Vacuum stress tensor in an Einstein universe: Finite-temperature effects

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The vacuum average of the stress-energy tensor for a massive scalar field in an Einstein universe is calculated. The renormalization adopted depends on the special circumstance that the WKB approximation is exact on the three-dimensional sphere. The finite results are purely nonperturbative and a renormalization of only the cosmological constant is called for. The finite-temperature corrections have also been considered with the aid of a theory of relativistic time-temperature Green's functions in static space-times. It is shown explicitly that the massless scalar gas is an ideal one satisfying $PV = E/3$.

I. INTRODUCTION

In this paper an exact calculation is presented of the vacuum stress tensor and effective Lagrangian of a massive quantized scalar field in a background Einstein-universe geometry. The same situation has been discussed by Streeruwitz,[1] but our results differ from his. The reason is probably the different regularization methods used. The Streeruwitz method is based on the Euler-Maclaurin formula applied to an eigenvalue summation together with a continuous Pauli-Villars regularization. Our method is somewhat simpler and we would like to argue in its favor at this point.

The Einstein universe is just about the simplest way of altering the metric properties of the world from the Minkowski form. It corresponds to replacing Euclidean 3-space by a 3-sphere, S^3, to give the space-time $R\times S^3$. Now the essential point is that not only is the harmonic analysis on spheres as well known as that on flat space, but also for the special case of S^3 the quantum-mechanical WKB approximation is exact, for free particles at least, just as for E^3. This immediately suggests that scalar quantum field theory on $R\times S^3$ should be very similar to that on Minkowski space-time since the quantum field theory Green's functions, etc., can be expressed as proper-time integrals over quantum-mechanical propagators on $R\times S^3$. We would expect the Green's functions to be, more or less, just the standard Minkowski ones; any differences are due solely to the different topology. (The Einstein universe is closed and compact.) This being so, the renormalization procedure will be essentially the same as in flat space. We thus anticipate that a renormalization of only the cosmological constant will be necessary and this is borne out by the detailed calculations. The Streeruwitz method leads in addition to a renormalization of the gravitational constant and of the coefficients of the quadratic parts of the action, which seems to be necessary for a *general* space-time.

In the next section we give the details of the calculation. In Sec. IV we extend the method to field theory at a finite temperature and discuss the thermodynamics of the system. The averages are evaluated using finite-temperature Green's functions, a subject of topical interest.

II. STRESS-ENERGY TENSOR AND GREEN'S FUNCTION

The quantity we wish to calculate is the average of the improved stress-energy tensor of the neutral scalar field. This is given by the averaged coincidence limit,

$$\langle \hat{T}_{\mu\nu}(x)\rangle = -\tfrac{1}{6}i \lim_{x'\to x} [4\nabla_\mu\nabla_{\nu'} - g_{\mu\nu'}(g^{\lambda\sigma'}\nabla_\lambda\nabla_{\sigma'} - \nabla_\rho\nabla^\rho - \nabla_{\rho'}\nabla^{\rho'} + \tfrac{1}{2}R) - g_{\mu\rho'}\nabla^{\rho'}\nabla_{\nu'} - g_{\nu'\sigma}\nabla^\sigma\nabla_\mu - \tfrac{1}{2}(R^\sigma_\mu g_{\sigma\nu'} + g_{\mu\rho'}R^{\mu'}_{\nu'})]G(x,x')\,, \quad (1)$$

where $G(x,x')$ is the Feynman Green's function on $R\times S^3$ satisfying the covariant equation

$$(\Box^2 + \kappa^2 + \tfrac{1}{6}R)G(x,x') = \delta(x,x'), \quad \Box^2 \equiv \nabla_\mu\nabla^\mu\,.$$

If taken directly the coincidence limit diverges, and so a systematic method of investigating this divergence is required. For example, we could see how the limit depends on the difference between x and x' as this becomes small This is the point-separation technique. Alternatively, we could use dimensional regularization perhaps turning S^3 into $S^{2\omega}$ or maybe R into R^d (or possibly both at once). However, we do not need to be so sophisticated. In order to see what to do we require the form of the Green's function, G, and so we make a slight detour to arrive at this quantity. There are in fact quite a few ways of deriving G. We take the following route in order to introduce some useful facts.

In the Schwinger-DeWitt approach, G is written as a proper-time integral,

$$G(x,x') = i\int_0^\infty d\tau e^{-i\kappa^2\tau}\langle x,\tau|x',0\rangle\,, \qquad (2)$$

where the "quantum-mechanical" propagator $\langle x,\tau|x',0\rangle$ satisfies a "Schrödinger equation"

$$\left(i\frac{\partial}{\partial\tau} - \Box^2 - \tfrac{1}{6}R\right)\langle x,\tau|x',0\rangle = i\delta(\tau)\delta(x,x')\,.$$

For the Einstein universe, a static space-time, we have the factored form

$$\langle x,\tau|x',0\rangle = (-4\pi i\tau)^{-1/2}\exp[-\tfrac{1}{4}i\tau^{-1}(t-t')^2] \times K(q,q',\tau)\,, \qquad (3)$$

where K is the propagator on S^3 satisfying

$$\left(i\frac{\partial}{\partial\tau} + \Delta_2 - \tfrac{1}{6}R\right)K(q,q',\tau) = i\delta(\tau)\delta(q,q')\,, \quad q,q'\in S^3\,.$$

Δ_2 is the Laplace-Beltrami operator on S^3 and K is a function of q and q' through the geodesic distance between the points q and q', $s(q,q')$. Then Δ_2 can be replaced by its radial part Δ_2^r,

$$\Delta_2^r = [\sin(s/a)]^{-1}\frac{d^2}{ds^2}\sin(s/a) + \tfrac{1}{6}R \quad (a = \text{radius of } S^3)\,.$$

With this, it can easily be shown[2] that K is given by

$$K(q,q',\tau) = (4\pi i\tau)^{3/2}[a\sin(s/a)]^{-1} \times \sum_{n=-\infty}^{\infty}(s+2\pi na)\exp\left[\frac{i}{4\tau}(s+2\pi na)^2\right],$$

which exhibits the basic structure of a WKB approximation, as mentioned in Sec. I. The factor $s/[a\sin(s/a)]$ is just the square root of the Van Vleck determinant and $s^2/4\tau$ is the classical action The sum over n produces the required periodicity on the sphere S^3, since it is a sum over the geodesics connecting q and q'.

