lower half-plane.

In Minkowski space these properties are equivalent since they each define the same propagator G. However, it is not clear *a priori* that unique covariant generalizations of these properties to curved space are possible, or that any such generalizations would be equivalent.

DeWitt[1] has studied the Feynman propagator in curved space-time by an extension of the proper-time formalism of Feynman and Schwinger. Rewriting (2) as a matrix equation

$$FG=-1, \qquad (3)$$

with F a matrix with space-time components $(\Box-m^2)g^{-1/2}\delta(x,x')$, DeWitt obtains a representation for G of the form

$$G=i\int_0^\infty e^{isF}ds, \qquad (4)$$

which incorporates the usual "$i\epsilon$ prescription." In terms of space-time components (4) corresponds to

$$G(x,x')=i\int_0^\infty ds\, e^{-im^2s}f(x,x';s), \qquad (5)$$

where f satisfies a "Schrödinger equation"

$$\Box f=-i\frac{\partial}{\partial s}f \qquad (6)$$

subject to initial conditions

$$f(x,x';0)=-g^{-1/2}\delta(x,x').$$

A virtue of this approach is that it gives rise to an interpretation in which $f(x,x';s)$ is the amplitude for a particle to propagate from a point x to a point x' in a proper time s. The representation (5) then exhibits $G(x,x')$ as the amplitude for the particle to propagate from x to x' in some proper time.

Sufficient conditions for a *given* Green's function to admit a representation of the form (5) are that, regarded as a function of m^2, G admits a Fourier transform, is analytic in the lower half m^2 plane, and tends to zero as $m^2 \to \infty$ along any ray in the lower half-plane. On the other hand, the above construction of the proper-time representation is entirely heuristic and any attempt at a more rigorous derivation will clearly require careful consideration of the boundary conditions to be imposed on G and of the global properties of the space-time manifold. A partial statement of our main result is that for a complete, nonsingular, manifold we can construct a propagator which exhibits covariant generalizations of properties (i)–(iii) and admits a proper-time representation of the form (5). Moreover, it is symmetric, and satisfies DeWitt's variational law for the change in G resulting from a small variation in the geometry,

$$\delta G = G\delta F G .$$

Under a restrictive set of assumptions we are able to show that this propagator is unqiue, and we indicate why we expect that the restrictions can be weakened. The construction of the Feynman propagator on singular manifolds is discussed briefly in Sec. V. We emphasize that we are concerned here with the question of whether, from the assumed existence of many solutions to (1), it is possible to pick one, to be called the Feynman propagator, in a reasonable way. We are not here concerned with the more subtle question of the global existence of solutions to (1). Thus we shall not proceed with total rigor in that we shall need to assume that certain quantities are reasonably well behaved; we shall, for example, assume it to be permissible to neglect surface terms when using the divergence theorem.

Finally, we should note that a unique Feynman propagator does not per se supply a unique physical interpretation, since the regularization scheme is not sacrosanct, and is to be invoked with regard to the physical considerations pertinent to a particular problem.[2]

II. A GENERALIZED WICK ROTATION

The replacement $t \to it$ is clearly beset with ambiguities if the metric is not analytic, and even in the analytic case there may well be ambiguities through choice of t, or the replacement may be ineffective if the coefficients in L involve functions of t. We have shown elsewhere[3] how the Wick rotation may be written in a manifestly covariant form on a static manifold. We now generalize this to a (C^∞) manifold $(\mathfrak{M}, g)$ which admits an everywhere timelike smooth vector field V, the integral curves of which have infinite proper length in the metric g. (This is a restriction both on the manifold and the vector field.) Then for $\lambda - 1$ real and positive

$$\tilde{g}(X, Y) = g(X, Y) + \lambda g(X, V) g(Y, V) ,$$

with V normalized such that $g(V, V) = -1$ is a positive-definite metric on a manifold $\tilde{\mathfrak{M}}$. For the theorems that follow we require $\mathfrak{M}$ to be topologically $\mathbb{R} \times \Sigma$, where Σ is a manifold without boundary. Thus, a necessary condition is that $\mathfrak{M}$ be globally hyperbolic.

