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Stress-tensor trace anomaly in a gravitational metric: Scalar fields*

Lowell S. Brown
Department of Physics, University of Washington, Seattle, Washington 98195
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We compute the stress-tensor vacuum expectation value of a massive, scalar quantum field that is coupled to the metric of an arbitrary classical gravitational field. The renormalized tensor is defined by a dimensionally continued, proper-time representation. The stress tensor is calculated for arbitrary dimension in a potentially conformal-invariant manner so that its trace is formally proportional to the square of the scalar-field mass with this trace vanishing as the scalar field becomes massless. However, the renormalized stress tensor violates this formal identity with its trace containing additional, anomalous terms. These finite-trace anomalies are intimately related to the infinite counterterms that must be put into the action to make the stress tensor finite.

I. INTRODUCTION AND SUMMARY

The study of the stress-energy tensor of quantized matter fields coupled to the metric of a classical gravitational field is interesting for several reasons. Hawking[1] discovered that collapsing stars alter the metric in such a way as to make quantum-mechanical particle production take place in a thermal distribution. The flow of energy and momentum of these particles is described by a vacuum expectation value of the matter-field stress tensor. This quantity is needed, in a more accurate approximation, for the calculation of the reaction back on the metric, since it will appear as a source driving the Einstein field equation. A similar reaction on the metric by quantized matter fields may be significant in producing isotropy in the early universe.[2] In addition to such specific processes, a clear understanding of the stress produced in a quantized field by a classical gravitational field[3] is, of course, necessary for the ultimate construction of a consistent quantum gravitational theory. The vacuum expectation value of the stress tensor is infinite; it must be regulated and renormalized. Recently, Deser, Duff, and Isham[4] showed that, in general, the trace of the renormalized tensor involves anomalous terms.[5] Here we shall examine the simplest situation in detail, that of a scalar field coupled to the metric of an arbitrary gravitational field.

We shall define the theory by a method of dimensional continuation. A Lagrange function for space-time of an arbitrary dimension will be chosen such that the trace of the stress tensor formally vanishes as the scalar-field mass vanishes. We shall compute the vacuum expectation value of the stress tensor and the corresponding one-loop effective-action functional by a dimensional continuation of the proper-time representation which was introduced by Schwinger[6] and which has been developed further by DeWitt.[3,7]

This proper-time method gives a specific solution to the scalar-field Green's function equation, that solution corresponding to the vacuum state when the metric is asymptotocally flat. In more general circumstances, the possibility exists of adding homogeneous solutions to the Green's function. However, this possibility should not alter our results on the counterterms that are necessary to renormalize the stress tensor or on the anomalous terms in its trace, since these are related to short-distance limits.[8]

Our new method is well defined and free of ambiguity. The infinite counterterm needed to renormalize the stress tensor is the metric variational derivative of the scalar counterterm which renormalized the one-loop action. This is a condition that must be met if the stress tensor is to be renormalized consistently. It guarantees that the stress-tensor counterterm is conserved. This counterterm has a vanishing trace when the scalar field mass vanishes. The renormalized stress tensor is defined without ambiguity. It is conserved and has well-defined trace anomalies. The stress tensor of a scalar field in a gravitational metric has also been computed by other authors using a point-separation technique for two-[9] and four[10]-dimensional space-time. This technique yields ambiguous terms involving $t^\mu t^\nu / t^\alpha t_\alpha$, where t^μ is the tangent vector to the geodesic used in the point-separation. This technique also produces[9,10] a stress tensor which is not conserved. If the ambiguous terms involving $t^\mu t^\nu / t^\alpha t_\alpha$ are discarded, a conserved stress tensor is obtained which has trace anomalies. These trace anomalies[9,10] agree precisely with those found in our work, including the values of their numerical coefficients.[11]