To separate the direct geodesic contribution ($n=0$) rewrite K,

$$K(q,q',\tau) = (4\pi i\tau)^{-3/2}(s/a)[\sin(s/a)]^{-1} \times\left[1 + 4\pi i a\sum_1^\infty ns^{-1}\sin\left(\frac{n\pi as}{\tau}\right)\exp\left(\frac{i}{\tau}\pi^2n^2a^2\right)\right] \times\exp\left(\frac{i}{4\tau}s^2\right). \qquad (4)$$

Compare this with the standard expansion in powers of τ,

$$K(q,q',\tau) = (4\pi i\tau)^{-3/2}(s/a)[\sin(s/a)]^{-1}\exp\left(\frac{i}{4\tau}s^2\right) \times\sum_{m=0}^\infty a_m(i\tau)^m + (\text{terms exponentially small})\,, \qquad (5)$$

and see that all the a_m ($m>0$) are zero. The second term in (4) is due to the indirect paths from q to q' (i.e., those *geodesics* that encircle S^3 at least once) and appears in the expansion (5) as the additional "terms exponentially small."

The Green's function G can now be constructed from (2), (3), and (4). We find a sum of standard Feynman Green's functions,

$$G(x,x',\kappa^2) = -\frac{\kappa^2}{8\pi a\sin(s/a)}\sum_{n=-\infty}^{\infty}(s+2\pi na)\frac{H_1^{(2)}(\kappa\sigma_n)}{\kappa\sigma_n}\,, \qquad (6)$$

with $\sigma_n^2 = (t-t')^2 - (s+2\pi na)^2 - i\epsilon$ and the usual choice of square root branches for σ_n.

We now return to the vacuum stress-energy density, (1), and examine the nature of the coincidence limit, $t\to t'$ and $s(q,q')\to 0$, bearing in mind the structure of G, Eq. (6).

It is easy to see that only the $n=0$ term in (6) will lead to divergences in the evaluation of (1). Furthermore, it is the only term that remains in the flat-space limit, $a\to\infty$, to give the ordinary Minkowski Green's function, and in the coincidence limit it diverges with the same singularity as in flat space. Thus a natural procedure to remove the infinities in $\langle\hat{T}_{\mu\nu}\rangle$, and other expressions involving G, would be simply to drop the $n=0$ term in G. This is the prescription adopted in the present work. We define a "renormalized" Green's function, G_{ren}, as the series (6) omitting the $n=0$ term. Then the renormalized $\langle\hat{T}_{\mu\nu}\rangle$ is given by (1) with G replaced by G_{ren}. Before actually calculating $\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}}$ we wish to make a few remarks on this subtraction method.

Firstly we should say that there is no reason to endow the renormalized Green's function with any propagating properties. In fact, the subtraction of the $n=0$ term violates the periodicity condition. This is not serious. A similar situation occurs in the theory of the Casimir effect.[3,4]

It is also necessary to remark that each term of the series (6) satisfies the homogeneous wave equation (if $t\neq t'$). Thus so does G_{ren} as well as G. This is important because we want the subtraction procedure to preserve those properties of $\langle\hat{T}_{\mu\nu}\rangle$ that depend on G satisfying the wave equation. For example, in the massless case[3] $\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}}$ should be traceless. If a term were subtracted that did not satisfy the wave equation this would not be true.

Of course this condition alone is not sufficient to say exactly how the subtraction of terms from G should be done. The $n=0$ term has to go, but apparently we could also remove any amount of the other terms. Such a possibility can be eliminated by the requirement that the subtraction be equivalent to a conventional renormalization of Einstein's equations. In a general space-time this latter is usually discussed in terms of the expansion (5),[5,6] and it is shown, one way or another, that only the a_0, a_1, and a_2 terms are involved. In our case a_1 and a_2 are zero and therefore a renormalization of only the cosmological constant is called for, from the a_0 term, i.e., from just the $n=0$ term in (6). A similar conclusion also follows from the dimensional regularization methods where the dimension-4 pole corresponds exactly to the $n=0$ term.

We now proceed to evaluate the renormalized $\langle\hat{T}_{\mu\nu}\rangle$. From (1) we have

$$\langle\hat{T}_{00}\rangle_{\rm ren}=i\nabla_0\nabla_0 G_{\rm ren}(x,x')\,\big|_{x'=x} \tag{7}$$

and

$$\langle\hat{T}_{ij}\rangle_{\rm ren}=-\tfrac{1}{3}g_{ij}[\langle\hat{T}_{00}\rangle_{\rm ren}+i\kappa^2 G_{\rm ren}(x,x)]\,. \tag{8}$$

However, instead of using (7) directly to determine $\langle\hat{T}_{00}\rangle_{\rm ren}$ we introduce at this point the expression for the renormalized effective Lagrangian $\mathcal{L}^{(1)}_{\rm ren}(x)$,

$$\mathcal{L}^{(1)}_{\rm ren}(x)=-\tfrac{1}{2}i\int_{\kappa^2}^{\infty}d\mu^2 G_{\rm ren}(x,x,\mu^2)\,. \tag{9}$$

It is easy to relate $\mathcal{L}^{(1)}_{\rm ren}$ and $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ by varying $\mathcal{L}^{(1)}_{\rm ren}$ with respect to a. This produces (see below) the trace of the spatial part of $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ and gives Eq. (8) with $\langle\hat{T}_{00}\rangle_{\rm ren}=-\mathcal{L}^{(1)}_{\rm ren}$, a result which can be checked by direct evaluation of (7) and (9).

The reason for using (9) to find $\langle\hat{T}_{00}\rangle_{\rm ren}$ is purely one of convenience in that all we then need for $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ is the coincidence limit of $G_{\rm ren}(x,x')$ and not of any of its derivatives.

For the limit in question we have

$$G_{\rm ren}(x,x,\mu^2)=-i\frac{\mu^2}{8\pi^2 a}\sum_{n=1}^{\infty}n^{-1}\frac{\partial}{\partial\mu}H_1^{(2)}(-i2\pi na\mu)\,.$$