We have

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \lambda V_\alpha V_\beta ,$$

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} + \frac{\lambda}{\lambda - 1} V^\alpha V^\beta ,$$

$$\tilde{g}^{1/2} = -i(\lambda - 1)^{1/2} g^{1/2} ,$$

and we take λ to be a complex parameter. We define $G_\lambda(x, x' \,|\, V)$ as the solution of the inhomogeneous Klein-Gordon equation relative to the "metric" $\tilde{g}$:

$$LG_\lambda(x, x' \,|\, V) \equiv [g^{-1/2}\partial_\alpha(g^{1/2}\tilde{g}^{\alpha\beta}\partial_\beta) - m^2] G_\lambda(x, x' \,|\, V) = -i(\lambda - 1)^{1/2} g^{-1/2}\delta(x, x') , \qquad (7)$$

with the boundary condition that $G_\lambda \to 0$ as $x \to \infty$. The amplitude $f_\lambda(x, x'; s \,|\, V)$ is defined relative to $\tilde{g}$ by analogy with (7) as the solution of

$$g^{-1/2}\partial_\alpha(g^{1/2}\tilde{g}^{\alpha\beta}\partial_\beta) f_\lambda = (\lambda - 1)^{-1/2}\frac{\partial}{\partial s} f_\lambda \qquad (8)$$

subject to

$$f_\lambda(x, x'; 0 \,|\, V) = g^{-1/2}\delta(x, x') , \quad f_\lambda \to 0 \text{ as } x \to \infty .$$

Since L_λ is an elliptic operator in the cut plane $|\arg(\lambda - 1)| < \pi$, it is reasonable to assume, for each V, the existence of solutions to (7) and (8). Our strategy will be to define a Feynman propagator as the limit as $\lambda \to 0^+$ (i.e., from above the cut) of $G_\lambda[V]$. For the consistency of this approach it is necessary to establish that, for a given manifold, the solutions to (7) and (8) with the stated boundary conditions are unique and analytic in λ and that the limits as $\lambda \to 0^+$ of $G_\lambda[V]$ and $f_\lambda[V]$ exist [if these limits exist in the sense of distributions, then they certainly satisfy (2) and (6)]. The uniqueness of G follows if we can show that

$$\lim_{\lambda \to 0^+} G_\lambda[V]$$

is independent of V.

The uniqueness and analyticity of $G_\lambda[V]$ and $f_\lambda[V]$ might be expected from the corresponding results in the static case. We consider this in Sec. III. In Secs. IV and V we consider the existence of the limit as $\lambda \to 0^+$ and its invariance under change of V. To see that it is not unreasonable to expect this construction to yield a unique propagator, even for a nonstatic manifold where there is no privileged choice for V, suppose that

$$\lim_{\lambda\to 0^+} G_\lambda(x,x'\,|\,V)$$

exists for arbitrary V. Then we have

$$L_\lambda[W](G_\lambda[W]-G_\lambda[V]) = (L_\lambda[V]-L_\lambda[W])G_\lambda[V],$$

where

$$L_\lambda[V]-L_\lambda[W] = \lambda(\lambda-1)^{-1}g^{-1/2}\partial_\alpha(V^\alpha V^\beta - W^\alpha W^\beta)\partial_\beta \equiv \lambda(\lambda-1)^{-1}X.$$

Thus (an asterisk denotes a convolution)

$$G_\lambda[W]-G_\lambda[V] = \lambda(\lambda-1)^{-1}G_\lambda[W]*XG_\lambda[V],$$

and in the limit $\lambda\to 0^+$ we see that G is independent of V provided the convolution integral exists. A similar result holds for $f_\lambda[V]$ if we start from the identity

$$\left(L_\lambda[W]-(\lambda-1)^{-1/2}\frac{\partial}{\partial s}\right)(f_\lambda[W]-f_\lambda[V]) = (L_\lambda[V]-L_\lambda[W])f_\lambda[V],$$

and note that $f_\lambda[W]$ is the appropriate inverse for the operator on the left-hand side. The existence of the convolution is difficult to prove since to estimate the integrals when V and W differ on a noncompact set requires a detailed knowledge of the behavior of $G_\lambda[V]$ or $f_\lambda[V]$. Thus we do not achieve this proof in the general case.