In basic outline, our method proceeds essentially as follows. We have dimensionally continued, proper-time integrals of the general form

$$I = \int_0^\infty i\, ds (is)^{-n/2} F(is; n), \tag{1.1}$$

where $F(is;n)$ is analytic but nonvanishing at $s=0$. This integral diverges in the limit $n\to 2$, and it must be renormalized. (The limit in four-dimensional space-time involves $n\to 4$ and different explicit powers of s in the integrand. We use the limit $n\to 2$ here and below in order to simplify the notation.) We do this renormalization by first introducing an arbitrary, auxiliary scale mass κ so that the integrand $F(is;n)$ has a fixed scale dimension appropriate to $n=2$. Thus, on introducing a temporary, intermediate, proper-time cutoff s_0, and integrating by parts, we have

$$\begin{aligned} n\to 2:\ I &= \kappa^2 \int_0^\infty i\,ds(\kappa^2 is)^{-n/2} F(is;n) \\ &= \kappa^2 \int_0^{s_0} i\,ds(\kappa^2 is)^{-n/2} F(is;n) + \int_{s_0}^\infty i\,ds(is)^{-1} F(is;2) \\ &= \kappa^2 \int_0^{s_0} i\,ds(\kappa^2 is)^{-n/2} F(is;n) - (\ln is_0) F(is_0;2) - \int_{s_0}^\infty i\,ds(\ln is)\frac{\partial}{\partial is}F(is;2). \end{aligned} \tag{1.2}$$

We expand $F(is;n)$ in the first integral above in a power series in s and integrate term by term under the assumption that $n<2$ so that the lower limit of the integration does not contribute. In the limit $n\to 2$, the dependence on the intermediate cutoff s_0 in this series must cancel the s_0 dependence which occurs in the lower limit of the remaining integral in Eq. (1.2). We can dispense with this temporary cutoff by setting $s_0=0$ and secure the dimensional continuation limit

$$\begin{aligned} n\to 2:\ I &= \frac{1}{1-n/2}F(0;2) - 2\frac{\partial}{\partial n}F(0;n)\Big|_{n=2} \\ &\quad - \int_0^\infty i\,ds(\ln\kappa^2 is)\frac{\partial}{\partial is}F(is;2). \end{aligned} \tag{1.3}$$

The renormalized integral is defined by deleting the pole at $n=2$:

$$\begin{aligned} I_{\rm ren} &= -2\frac{\partial}{\partial n}F(0;n)\Big|_{n=2} \\ &\quad - \int_0^\infty i\,ds(\ln\kappa^2 is)\frac{\partial}{\partial is}F(is;2). \end{aligned} \tag{1.4}$$

We can now see how the anomaly arises in the trace of the stress-tensor vacuum expectation value. This tensor has a dimensionally continued, proper-time representation of the sort that we have just discussed with $F(is;n)$ replaced by a weight $T^{\mu\nu}(x;is;n)$. Thus the renormalized tensor is given by

$$\langle T^{\mu\nu}\rangle_{\rm ren} = \langle T^{\mu\nu}\rangle - \frac{1}{1-n/2}T^{\mu\nu}(x;0;2). \tag{1.5}$$

The weight may be written in the form

$$T^{\mu\nu}(x;is;n) = g^{\mu\nu}A + B^{\mu\nu}, \tag{1.6}$$

where the quantity $B^{\mu\nu}$ does not contain an explicit factor of $g^{\mu\nu}$. Let us work in the conformally invariant theory where the scalar-field mass vanishes, and where

$$0 = g_{\mu\nu}T^{\mu\nu}(x;is;n) = nA + g_{\mu\nu}B^{\mu\nu}. \tag{1.7}$$

With $n\neq 2$, the unrenormalized tensor has a vanishing trace. Hence taking the trace of Eq. (1.5) and then taking the limit $n\to 2$ yields

$$\begin{aligned} g_{\mu\nu}\langle T^{\mu\nu}\rangle_{\rm ren} &= 0 - \frac{1}{1-n/2}g_{\mu\nu}T^{\mu\nu}(x;0;n=2) \\ &= -\frac{1}{1-n/2}(nA-2A) \\ &= 2A. \end{aligned} \tag{1.8}$$