In order to obtain the $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ in terms of the $\mathcal{L}^{(1)}_{\rm ren}$ we firstly note that because of the symmetries of the space, and of the dynamics, $\langle\hat{T}_{i0}\rangle_{\rm ren}$ will be zero and $\langle\hat{T}_{ij}\rangle_{\rm ren}$ will be proportional to g_{ij}. Equations (7) and (8) say this explicitly and so all we need to find $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ is the trace of the spatial part, $T=g^{ij}\langle\hat{T}_{ij}\rangle_{\rm ren}$. In general[5] $\langle\hat{T}_{\mu\nu}\rangle$ is given by varying the effective action with respect to the metric $g^{\mu\nu}$. In our case this variation reduces to one of only the radius a. Thus we have

$$T=2K^{5/2}\frac{d}{dK}(K^{-3/2}\mathcal{L}^{(1)}_{\rm ren})\,,$$

where $K\equiv a^{-2}$. For $\mathcal{L}^{(1)}_{\rm ren}$ we use expression (9) and we further note that because $G_{\rm ren}(x,x,\mu^2)$ is μ^2 times a function of $\mu K^{-1/2}$ the derivative with respect to K can be turned into one with respect to μ. This allows a partial integration and a simple calculation produces Eq. (8) with $\langle\hat{T}_{00}\rangle_{\rm ren}$ equal to $-\mathcal{L}^{(1)}_{\rm ren}$, as promised. It is then straightforward to derive the expressions

$$\langle\hat{T}_{00}\rangle_{\rm ren}=\frac{\kappa^2}{4\pi^3a^2}\sum_1^{\infty}n^{-1}K_1(2\pi na\kappa)+\frac{3\kappa^2}{8\pi^4a^2}\sum_1^{\infty}n^{-2}K_2(2\pi na\kappa) \tag{10}$$

and

$$\langle\hat{T}^{\mu}_{\mu}\rangle_{\rm ren}=-\frac{\kappa^4}{4\pi^2}\sum_1^{\infty}[K_0(2\pi na\kappa)+K_2(2\pi na\kappa)]\equiv T\,. \tag{11}$$

The $K(z)$ are modified Bessel functions.

III. SELF-CONSISTENT EINSTEIN EQUATIONS

Equations (10) and (11) constitute one of the basic results of the present paper, but if the situation is taken seriously one should consider the effective Einstein equations,

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\lambda g_{\mu\nu}=-8\pi G\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}\,, \tag{12}$$

to see if they are self-consistent with the choice of an Einstein-universe geometry.

When the quantities for an Einstein universe are substituted into the left-hand side of (12) we find the single self-consistency condition,

$$\frac{6}{a^2}=R=8\pi G(4\langle\hat{T}^0_0\rangle_{\rm ren}-T)\,.$$

Substitution of the results (10) and (11) yields an equation the solutions of which, in terms of a and κ, do not seem particularly relevant physically. However, self-consistent solutions are *possible* and this is, perhaps, the only useful conclusion we can draw. If κ is zero the self-consistent solution is $a\sim10^{-34}$ cm.

IV. FINITE-TEMPERATURE CORRECTIONS

Instead of taking the average over the vacuum it is of interest to use a state describing the thermal equilibrium of the φ field at temperature T. The vacuum case is regained at $T=0$.

Since there is no self-interaction of the φ quanta in the system as we have described it, we have to assume that thermal equilibrium has been achieved by some unspecified process. The case which interests us ultimately in this section is that which has the closest analogy with ordinary blackbody radiation. Then we allow an arbitrary number of φ quanta and the chemical potential vanishes.

For simplicity we are also going to take the massless limit, $\kappa = 0$. Thus we shall be evaluating the temperature correction to the vacuum calculation of our earlier work.[3] However, for a little while we wish to consider the more general case.

The relevant averages are obtained by replacing the Feynman Green's function G in the preceding sections by the finite-temperature Green's function, $G_{(\beta,\mu)}$, defined by the statistical average

$$G_{(\beta,\mu)}(x,x') = i\,\mathrm{tr}(\hat{\rho}(\beta_0,\mu^0)T\{\hat{\varphi}(x)\hat{\varphi}^\dagger(x')\}) \equiv i\langle T\{\hat{\varphi}(x)\hat{\varphi}^\dagger(x')\}\rangle_T . \qquad (13)$$

The symbol $T\{\ \}$ indicates the usual time-ordered product.

The quantum density operator $\hat{\rho}$ is that for a grand canonical ensemble. Generally, and this is our definition of equilibrium,

$$\hat{\rho}(\beta_0,\mu^0) = \exp\left[\beta_0\left(\Omega_0 + \sum_i \mu_i^0 \hat{N}_i - \hat{H}\right)\right], \qquad (14)$$

where $\hat{H}$ is the second-quantized Hamiltonian and the $\hat{N}_i$ are conserved operators. The μ_i^0 are the corresponding chemical potentials and Ω_0 is the thermodynamic potential. The zero subscript or superscript indicates that the relevant quantity is a constant and distinguishes it from the corresponding "local" quantity.

What the $\hat{N}_i$ are depends on the system under investigation.[7] If there are no conserved objects, apart from mechanical ones, then effectively all the μ_i^0 are zero and we have, for example, a blackbody radiation situation. This would be the case if φ were a real field. For a complex field an $\hat{N}_i$ would be the "charge," or the number of particles minus the number of antiparticles, $\hat{N}_+ - \hat{N}_-$.

Let us now specifically work with a static space-time. Without loss of generality we write its metric in the form

$$ds^2 = g_{00}dt^2 - g_{ij}dx^i dx^j ,$$

where the $g_{\mu\nu}$ are independent of the time $t, = x^0$.

The total entropy S is defined by $S = -k\langle \ln\hat{\rho}\rangle_T$ and (14) produces the standard thermostatic identity

$$E_0 = T_0 S + \Omega_0 + \sum \mu_i^0 N_i$$

or (15)

$$F = \Omega_0 + \sum \mu_i^0 N_i ,$$

with $E_0 = \langle\hat{H}\rangle_T$ and $N_i = \langle\hat{N}_i\rangle_T$. The following analysis is more or less standard.[8,9]

The normalization condition $\langle\hat{1}\rangle_T = 1$ is differentiated with respect to the temperature, chemical potentials, and the metric "parameters" $g_{\mu\nu}(x)$. After (15) has been used, slight manipulation yields the thermodynamic identity for the grand canonical potential,

$$\delta\Omega_0 = -S\delta T_0 - \sum N_i \delta\mu_i^0 + \int \left\langle \frac{\delta\hat{H}}{\delta g^{\mu\nu}(x)}\right\rangle_T \delta g^{\mu\nu}(x) d^3x . \qquad (16)$$

Here d^3x is $dx^1dx^2dx^3$. The invariant volume element on the hypersurface, $t = \text{constant}$, is $d\Sigma = (-g)^{1/2}(g_{00})^{-1/2}d^3x$.