For a given vector field V, we can see how our prescription establishes or generalizes the properties of the Feynman propagator discussed in Sec. I. Our prescription for the construction of $G[V]$ may itself be regarded as a generalization of property (i). To generalize property (ii) consider the "geodetic interval" $\tilde\sigma(x,x')$ relative to $\tilde g$ which is related to the geodetic interval $\sigma(x,x')$ in $(\mathfrak{M},g)$ by[4]

$$\tilde\sigma(x,x') = \tfrac12(\tau'-\tau)\int_x^{x'} d\tau''\tilde g_{\alpha\beta}\frac{dx^\alpha}{d\tau''}\frac{dx^\beta}{d\tau''} = \sigma(x,x') + \tfrac12\lambda(\tau'-\tau)\int_x^{x'} d\tau''\left(V_\alpha\frac{dx^\alpha}{d\tau''}\right)^2,$$

where τ denotes proper time. The expression multiplying λ is positive-definite. Consider then $G_\lambda(x,x')$ as a function of $\tilde\sigma(x,x')$ and *any* seven other variables which are functions of the eight coordinates, Z_J $(J=1,\ldots,7)$ say. Then in the limit $\lambda\to 0^+$ we obtain

$$G_\lambda(\tilde\sigma(x,x'),Z_J\,|\,V)\to G(\sigma(x,x')+i\epsilon, Z_J\,|\,V),$$

which expresses property (ii).

To derive the relation between f_λ and G_λ we note that

$$\begin{aligned}(\tilde\Box-m^2)i\int_0^\infty ds\,e^{-(\lambda-1)^{1/2}m^2 s}f_\lambda(x,x';s\,|\,V) &= i\int_0^\infty ds\left[e^{-(\lambda-1)^{1/2}m^2 s}\tilde\Box f_\lambda + (\lambda-1)^{-1/2}f_\lambda\frac{\partial}{\partial s}e^{-(\lambda-1)^{1/2}m^2 s}\right]\\ &= i(\lambda-1)^{-1/2}(f_\lambda e^{-(\lambda-1)^{1/2}m^2 s})_0^\infty + i\int_0^\infty ds\,e^{-(\lambda-1)^{1/2}m^2 s}\left[\tilde\Box-(\lambda-1)^{-1/2}\frac{\partial}{\partial s}\right]f_\lambda\\ &= i(\lambda-1)^{-1/2}g^{-1/2}\delta(x,x'),\end{aligned} \tag{9}$$

where we neglect a possible contribution from the upper limit of the integrated term in (9) since the vanishing of this contribution is a weaker condition than the existence of the integral of f. Thus, provided the integral exists, it will follow from the uniqueness of the solution to (7) that

$$G_\lambda(x,x'\,|\,V) = i\int_0^\infty ds\,e^{-(\lambda-1)^{1/2}m^2 s}f_\lambda(x,x';s\,|\,V). \tag{10}$$

Continuing to $\lambda\to 0^+$ we see that $G_\lambda[V]$ is analytic in the lower half m^2 plane. This establishes property (iii) and gives the proper-time representation of $G_\lambda[V]$.

The symmetry of $G_\lambda(x,x'\,|\,V)$ is a standard property of Green's functions for elliptic operators (provided we are allowed to discard surface terms on integrating by parts) and will be preserved in the limit. The same remark applies, *mutatis mutandis*, to the DeWitt variational law.

III. UNIQUENESS AND ANALYTICITY OF $G_\lambda[V]$

The uniqueness theorems for elliptic equations can be adapted to (7) for a fixed vector field V. Consider the homogeneous equation corresponding to (7),

$$L_\lambda u = 0 \tag{11}$$

for $|\arg(\lambda-1)|<\pi$, and assume u to vanish sufficiently rapidly as $x\to\infty$ to permit integration by parts as required in the following. Clearly

$$\int_{\mathfrak{M}} u^* L_\lambda u = 0,$$

and, using the divergence theorem, this becomes

$$\int_{\tilde{\mathfrak{M}}} (\tilde{g}^{\alpha\beta} u^*_{;\alpha} u_{;\beta} + m^2 u^* u) = 0 . \tag{12}$$

If $\lambda - 1$ is real and positive then $\tilde{g}$ is positive-definite and trivially $u = 0$. If $\lambda \neq \lambda^*$ then subtracting (12) from its complex conjugate yields

$$[(\lambda - 1)^{-1} - (\lambda^* - 1)^{-1}] \int_{\tilde{\mathfrak{M}}} |V^\alpha u_{;\alpha}|^2 = 0 .$$

Thus u is a constant along the integral curves of V and since $u \to 0$ as $x \to \infty$, u must vanish everywhere. This establishes the uniqueness of $G_\lambda[V]$ on the cut λ plane since any two solutions of (7) differ by a solution of (11).