This trace anomaly is directly related to the infinite scalar counterterm that is needed in the action to make the stress-tensor finite. This can be seen as follows. The vacuum expectation value of the stress tensor can be expressed as the metric variation of an effective-action functional,[8]

$$\tfrac{1}{2}[-g(x)]^{1/2}\langle T^{\mu\nu}(x)\rangle = \frac{\delta}{\delta g_{\mu\nu}(x)}W_1[g_{\alpha\beta}], \tag{1.9}$$

where g is the determinant of the metric $g_{\alpha\beta}$. The effective action W_1 corresponds to a single closed-loop vacuum graph with the scalar field propagating in the background metric $g_{\mu\nu}$. It has a proper-time representation of the form

$$W_1[g_{\alpha\beta}] = \frac{1}{1-n/2}W[0;2;g_{\alpha\beta}] + W_{1\,\rm ren}[g_{\alpha\beta}], \tag{1.10}$$

with

$$\begin{aligned} W_{1\,\rm ren}[g_{\alpha\beta}] &= -2\frac{\partial}{\partial n}W[0;n;g_{\alpha\beta}]\Big|_{n=2} \\ &\quad - \int_0^\infty i\,ds(\ln\kappa^2 is)\frac{\partial}{\partial is}W[is;2;g_{\alpha\beta}]. \end{aligned} \tag{1.11}$$

We should first note a general feature of the dimensional regularization method: The value of the auxiliary scale mass κ is quite arbitrary, but a change in it is precisely compensated by a finite change in the renormalization counterterm. Thus, a change $\kappa \to \kappa'$ in Eq. (1.11) gives the integral of a total derivative which is precisely compensated by a change

$$\frac{1}{1-n/2} W[0;2;g_{\alpha\beta}] \to \left[\frac{1}{1-n/2} - \ln\left(\frac{\kappa'}{\kappa}\right)^2\right] \times W[0;2;g_{\alpha\beta}] \qquad (1.12)$$

in the counterterm in Eq. (1.10). We now assume that the scalar field is massless so that we have the conformal invariance

$$W[is;2;\lambda^2 g_{\alpha\beta}] = W[\lambda^{-2} is;2;g_{\alpha\beta}] \qquad (1.13)$$

under *constant* scale transformations λ. On changing the integration variable in Eq. (1.11) to $s' = \lambda^{-2}s$, we see that this scale transformation is equivalent to an alteration of the auxiliary mass $\kappa \to \lambda\kappa$, an alteration which is equivalent to a finite change in the counterterm. Hence

$$W_1[\lambda^2 g_{\alpha\beta}] = W_1[g_{\alpha\beta}] + (\ln\lambda^2) W[0;2;g_{\alpha\beta}]. \qquad (1.14)$$

It follows from Eq. (1.9) that this implies the trace identity

$$\int (d^2x)\sqrt{-g}\, g_{\mu\nu}\langle T^{\mu\nu}\rangle = \frac{\partial}{\partial\lambda} W_1[\lambda^2 g_{\alpha\beta}]\Big|_{\lambda=1} = 2W[0;2;g_{\alpha\beta}]. \qquad (1.15)$$

Since the scalar field was assumed to be massless, this trace vanishes when computed in a formal way. We find that there is a trace anomaly, and that the space-time integral of this finite anomaly is proportional to the scalar counterterm which is needed to renormalize the stress tensor.

We review the techniques that are needed in Sec. II. First we explain how a Lagrange function for a scalar field ϕ can be constructed which preserves conformal invariance in a space-time of arbitrary dimensionality. On adding a mass term to this Lagrange function we have

$$\mathcal{L} = -\tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - \tfrac{1}{2}\xi R\phi^2 - \tfrac{1}{2}m^2\phi^2, \qquad (2.4)$$

where R is the curvature scalar and where

$$\xi = \frac{n-2}{4(n-1)}, \qquad (2.5)$$

with n the dimensionality of the space-time. The stress tensor derived from the corresponding action obeys the formal trace identity

$$g_{\mu\nu}T^{\mu\nu} = -m^2\phi^2 \qquad (2.9)$$

for arbitrary dimensionality n. [To be precise about the notation, we use a metric $g_{\mu\nu}$ with signature $(-+++)$, and a curvature tensor defined by

$$R^{\lambda}_{\mu\kappa\nu} = \Gamma^{\lambda}_{\mu\nu,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\kappa} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\kappa\sigma} - \Gamma^{\sigma}_{\mu\kappa}\Gamma^{\lambda}_{\nu\sigma}. \qquad (1.16)$$