We now rewrite (16) as

$$\delta\Omega_0 = -S\delta T_0 - \sum N_i\delta\mu_i^0 - \tfrac{1}{2}\int A_{ij}\delta g^{ij} g_{00}{}^{1/2} d\Sigma , \qquad (17)$$

where

$$A_{ij} \equiv -2(-g)^{-1/2}\left\langle \frac{\delta\hat{H}}{\delta g^{ij}}\right\rangle_T ,$$

and we have restricted the variation to that of the g^{ij} only. For the special variation

$$\delta g^{ij}(x) = -\tfrac{2}{3}(-g)^{-1/2} g^{ij}\delta f(x)$$

(a conformal scaling), we find that

$$\left[\frac{\delta\Omega_0}{\delta f(x)}\right]_{T_0,\mu^0} = -P(x) ,$$

where the "pressure" P is defined by

$$P \equiv -\tfrac{1}{3} g^{ij} A_{ij} .$$

This equation is the analog of the conventional thermodynamic relation

$$\left(\frac{d\Omega}{dV}\right)_{T,\mu} = -P ,$$

which is appreciated if we note that

$$\delta V \sim \delta \int (-g)^{1/2} d^3x = \int \delta f\, d^3x .$$

Instead of the total quantities E_0, N_i, S, etc., it is useful in the inhomogeneous situation to use specific quantities, or densities. Thus we write

$$E_0 = \int \langle \hat{T}_0^0\rangle_T (g_{00})^{1/2} d\Sigma \equiv \int \eta g_{00}{}^{1/2} d\Sigma ,$$

$$N_i = \int \langle \hat{j}_i^0\rangle_T d\Sigma \equiv \int n_i d\Sigma$$

(where j_i^0 is the density of the ith conserved quantity). In equilibrium

$$S = \int s\, d\Sigma$$

[cf. Ref. 10, Eqs. (157) and (176)] and

$$\Omega_0 \equiv -\int P' g_{00}{}^{1/2} d\Sigma \,. \tag{18}$$

The thermodynamic identity (15) then reads

$$\eta = Ts - P' + \sum \mu_i n_i \,, \tag{19}$$

where $T = T_0(g_{00})^{-1/2}$ and $\mu_i = \mu_i^0(g_{00})^{-1/2}$ are the local (invariant) temperature and chemical potentials, respectively.[9,11]

The identity (17) in terms of densities is

$$\delta P' = s\delta T + \sum n_i \delta\mu_i + \tfrac{1}{2}(A_{ij} + P' g_{ij})\delta g^{ij} \,, \tag{20}$$

and we can also find the local relation

$$\delta\eta = T\delta s + \sum \mu_i \delta n_i - \tfrac{1}{2}(A_{ij} + P' g_{ij})\delta g^{ij} \tag{21}$$

corresponding to the total expression

$$\delta E_0 = T_0 \delta S + \sum \mu_i^0 \delta N_i - \tfrac{1}{2}\int A_{ij}\delta g^{ij} g_{00}{}^{1/2} d\Sigma \,. \tag{22}$$

Perhaps we should now interject some remarks on the nature of the thermodynamical system we are considering.

Classical thermodynamics mainly confines itself to homogeneous systems (within one phase), and it is then usual to choose the volume of the system as one of the state variables (or "deformation coordinates" in Buchdahl's terminology[12]). This is not possible for inhomogeneous systems (e.g. Ref. 12, p. 146) and the choice of coordinates then depends strongly on the particular system.

The system can be defined by the quantum Hamiltonian (see the standard discussion in Ref. 9, p. 108, and Ref. 13, Sec. 2.6) which in turn, for us, depends on the metric, $g_{\mu\nu}$, and the region, $\mathfrak{M}$, of three-dimensional space occupied, by definition, by the system. (We do not delimit the range of t.)

For the sake of having something specific in mind it is helpful to consider the perfect-gas (i.e., free-field) case. Then $\hat{H}$ is given by[14]

$$\hat{H} = \tfrac{1}{2}\sum_k \omega_k(\{a_k^\dagger, a_k\} + \{b_k^\dagger, b_k\})$$

$$= \sum_k \omega_k[(\hat{N}_k^+ + \hat{N}_k^-) + \delta(k,k)] \,, \tag{23}$$

where the $\omega_k{}^2$ are the eigenvalues of the elliptic equation,

$$(-g)^{-1/2} g_{00}\partial_i[(-g)^{+1/2} g^{ij}\partial_j \varphi_k] + g_{00}(\kappa^2 + \tfrac{1}{6}R)\varphi_k = \omega_k{}^2 \varphi_k \,, \tag{24}$$

subject to the vanishing of φ, say, on $\partial\mathfrak{M}$, the boundary of $\mathfrak{M}$. If $\mathfrak{M}$ is closed, $\partial\mathfrak{M} = \emptyset$ and the boundary condition becomes a periodicity requirement.

Note that the ω_k incorporate the red-shift factor,[15] $(g_{00})^{1/2}$. All the mode energies ω_k are greater than some non-negative minimum, $\min(\omega_k)$.

In general the ω_k will depend on $g_{\mu\nu}$ and on $\partial\mathfrak{M}$ (we can consider $\mathfrak{M}$ to be known when g_{ij} and $\partial\mathfrak{M}$ are given), and the system can be varied by changing either g_{ij} or $\partial\mathfrak{M}$ or both. We have so far considered only the former variation since we are later interested in the Einstein universe, which has no spatial boundary, $\partial S^3 = \emptyset$. However, it is the variation of $\partial\mathfrak{M}$ that corresponds most closely to the classical situation where the extension of the system is varied by altering its boundaries, and one equates the work done to the sum of the products of the pressure at $\partial\mathfrak{M}$ with the small changes in volume over $\partial\mathfrak{M}$ (e.g. Ref. 13, p. 77). $\mathfrak{M}$ would be specified by saying that, for example, the coordinates in some particular system run over invariantly specified ranges. Thus one might think of an angular parameter going from 0 to $\pi/2$. Varying $\partial\mathfrak{M}$ corresponds to varying these ranges.

The invariant volume of $\mathfrak{M}$ is given (including the $g_{00}{}^{1/2}$ term) by

$$V \equiv |\mathfrak{M}| = \int_{\mathfrak{M}} (-g)^{1/2} d^3x \,,$$

and, in general, does not specify $\mathfrak{M}$ completely (as it effectively would in the classical homogeneous case). $|\mathfrak{M}|$ will change owing to changes in either $\partial\mathfrak{M}$ or $g_{\mu\nu}$. These will be independent changes, the former being given, roughly speaking, by changes in the integration limits and the latter by changes in $(-g)^{1/2}$. There will be concomitant concepts of pressure for these distinct variations, for homogeneous systems. In the inhomogeneous case it is not enough to consider changes in just $|\mathfrak{M}|$. A more detailed local description must be used. We will not enter into exactly what restrictions must be placed on $\mathfrak{M}$ and $g_{\mu\nu}$ so that the system is thermodynamically homogeneous. Certainly a sufficient condition is that $\mathfrak{M}$ should be a homogeneous space $(G/H)/\Gamma$, e.g., S^3/Γ. In fact, let us discuss the case $\mathfrak{M} = S^3$ a little more closely since it is for this we explicitly calculate later.