From this result we can deduce also the analyticity of G_λ, for, differentiating (7) with respect to λ^*, we obtain

$$L_\lambda \frac{\partial G_\lambda}{\partial \lambda^*} = 0 ,$$

which has the unique solution $[\,|\arg(\lambda - 1)| < \pi]$

$$\frac{\partial G_\lambda}{\partial \lambda^*} = 0 ,$$

and these are the Cauchy-Riemann relations.

The uniqueness of the amplitude f_λ (again for fixed V) may be developed in close analogy. Suppose that $v(x, x'; s)$ satisfies

$$\tilde{\Box} v = (\lambda - 1)^{-1/2} \frac{\partial v}{\partial s}$$

subject to $v(x, x'; s) \to 0$ as $x \to \infty$ sufficiently rapidly to allow surface terms to be discarded, and that $v(x, x'; 0) = 0$. Clearly

$$\int_0^s ds' \int_{\tilde{\mathfrak{M}}} v^* \left[(\lambda - 1)^{1/2} \tilde{\Box} - \frac{\partial}{\partial s'}\right] v = 0 .$$

The divergence theorem yields

$$\int_0^s ds' \int_{\tilde{\mathfrak{M}}} \left[(\lambda - 1)^{1/2} \tilde{g}^{\alpha\beta} v^*_{;\alpha} v_{;\beta} + v^* \frac{\partial v}{\partial s}\right] = 0 .$$

Adding this to its complex conjugate and writing $\lambda = 1 + R^2 e^{2i\theta}$ with $|\theta| < \pi/2$, we obtain

$$\int_0^s ds' \int_{\tilde{\mathfrak{M}}} \left[2R\cos\theta (g^{\alpha\beta} + V^\alpha V^\beta) v^*_{;\alpha} v_{;\beta} + 2R^{-1} \cos\theta \, |V^\alpha v_{;\alpha}|^2 + \frac{\partial}{\partial s'} |v|^2\right] = 0 .$$

Assuming that we can interchange the order of integration, we find that

$$\int_{\tilde{\mathfrak{M}}} \left\{ \int_0^s ds' [2R\cos\theta (g^{\alpha\beta} + V^\alpha V^\beta) v^*_{;\alpha} v_{;\beta} + 2R^{-1}\cos\theta \, |V^\alpha v_{;\alpha}|^2] + |v(s)|^2 \right\} = 0 .$$

Each term is non-negative, hence $v(x, x'; s) = 0$ as before. The uniqueness and analyticity of f_λ follow from this result in precise analogy with the previous case.

IV. EXISTENCE OF $G[V]$

There are two reasons why the limit $\lambda \to 0^+$ may give trouble. It could happen that the regular part of $G_\lambda[V]$ may not have a limit. For example, it could be of the form $\lambda^{-1} \times$ (C^∞ function regular at $\lambda = 0$). Such a situation is conceivable since the method need not yield a Green's function for the hyperbolic equation at all. The alternative possibility is that the singular part of $G_\lambda[V]$ does not have a limit (as a distribution). In certain cases we can show that neither problem arises by means of an explicit construction of the Green's function, as the sum of an infinite series, which we carry out in this section (subject to questions of convergence which we do not discuss). In this way we see that there is definitely a (nonempty) class of manifolds for which the method is valid. In other cases we shall see that the second possibility does not occur and we shall simply disregard the first possibility; this is equivalent to our assumption that the procedure yields *a* Feynman propagator and that the problem is one of uniqueness only.

We have previously noted the interpretation of (6) as a Schrödinger equation. We assume, in the first instance, that the corresponding dynamical system does not contain caustics for initial data representing a point source at $x = x'$. Since the classical action for this dynamical system is the geodetic interval σ, the required condition is that $(\mathfrak{M}, g)$ can be covered by a normal coordinate system, the regularity of normal coordinates and of the second variation of the action being here one and the same. Thus $(\mathfrak{M}, g)$ is a geodesically convex manifold. Note that we require this condition even though we work with $\tilde{\sigma}$, for complex λ, not σ itself.