Commas denote an ordinary derivative; semicolons denote a covariant derivative.] In the remainder of Sec. II we discuss the proper-time representation of the scalar-field Green's function.

In Sec. III we derive the dimensionally regularized, proper-time representation of the one-loop action functional $W_1[g_{\alpha\beta}]$. It is expressed as the space-time integral of an effective Lagrange function,

$$W_1[g_{\alpha\beta}] = \int (d^n x)\sqrt{-g}\,\mathcal{L}_1[x;g_{\alpha\beta}]. \qquad (3.6)$$

In space-time of two dimensions, the effective Lagrangian is renormalized on writing

$$\mathcal{L}_1^{(n=2)} = \left(\frac{1}{2-n} + L_2\right)\mathcal{a}_2 + \mathcal{L}_{1\,\text{ren}}^{(n=2)}, \qquad (1.17)$$

where

$$\mathcal{a}_2 = \frac{1}{4\pi}\frac{\partial}{\partial is}[e^{-m^2 is}F(x,x;is;2)]\Big|_{s=0} = \frac{1}{4\pi}(\tfrac{1}{6}R - m^2), \qquad (1.18)$$

and where $\mathcal{L}_{1\,\text{ren}}^{(n=2)}$ is a finite quantity with a proper-time representation

$$\mathcal{L}_{1\,\text{ren}}^{(n=2)} = \tfrac{1}{2}\mathcal{a}_2 + \frac{1}{8\pi}\left\{\tfrac{1}{2}R - \int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is;2)]\right\}. \qquad (1.19)$$

The biscalar $F(x,x';is;n)$ is a weight in the proper-time representation of the Green's function. The constant L_2 in the infinite counterterm in Eq. (1.17) arises from the derivative with respect to n of dimensional-dependent factors such as $(4\pi)^{-n/2}$. Note that the Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R \qquad (1.20)$$

vanishes in two dimensions. Hence, the quantity

$$\mathcal{g}_2 = \int (d^2x)\sqrt{-g}\,R \qquad (1.21)$$

is a topological invariant in the sense that its metric variation vanishes identically. Accordingly, the scalar curvature contribution to the infinite counterterm in Eq. (1.17) can be omitted from the infinite renormalization to the action. The action can be rendered finite by writing

$$W_1^{(n=2)} = -\frac{1}{4\pi}\left(\frac{1}{2-n}+L_2\right)m^2\int(d^2x)\sqrt{-g} + W_{1\,\mathrm{ren}}^{(n=2)}, \qquad (1.22)$$

where $W_{1\,\mathrm{ren}}^{(n=2)}$ is the space-time integral of the renormalized effective Lagrangian $\mathcal{L}_{1\,\mathrm{ren}}^{(n=2)}$. This renormalization corresponds to a renormalization of the cosmological term in the Einstein Lagrange function. No renormalization is necessary for the massless theory in two dimensions.

In space-time of four dimensions, the effective Lagrangian is renormalized on writing

$$\mathcal{L}_1^{(n=4)} = \left(\frac{1}{4-n}+L_4\right)\mathcal{A}_4 + \mathcal{L}_{1\,\mathrm{ren}}^{(n=4)}, \qquad (1.23)$$

where

$$\mathcal{A}_4 = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2\left[e^{-m^2 is}F(x,x;is;4)\right]\bigg|_{s=0}$$
$$= \frac{1}{(4\pi)^2}\left[\tfrac{1}{180}(R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} - R_{\mu\nu}R^{\mu\nu} + R_{,\mu}{}^{;\mu}) + \tfrac{1}{2}m^4\right], \qquad (1.24)$$

and where

$$\mathcal{L}_{1\,\mathrm{ren}}^{(n=4)} = \tfrac{1}{4}\mathcal{A}_4 - \frac{1}{2(4\pi)^2}\left\{\frac{m^2}{18}R - \tfrac{1}{108}R_{,\sigma}{}^{;\sigma} + \tfrac{1}{2}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3\left[e^{-m^2 is}F(x,x;is;4)\right]\right\}. \qquad (1.25)$$