Since $\partial S^3 = \emptyset$ we cannot think of the change in internal energy as being equal to the work done by an external system at a varying boundary. Rather, the change must be considered as spread out over all of $\mathfrak{M}$ much as the change in surface energy of a soap bubble.

Because of the properties of S^3, A_{ij} must be proportional to g_{ij} and so the final term of (22) is (for simplicity we set g_{00} equal to unity)

$$\tfrac{1}{2}P\int_{S^3} g_{ij}\delta g^{ij}d\Sigma = -P\delta\int_{S^3} d\Sigma = -P\delta V,$$

where P is the constant pressure, $A_{ij}=-Pg_{ij}$. This shows that the change in total internal energy E_0 due to the deformation of S^3 depends on only the change in invariant volume and not on exactly how the deformation is made. In fact, we might take this property as the definition of a homogeneous system.

It is also worth pointing out that even for homogeneous sytems the final term in the local equation (21) does not appear to vanish. In ordinary thermodynamics the extensive nature of the total energy implies that the energy density is explicitly independent of the volume. This gives the Gibbs-Duhem relation and $\Omega=-PV$. As will turn out in the actual calculation E_0 is not extensive here, even for a homogeneous static situation. Thus $P'\neq P$.

Our discussion is incomplete until we have given the relation between the A_{ij} and the averages $\langle \hat{T}_{\mu\nu}\rangle_T$. In particular, is P equal to the field pressure $-\frac{1}{3}g^{ij}\langle \hat{T}_{ij}\rangle_T$? For the Einstein case it is equal and we shall not investigate this, admittedly vital, matter further for the general situation.

V. FINITE-TEMPERATURE GREEN'S FUNCTIONS IN STATIC GEOMETRIES

After this slight digression on thermodynamic matters we return to an analysis of the Green's function $G_{(\beta,\mu)}$, Eq. (15).

The theory of finite-temperature Green's functions is well developed for the nonrelativistic situation[16] and an important feature is the quasiperiodicity in imaginary time. If this property is to be carried over directly into the relativistic regime the conserved operators $\hat{N}_i$ must have certain commutation relations with the field operator $\hat{\varphi}$, because these relations play a vital role in the nonrelativistic proof of quasiperiodicity.

Charge would be one typical operator with the required properties

$$\exp[-\beta_0\mu^0(\hat{N}_+-\hat{N}_-)]\,\hat{\varphi}\,\exp[\beta_0\mu^0(\hat{N}_+-\hat{N}_-)] = \exp(\beta_0\mu^0)\hat{\varphi}\,.$$

The nonrelativistic proof of the quasiperiodicity of $G_{(\beta,\mu)}$ can now be paralleled *exactly*, leading firstly to the boundary condition

$$\langle\hat{\varphi}(t)\hat{\varphi}^\dagger(t')\rangle_T = \exp(-\beta_0\mu^0)\langle\hat{\varphi}^\dagger(t')\hat{\varphi}(t+i\beta_0)\rangle_T\,.$$

Thus, if we write the finite-temperature Wightman and Pauli-Jordan functions as Fourier integrals,

$$\begin{aligned}\langle\hat{\varphi}(t)\hat{\varphi}^\dagger(t')\rangle_T &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,g^{(+)}(\omega)e^{-i\omega(t-t')}\,,\\ \langle\hat{\varphi}^\dagger(t')\hat{\varphi}(t)\rangle_T &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,g^{(-)}(\omega)e^{-i\omega(t-t')}\,,\\ \langle[\hat{\varphi}(t),\hat{\varphi}^\dagger(t')]\rangle_T &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,c(\omega)e^{-i\omega(t-t')}\,,\end{aligned}\qquad(25)$$

we find the relation (cf. Ref. 16)

$$g^{(\pm)}(\omega) = \pm c(\omega)(1-e^{\mp\beta_0(\omega-\mu^0)})^{-1}\,.\qquad(26)$$

Thus the averaged time-ordered product is determined by the commutator average. This has the advantage that in the free-field case, since the commutator is a c number, its statistical average is the same as its vacuum expectation value. The following analysis is based on the work of Brown and Maclay in their elegant paper[4] on temperature corrections to the Casimir effect.

It is easy to show from (25) and (26) that, in general,

$$\langle T\{\hat{\varphi}(t)\hat{\varphi}^\dagger(t')\}\rangle_T = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{\theta(t-t')}{1-\exp[-\beta_0(\omega-\mu^0)]}-\frac{\theta(t'-t)}{1-\exp[\beta_0(\omega-\mu^0)]}\right\}c(\omega)e^{-i\omega(t-t')}\,.$$

If the vacuum average ($T_0=0$, $\mu^0=0$) is extracted we then find, for free fields,

$$\langle T\{\hat{\varphi}(t)\hat{\varphi}^\dagger(t')\}\rangle_T = \langle T\{\hat{\varphi}(t)\hat{\varphi}^\dagger(t')\}\rangle_0 + \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{\theta(\omega)\exp[-\beta_0(\omega-\mu^0)]}{1-\exp[-\beta_0(\omega-\mu^0)]}-\frac{\theta(-\omega)\exp[\beta_0(\omega-\mu^0)]}{1-\exp[\beta_0(\omega-\mu^0)]}\right\}c(\omega)e^{-i\omega(t-t')}\,,\qquad(27)$$

and the idea now is to expand the denominators in powers of $\exp[\mp\beta_0(\omega-\mu^0)]$. To do this we need some restriction on the relative sizes of ω and μ^0. This comes about as follows.

The spectral density $c(\omega)$ is zero if $|\omega|$ is less than the minimum single-particle excitation energy in the system. Further, $|\mu^0|$ is also smaller than this minimum. That is to say, μ^0 lies in the energy gap.