The method we use is an adaption of the Riesz method of fractional potentials,[5] the modification consisting in the choice of certain Bessel functions as kernels rather than powers of the geodesic distance. This allows us to control the behavior at infinity, and so to construct a Green's function rather than a parametrix.[5] It is closely related to DeWitt's method.

For reasons which will become apparent later we shall work with n-dimensional manifolds. Let

$$G_{\alpha,2k}(x,x')=g(\alpha,2k)K_{(n-\alpha-2k)/2}(2^{1/2}m\tilde{\sigma}^{1/2})\times\left(\frac{2\tilde{\sigma}}{m^2}\right)^{(\alpha-n-2k)/4},$$

where

$$g(\alpha,2k)=2^{(2-\alpha-n-2k)/2}e^{i\pi(n-\alpha-2k)/2}\pi^{1-n/2}\frac{1}{\Gamma(\alpha/2)},$$

and assume that we can write the fractional potential $\Omega^{(\alpha)}(x,x')$ as a series in $G_{\alpha,2k}(x,x')$:

$$\Omega^{(\alpha)}(x,x')=\sum_{k=0}^{\infty}G_{\alpha,2k}(x,x')\Omega_k(x,x')\left(\frac{-i}{\lambda-1}\right). \quad (13)$$

The coefficients Ω_k are to be chosen so that

$$L\Omega^{(\alpha+2)}(x,x')=\Omega^{(\alpha)}(x,x').$$

For $\alpha<0$, $\Omega^{(\alpha)}$ is a regular function at $\tilde{\sigma}=0$ [assuming the series (13) to converge] and an analytic function of α. We shall show that analytic continuation to $\alpha=2$ then defines the distribution $\Omega^{(2)}$ as a Green's "function" for all λ in the cut λ plane.

By direct calculation, using the recurrence relations for the Macdonald functions

$$K_{\nu-1}(z)-K_{\nu+1}(z)=-\frac{2\nu}{z}K_\nu(z),$$

$$K_{\nu-1}(z)+K_{\nu+1}(z)=-2K_\nu'(z)$$

and the defining equation for $\tilde{\sigma}$,

$$\tilde{g}^{\alpha\beta}\partial_\alpha\tilde{\sigma}\partial_\beta\tilde{\sigma}=2\tilde{\sigma},$$

we find that

$$L\Omega^{(\alpha+2)}(x,x')=\left(\frac{-i}{\lambda-1}\right)\sum_{k=0}^{\infty}\frac{1}{\alpha}G_{\alpha,2k}(x,x')\left[\alpha\Omega_k(x,x')+(2k-n+\tilde{\Box}\tilde{\sigma})\Omega_k(x,x')\,4\tilde{\sigma}\frac{d}{d\tilde{\sigma}}\Omega_k(x,x')+\tilde{\Box}\Omega_{k-1}(x,x')\right].$$

Thus, setting

$$4\tilde{\sigma}\frac{d\Omega_k}{d\tilde{\sigma}}+\Omega_k(2k-n+\tilde{\Box}\tilde{\sigma})+\tilde{\Box}\Omega_{k-1}=0, \quad (14)$$

we have

$$L\Omega^{(\alpha+2)}(x,x')=\Omega^{(\alpha)}(x,x')$$

as required.

The coefficients have been chosen so that as $\tilde{\sigma}\to0$ we regain the usual series in powers of $\tilde{\sigma}$ by taking the asymptotic form of the Macdonald functions

$$K_\nu(z)\sim\tfrac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\quad(\mathrm{Re}\,\nu>0).$$

Hence the analytic continuation of $\Omega^{(\alpha)}$ to $\alpha=0$ gives $-i(\lambda-1)^{-1}\delta(x,x')$ [the factor of $\Gamma(\alpha/2)$ removes the apparently regular terms], and it follows that the analytic continuation of $\Omega^{(\alpha+2)}$ is a Green's function $G_\lambda[V]$ which vanishes at infinity.