We can see this most easily in the free-field case for which the lowest excitation energy is the lowest eigenvalue of (24), $\min(\omega_k)$. The expression for $c(\omega)$ in terms of the mode functions φ_k of (24) is

$$c(\omega,x,x') = 2\pi i\sum_k \omega_k^{-1}[\varphi_k(x)\varphi_k^*(x')\delta(\omega-\omega_k) - \varphi_k^*(x)\varphi_k(x')\delta(\omega+\omega_k)]\,.$$

The thermodynamic grand potential Ω_0 is given by

$$\Omega_0 = -\beta_0^{-1} \sum_k \ln \sum_{n_k^+=0}^{\infty} \sum_{n_k^-=0}^{\infty} \exp[n_k^+ \beta_0(\mu^0 - \omega_k)] \times \exp[-n_k^- \beta_0(\mu^0 + \omega_k)] ,$$

which is, of course, just the sum of the particle and antiparticle potentials. The geometric series converge only if $|\mu^0| < \min(\omega_k)$ (cf. Ref. 9, Sec. 53). In Minkowski space-time $\min(\omega_k) = \kappa$ and this result generalizes the usual relativistic restriction on Bose-Einstein chemical potentials.[10] [We conjecture that the same result is true in the interacting case (although this may be obvious).]

If now the denominators in (27) are expanded we obtain the finite-temperature Green's function $G_{(\beta,\mu)}$ as an image sum of *vacuum* Green functions,

$$G_{(\beta,\mu)}(x,x') = \sum_{m=-\infty}^{\infty} e^{-m\beta_0\mu^0} G(x, x' - im\beta_0\lambda) , \tag{28}$$

where λ is the timelike unit vector $(1, 0, 0, 0)$.

This result shows, amongst other things, that the finite-temperature Feynman Green's function is quasiperiodic in imaginary time with period β_0 and real "phase factor" $\exp(-\beta_0\mu^0)$. It generalizes Eq. (80) of Ref. 4 to a noninteracting Bose system with nonzero charge chemical potential in a static space-time. Gibbons and Perry[18] have also shown this periodicity in static spaces but our result, (28), goes a little further. More comments on this expression will be found in the final section.

At this point we cease discussion of the general case and specialize to a neutral, massless field in the static space $R \times S^3$. This system can be thought of as a scalar photon gas in an Einstein universe and we seek its internal energy density $\langle \hat{T}_{00} \rangle_T$. Since $g_{00} = 1$ we do not have to distinguish between β_0, μ^0 and β, μ and in addition μ is assumed zero [even though $\min(\omega_k) \neq 0$].

From expression (6) for G it is seen that the averages we need, say Eq. (1) with G replaced by $G_{(\beta,0)}$, will appear as double sums over m and n of certain coincidence limits, $\sum_{m,n=-\infty}^{\infty} C_{mn}$. The renormalization ansatz is to drop the $n = 0 = m$ term, leaving the contributions

$$\sum_{-\infty}^{\infty}{}' C_{0n}, \quad \sum_{-\infty}^{\infty}{}' C_{m0}$$

and

$$\sum_{-\infty}^{\infty}{}' \sum_{-\infty}^{\infty}{}' C_{mn} .$$

The first of these contributions is temperature independent and is just the vacuum average calculated before.[3] This is the "Casimir term." The second contribution corresponds to blackbody radiation in infinite space as we shall see, and the final double sum is the correction term. Correspondingly we thus write[4] for the renormalized average of the stress-energy tensor

$$\langle \hat{T}_{\mu\nu} \rangle_T = \langle \hat{T}_{\mu\nu} \rangle_0 + \langle \hat{T}_{\mu\nu} \rangle_T^{\infty} + \langle \hat{T}_{\mu\nu} \rangle_T^{a} . \tag{29}$$

We can restrict ourselves to $\langle \hat{T}_{00} \rangle_T$ because $\langle \hat{T}_{ij} \rangle_T$ is given by $\langle \hat{T}_{ij} \rangle_T = -\frac{1}{3} g_{ij} \langle \hat{T}_{00} \rangle_T$ just as in Eq. (8) for $\kappa = 0$. $\langle \hat{T}_{00} \rangle_T$ can be easily found from

$$\langle \hat{T}_{00}(x) \rangle_T = i \nabla_0 \nabla_0 G_{(\beta)}(x, x') \big|_{x'=x}$$

with

$$G_{(\beta)}(x,x') = \sum_{m=-\infty}^{\infty} G(x, x' - im\beta\lambda) ,$$

where

$$G = -\frac{i}{4\pi^2 a} \frac{1}{\sin(s/a)} \sum_{n=-\infty}^{\infty} \frac{s + 2\pi n a}{\sigma_n^2}$$

is (6) for $\kappa = 0$. The renormalization ansatz is understood.

The three terms in (29) are readily evaluated and we find for the Casimir contribution the value[3]

$$\langle \hat{T}_{00} \rangle_0 = 3(16\pi^6 a^4)^{-1} \zeta(4) = (480\pi^2 a^4)^{-1} \tag{30}$$

first given by Ford.[17] The blackbody term is one half of the standard Planck expression,

$$\langle \hat{T}_{00} \rangle_T^{\infty} = \frac{\pi^2}{30} (kT)^4 , \tag{31}$$

as expected.

The correction to the energy density can be written as the sum of two terms corresponding to (15) (cf. Ref. 4)

$$\langle \hat{T}_{00} \rangle_T^{a} = V^{-1}(F' + TS') , \tag{32}$$

where $V = 2\pi^2 a^3$ is the volume of S^3. F' and S' are the corrections to the total free energy and entropy, respectively, of the field at temperature T in the Einstein universe and are given by

$$F' = \frac{4}{\pi^4 a} \sum_{m,n=1}^{\infty} \frac{\xi^4 (12\xi^2 n^2 - m^2)}{(m^2 + 4\xi^2 n^2)^3} \equiv a^{-1} f(\xi) \tag{33}$$

and

$$S' = -\left(\frac{dF'}{dT}\right)_V = -k\pi \frac{df}{d\xi} , \tag{34}$$

where ξ is the dimensionless parameter $kT\pi a$. Equation (34) agrees with (16).

In this way we can establish the thermodynamical properties of the gas of φ quanta since the complete energy density $\langle\hat{T}_{00}\rangle_T$, obtained by adding (30), (31), and (32), can obviously be written as $V^{-1}E = V^{-1}(F+TS)$, where F and S will be the total free energy and entropy. The Casimir entropy is of course zero.

As a simple example it is straightforward to check that the gas satisfies the relation $PV=\frac{1}{3}E$, where the pressure P is $-(dF/dV)_T$ in accordance with the considerations in Sec. IV. This result also follows from the tracelessness of $\langle T_{\mu\nu}\rangle_T$, or from noting that $\omega_k \sim a^{-1} \sim V^{-1/3}$.

We can also note that the energy density depends on the volume via ξ. Hence the energy is not extensive in this sense and so the P' of (18) is not equal to P.