The recurrence relations (14) for Ω_k are the usual ones; we have[1,5]

$$\Omega_0(x,x')=\tilde{\Delta}^{1/2}(x,x')\equiv[\tilde{g}^{-1/2}(x)\det(\tilde{\sigma}_{;\alpha\beta'})\tilde{g}^{-1/2}(x')]^{1/2},$$

and provided $\tilde{\Delta}\neq0$, i.e., provided there are no caustics, we find that

$$\Omega_k(x,x')=-\tfrac{1}{2}\Omega_0\left(\frac{\tilde{\sigma}}{2}\right)^{-k/2}\int_0^{\tilde{\sigma}}\left(\frac{\zeta}{2}\right)^{k/2-1/2}[\Omega_0(\zeta)]^{-1}\times\tilde{\Box}\,\Omega_{k-1}(\zeta)d\zeta. \quad (15)$$

Note that with the existence of caustics, the integral in (15) is not independent of path (in the complex $\tilde{\sigma}$ plane); hence the Ω_k's are not uniquely determined. Since (13) is always locally a C^∞ parametrix, this must correspond to the possible addition of a C^∞ function. We shall return to this point.

If the series converges uniformly with respect to λ, then continuing to $\lambda=0$ we see that the resulting Green's function is independent of V. Thus we have constructed the Feynman propagator G.

If the series does not converge (which is the general case if the manifold is not analytic) then this construction provides a C^∞ parametrix, and we can write $G_\lambda[V]$ in the form

$$G_\lambda^{(\alpha)}[V]=\sum_{k=0}^{N}C_{\alpha,2k}(x,x')\Omega_k(x,x')\left(\frac{-i}{\lambda-1}\right)+R_N^{(\alpha)}(x,x';\lambda).$$

In order to continue to $\lambda=0$ we have to assume that the remainder $R_N^{(\alpha)}$ does not have a singularity at $\lambda=0$. Furthermore, we do not at this stage obtain a uniqueness theorem for G. Note that we can continue to $\lambda=0$ before or after taking $\alpha\to2$ since analytic continuation in α *defines* the distributions $K_n((\zeta+i\epsilon)^{1/2})(\zeta+i\epsilon)^{-n}$.[6]

It is of interest to note that the construction yields a natural method of regularization: We choose $\mathrm{Re}\,\alpha$ sufficiently large that the coincident limit $x\to x'$ exists, and then analytically continue quantities such as $G^{(\alpha)}(x,x)$ to $\alpha=2$. The infinities of the theory appear as poles in the complex α plane. This is essentially the method of Salam and Strathdee.[7] To make the connection, observe that

$$G^{(\nu)}(x,x')=\frac{e^{\nu\pi i}}{\Gamma(\nu)}\int_0^\infty ds\, s^{\nu-1}e^{-im^2 s}f(x,x';s),$$

where $\nu=\alpha/2$. Since α and n occur explicitly in the combination $\alpha-n$ one might attempt to make the analytic continuation in n, rather than in α, as in the method of dimensional regularization. However, the dimension of the space occurs also in $\tilde{\sigma}$, and it is not clear whether the extension of $\tilde{\sigma}$ to n dimensions can be made arbitrarily. Thus dimensional regularization is not a natural procedure in curved space, although in situations of obvious symmetry one might expect it to yield the same result as the method outlined above.

V. INVARIANCE UNDER CHANGE OF V

In the general case there will be caustics and we cannot construct a Green's function (in any region of the complex λ plane) by the above method. However, locally, for x, x' sufficiently close, the above construction yields a C^∞ parametrix. If we now assume the existence of a Green's function $G_\lambda[V]$, this will differ from $\Omega^{(2)}$ locally by a C^∞ function. This C^∞ function may contribute to $\operatorname{Re}G(x,x)$ and hence give rise to pair production.[8]

We have already shown in Sec. II that the uniqueness problem for G now amounts to an examination of the convolution of distributions

$$\frac{\lambda}{\lambda-1}G_\lambda[W]*XG_\lambda[V]$$

as $\lambda\to 0^+$. These distributions are defined by

$$(\phi(x),G_\lambda(x,y))$$
$$=\underset{\alpha\to 2}{\text{analytic continuation}}\int\phi(x)G_\lambda^{(\alpha)}(x,y)dx$$

for $\phi(x)$, a C^∞ test function with compact support. $G_\lambda^{(\alpha)}$ differs from $\Omega^{(\alpha)}$ of Sec. IV by a C^∞ function which we assume to be well behaved in the limit $\lambda\to 0^+$. Thus, writing $G_\lambda(\phi,y)$ for $(\phi(x),G_\lambda(x,y))$ and $G_\lambda(\phi,\psi)$ for

$$(\phi(y),(\psi(x),G_\lambda(y,x)))$$

we have

$$G_\lambda(\phi,\psi\,|\,W)-G_\lambda(\phi,\psi\,|\,V)$$
$$=\frac{\lambda}{\lambda-1}G_\lambda(\phi,y\,|\,W)*X_yG_\lambda(y,\psi\,|\,V).$$

(i) If V and W differ only on a compact set, then the right-hand side of this expression is an integral of continuous functions over a compact set, and hence exists as λ approaches the real axis for all $\lambda\neq 1$. In the limit $\lambda\to 0^+$ we find that $G(\phi,\psi\,|\,W)=G(\phi,\psi\,|\,V)$ for arbitrary ϕ, ψ, and hence $G(x,x'\,|\,W)=G(x,x'\,|\,V)$.

(ii) If the manifold is analytic, and if x' is such that there is a tube of rays through x' which can be continued to infinite length without passing through a caustic, then in this domain, the explicit expression for G_λ in Sec. IV is valid. Thus, by inspection, $G_\lambda[V]\to G_\lambda[W]$ as $\lambda\to 0^+$ in this region, and hence, by analytic continuation, throughout $\mathfrak{M}$. The restriction to analytic manifolds clearly cannot be removed from the proof. Even in the analytic case this does not in fact complete the theory, since the two-dimensional Einstein universe provides an example in which for all x' there are caustics in all (timelike) directions.

We have not been able to specify sufficient conditions which are less restrictive. The problem is not the singular part of the Green's function (we have shown that the parametrix is independent of V) but the fact that we do not know enough about the regular part to bound integrals over noncompact sets in the way that would be possible if the metric were positive-definite. It is clear that some restriction of V and W is necessary to ensure convergence of the convolution integral, and it is possible that in certain cases V and W may differ on a noncompact set and yet the integral can still exist.

VI. DISCUSSION

We have shown that the methods for the construction of a Feynman propagator in the literature agree where they are applicable, and we have indicated the extent to which the construction can be extended. It is physically desirable that the conditions on $\mathfrak{M}$ be relaxed, in particular to include manifolds with boundary. To do this we obviously need to add further boundary conditions for G_λ. In the theory of the Wiener process,[9] $G=0$ or $\partial G/\partial n=0$ have natural interpretations as absorption and reflection boundary conditions, respectively. Insofar as solutions of the Schrödinger equation (6) can be constructed from path integrals[10] (which appear to be given meaning through analytic continuation), we might feel that these boundary conditions can be carried over in the same spirit as absorption and reflection conditions. Furthermore, if a singularity of $\mathfrak{M}$ corresponds to a boundary, in the manner of Schmidt's construction for example,[11] one might anticipate the same boundary conditions to be natural and possible on a singular manifold.

It is interesting that in their treatment of black-hole radiance, Hartle and Hawking[12] present a derivation that depends essentially only on the analytic properties of the Kruskal manifold. It seems that almost any Green's function obtained by analytic continuation of an elliptic equation would manifest the same analyticity properties and hence give the same answer. This follows because any Green's function analytic in Kruskal time is, by the nature of the coordinate transformation, periodic in Schwarzschild time, and hence a finite-temperature Green's function in Schwarzschild coordinates. In this case, Hartle and Hawking fix the indetermination in G by requiring it to correspond to a Fock space associated with excitations of the field on the horizon; this is the same condition as requiring the propagator to vanish as Kruskal time $\to i\infty$.

In the Robertson-Walker solutions it is possible to construct a unique Green's function with the property $G_\lambda(x,x')\to 0$ (or $\partial G/\partial n\to 0$) as $t\to 0$ and as $t\to\infty$, such that $G(x,x')$ tends to the usual Feynman function when both t and t' are large. An interesting property of this Green's function is that it does not give rise to, and cannot be constructed from, a Fock space at $t=0$.

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