From (33) it is possible to find the behavior of $\langle\hat{T}_{00}\rangle_T$ for large and small ξ. For large ξ, corresponding to a high temperature or to a big radius a, we find that the correction term tends to minus the Casimir energy, up to exponentially small additions, while for small ξ it approaches minus the Planck density, again up to exponential corrections. Thus in the two limits $\langle\hat{T}_{00}\rangle_T$ is the Casimir term for low temperatures and the Planck term for high ones.

Brown and Maclay[4] found that for small ξ there was an extra term that went like T^3. This is due to the different geometry. In the situation there considered (the standard slab configuration of the Casimir effect) space is infinitely extended parallel to the plates. For us space is compact.

In accordance with our discussion in Sec. III we might think of using $\langle\hat{T}_{\mu\nu}\rangle_T$ on the right-hand side of Einstein's equations and demanding self-consistency. This would yield the radius as a function of the temperature. The results will be given at another time if deemed to be of sufficient physical interest.

VI. COMMENTS AND EXTENSIONS

Our results, (10) and (11), for the vacuum-averaged stress-energy tensor differ from the expression derived by Streeruwitz.[1] In particular they cannot be expanded in inverse powers of $(\kappa a)^2$. Our values come entirely from the nonperturbative, indirect-path contribution in (4), or (5) or (6). We do not claim that this will be a feature of the theory in a general space-time.

Incidentally it might be thought that if we had chosen the open space section H^3 instead of the compact one, S^3, our renormalization ansatz would have given zero since there is only one geodesic on H^3 connecting two given points. However, the quantum-mechanical propagator on H^3 is still of the form (4), with $a \to ia$, and there is again a nonperturbative contribution coming from the "imaginary geodesics" $n \geq 1$. This suggests that even in a simple convex space-time (i.e., one with at most one real geodesic connecting any two points) a perturbation expansion should not be relied upon. In other words, there might be additional terms arising from multiple *imaginary* geodesics.

We have discussed the effect of averaging over a state in thermal equilibrium and have shown, as expected, that for the massless case the scalar gas is an ideal quantum gas[19] satisfying $PV=\frac{1}{3}E$. An explicit expression for the free energy has been found.

We would now like to make some further comments on the theory of finite-temperature Green's functions and in particular on Eq. (28).

This equation bears more than a formal similarity to the expression for the Green's function on a multiply-connected space. As shown by Laidlaw and DeWitt[20] and Schulman[21] and more formally by Dowker[22] the quantum-mechanical propagator K on a multiply-connected configuration space $\mathfrak{M} = \tilde{\mathfrak{M}}/\Gamma$ is given by the sum

$$K(q''\,|\,q';\tau) = \sum_{\gamma\in\Gamma} a(\gamma)\tilde{K}(\tilde{q}_0''\,|\,\tilde{q}_0'\gamma;\tau)\,, \tag{35}$$

where $\tilde{K}$ is the propagator on the simply connected universal covering space $\tilde{\mathfrak{M}}$, and the multipliers $a(\gamma)$ comprise a unitary, one-dimensional (for scalar quantum mechanics) representation of the fundamental group of $\mathfrak{M}$, $\pi_1(\mathfrak{M}) = \Gamma$.

Pictorially $\tilde{\mathfrak{M}}$ can be thought of as made up of $|\Gamma|$ copies of $\mathfrak{M}$ ($|\Gamma|$ is the order of the discrete symmetry group Γ). Each point q of $\mathfrak{M}$ then corresponds to $|\Gamma|$ points ("pre-images") in $\tilde{\mathfrak{M}}$ which can be represented by $\tilde{q}_0\gamma$, for all γ belonging to Γ. $\tilde{q}_0$ is one, arbitrarily selected, but then fixed pre-image of q and we have assumed the group action of Γ on $\tilde{\mathfrak{M}}$ to be on the right. This last is only a notational point.

The equation $\mathfrak{M} = \tilde{\mathfrak{M}}/\Gamma$ simply means that all the points $\tilde{q}_0\gamma$ or $\tilde{\mathfrak{M}}$ are identified as the single point q of $\mathfrak{M}$. Equation (35) says that the propagator on $\mathfrak{M}$ to go from q' to q'' is obtained from that on $\tilde{\mathfrak{M}}$ by adding the partial amplitudes to go from each of the pre-images of the initial point to one selected pre-image of the final point. (It should be said here that we are assuming that the dynamics as well as the manifold $\tilde{\mathfrak{M}}$ is invariant under Γ.)

The continuous paths from $\tilde{q}_0'\gamma$ to $\tilde{q}_0''$ for the different γ belong to distinct homotopy classes and the Feynman path-integral formulation is a natural one to use in this situation. References 20 and 21 use such an approach to show that the phase factors $a(\gamma)$ form a unitary representation of the first homotopy group of $\mathfrak{M}$, $\pi_1(\mathfrak{M})$, which is here isomorphic to Γ. A different attitude is taken in Ref. 22 in that no use is made of path summation, and $a(\gamma)$ is simply a multiplier which comes in quite naturally when quantum mechanics on $\tilde{\mathfrak{M}}$ is projected down to give $|\Gamma|$ copies on the $|\Gamma|$ "fibers" $\mathfrak{M}$. In this way we can make contact with an important body of mathematical lore associated with the Selberg trace formula. Since this is not directly relevant to our present rather discursive preliminary discussion we shall not consider the deeper theory of Eq. (35) further here.

Usually $\mathfrak{M}$ is a positive-definite Riemannian space but in certain circumstances one can use a continuation procedure, like Hadamard,[23] to take us to a space with an indefinite, e.g. Minkowskian, metric. We have used this idea earlier[24] in conjunction with Eq. (2) to derive a Feynman Green's function on de Sitter space.

The method clearly works for any reasonable static space-time, as indicated in Ref. 24. Basically the only difference between the theory in such a static space and that in flat space is the change in the spatial modes. For this reason the details of the continuation process for this case were not given in Ref. 24 as being obvious and are not given here either for the same reason.

Thus, given a suitable static space-time, we can construct the propagator on the corresponding Euclidean-signatured space, using (35) if this happens to be multiply-connected, and then substitute this K_E into Eq. (2) for the Euclidean Green's function G_E which can finally be continued to the physical signature.[24]

It is within this scheme that we should like to interpret Eq. (28). The correspondence is immediate. Equation (28) says that if the "time" axis T_E in the Euclidean-signatured space $\mathfrak{M}_E$ has the fundamental group $\pi_1(T_E)=Z_\infty$ then the Green's function in the Minkowski-signatured space $\mathfrak{M}$ is just a finite-temperature Green's function on $\mathfrak{M}$. Z_∞ is the infinite cyclic group and the action of γ on q_0' in (35) corresponds to the addition of $m\beta$ to it' in Eq. (28).

It should be emphasized that this result is a purely mathematical one. There is no obligation to give physical significance to the Fock space in which the thermal averages are taken. Other arguments will be needed for this.

Why $\pi_1(T_E)$ should be Z_∞ is another question. A very interesting example is given by Gibbons and Perry[18] in connection with the blackhole Schwarzschild geometry. The corresponding Euclidean-signatured space has a singularity at $r=2M$ (M = source mass) which can be removed if the simply connected time axis $\tilde{T}_E$ is turned into $T_E=\tilde{T}_E/Z_\infty$, a one-torus with period $8\pi M$. This is seen if the Kruskal coordinates U and V are used in place of t and r, for then U and V are seen to be periodic functions of t_E $(=-it)$, as emphasized in Ref. 18.

A slightly different way of saying this is to write the metric in terms of t_E and $|V|$ which play, respectively, the roles of an angle and of a cylindrical coordinate,

$$-ds_E{}^2=\frac{32M^3}{r}\,e^{-r/2M}\left[d\,|V|^2+|V|^2 d\left(\frac{t_E}{4M}\right)^2\right]$$

$$+\,r^2(d\theta^2+\sin^2\theta d\varphi^2)\,,$$

$$|V|=e^{r/4M}\left(\frac{r}{2M}-1\right)^{1/2}.$$

It is apparent from this form that unless we identify the points t_E and $t_E+8\pi M$ there will be a true singularity (i.e., an infinite curvature) of the conical type[25] at the "origin" $|V|=0$, i.e., at the horizon $r=2M$. According to Gibbons this is the reason why the black hole seems endowed with a temperature T_0 given by $\beta_0=1/kT_0=8\pi M$.

When the central source is charged the situation is akin to that of the Aharonov-Bohm effect or to that of the magnetic monopole since there is a singularity axis carrying a flux of electromagnetic (here electric) field. In the usual theory the phase factors $a(\gamma)$ in (35) are just $\exp(im\Phi)$, where Φ is the flux through the axis and m is the winding number that labels the elements of $\Gamma=Z_\infty$ in the standard way. (We have assumed the test particle to have unit charge.) A nice discussion of this situation is given by Schulman in Ref. 21.

In the black-hole case Φ is imaginary, due to the continuation, and equals $iQ(kT)^{-1}/[M+(M^2+Q^2)^{1/2}]$, where Q is the central charge and T is the black-hole temperature now given by

$$2\pi kT=(M^2-Q^2)^{1/2}/[M+(M^2-Q^2)^{1/2}]^2.$$

In this way Eq. (35) becomes identical to (28), with the chemical potential μ^0 equaling $Q/[M-(M^2-Q^2)^{1/2}]$. This value is due to Gibbons and Perry.[18]

If these ideas are taken seriously one might think of writing down conditions for the chemical equilibrium of a system of black holes. For example, if we equate the chemical potentials for several black holes we find the known equilibrium condition $|Q_i|=M_i$.[26]

Similar considerations involving a singularity axis having a flux of angular momentum hold for the Kerr metric. The details will be elaborated elsewhere.

Strictly speaking, the situation for the Reissner-Nordström and Kerr metrics is stationary rather than static and more care is needed when setting up Fock space.[27] The appearance of a chemical potential is symptomatic of the possibility of paradoxes of the Klein type.

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[1]E. Streeruwitz, Phys. Rev. D 11, 3378 (1975). Streeruwitz (private communication has pointed out that part of the difference is due simply to his choice of the minimally coupled Klein-Gordon equation as opposed to the conformal one.
[2]J. S. Dowker, J. Phys. A 3, 451 (1970); Ann. Phys. (N.Y.) 62, 361 (1971); L. S. Schulman, Phys. Rev. 176, 1558 (1968).
[3]J. S. Dowker and R. Critchley, J. Phys. A 9, 535 (1976).
[4]L. S. Brown and G. J. Maclay, Phys. Rev. 184, 1272 (1969).
[5]B. S. DeWitt, Phys. Rep. 19C, 296 (1975).
[6]S. Christensen, Phys. Rev. D 14, 2490 (1976).
[7]R. Dashen, S.-k. Ma, and H. J. Bernstein, Phys. Rev. 187, 345 (1969).
[8]D. ter Haar, *Elements of Statistical Mechanics* (Holt-Rinehart and Winston, New York, 1961).
[9]L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, London, 1958).
[10]J. Ehlers, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic, New York, 1971).
[11]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford U.P., Oxford, 1934); N. L. Balazs, Astrophys. J. 128, 398 (1958).
[12]H. A. Buchdahl, *The Concepts of Classical Thermodynamics* (Cambridge U.P., Cambridge, England, 1966).
[13]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
[14]S. A. Fulling, Phys. Rev. D 7, 2850 (1973); D. Kramer and K.-H. Lotze, Acta Phys. Polon. B5, 437 (1974).
[15]O. Halpern and G. Heller, Phys. Rev. 48, 434 (1935).
[16]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971); L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[17]L. H. Ford, Phys. Rev. D 11, 3370 (1975).
[18]G. Gibbons and M. J. Perry, DAMTP Cambridge report, 1976 (unpublished).
[19]H. Einbinder, Phys. Rev. 74, 805 (1948); 76, 410 (1949); P. T. Landsberg, *Thermodynamics* (Interscience, New York, 1961).
[20]C. Morette-DeWitt and M. G. G. Laidlaw, Phys. Rev. D 3, 1375 (1971).
[21]L. S. Schulman, J. Math. Phys. 12, 304 (1971).
[22]J. S. Dowker, J. Phys. A 5, 936 (1972).
[23]J. Hadamard, *Lectures on Cauchy's Problem* (Yale U.P., New Haven, 1923).
[24]J. S. Dowker and R. Critchley, Phys. Rev. D 13, 224 (1976).
[25]J. S. Dowker, Nuovo Cimento 52B, 129 (1967); J. Phys. A 2, 267 (1969); D. A. Feinblum, J. Math. Phys. 11, 2713 (1970).
[26]See e.g. W. B. Bonnor and J. P. Ward, Commun. Math. Phys. 28, 323 (1972); W. Israel and J. T. J. Spanos, Lett. Nuovo Cimento 7, 245 (1973).
[27]S. A. Fulling, Phys. Rev. D 14, 1939 (1976).