The nature of the renormalization term $\mathcal{A}_4$ is clarified if we consider its response to a conformal transformation,

$$g_{\mu\nu}(x) \to \lambda(x)^2 g_{\mu\nu}(x). \qquad (1.26)$$

The Weyl tensor

$$C^\lambda_{\mu\kappa\nu} = R^\lambda_{\mu\kappa\nu} - \tfrac{1}{2}(\delta^\lambda_\kappa R_{\mu\nu} - \delta^\lambda_\nu R_{\mu\kappa} - g_{\mu\kappa}R^\lambda_\nu + g_{\mu\nu}R^\lambda_\kappa) + \tfrac{1}{6}R(\delta^\lambda_\kappa g_{\mu\nu} - \delta^\lambda_\nu g_{\mu\kappa}) \qquad (1.27)$$

is not altered by this transformation. We note that in four dimensions the integral

$$\mathcal{G}_4 = \int(d^4x)\sqrt{-g}\,G, \qquad (1.28)$$

with

$$G = R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \qquad (1.29)$$

in a topological invariant. Expressing $\mathcal{A}_4$ in terms of these quantities yields

$$\mathcal{A}_4 = \frac{1}{(4\pi)^2}\left[\tfrac{1}{120}(C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} - \tfrac{1}{3}G) + \tfrac{1}{180}R_{,\mu}{}^{;\mu} + \tfrac{1}{2}m^4\right]. \qquad (1.30)$$

A total divergence as well as a topologically invariant quantity can be omitted in renormalizing the action. Hence

$$W_1^{(n=4)} = \frac{1}{(4\pi)^2}\left(\frac{1}{4-n}+L_4\right)\times\int(d^4x)\sqrt{-g}\,(\tfrac{1}{120}C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} + \tfrac{1}{2}m^4) + W_{1\,\mathrm{ren}}^{(n=4)}, \qquad (1.31a)$$

where $W_{1\,\mathrm{ren}}^{(n=4)}$ is the space-time integral of $\mathcal{L}_{1\,\mathrm{ren}}^{(n=4)}$. The massless theory is renormalized with the square of the Weyl tensor, a quantity which is invariant under conformal transformation. The massive theory has an additional infinite counterterm that corresponds to a renormalization of the cosmological term in the Einstein Lagrange function. Although the combination m^2R is of the proper scale dimension, it does not appear in the renormalization. The renormalization relation can be written in a simpler, equivalent form by again using the definitions of the Weyl tensor and the topologically invariant character of G,

$$W_1^{(n=4)} = \frac{1}{(4\pi)^2}\left(\frac{1}{4-n}+L_4\right)\times\int(d^4x)\sqrt{-g}\left[\tfrac{1}{180}(3R_{\mu\nu}R^{\mu\nu} - R^2) + \tfrac{1}{2}m^4\right] + W_{1\,\mathrm{ren}}^{(n=4)}. \qquad (1.31b)$$

In Sec. III we also derive the dimensionally continued, proper-time representation for the vacuum expectation value of the stress-energy tensor. In two dimensions we find that

$$\langle T^{\mu\nu}\rangle^{(n=2)} = 2\left(\frac{1}{2-n}+L_2\right)\frac{1}{4\pi}T^{\mu\nu}(x;0;2) + \langle T^{\mu\nu}\rangle_{\mathrm{ren}}^{(n=2)}, \qquad (1.32)$$

where

$$T^{\mu\nu}(x;0;2) = -\tfrac{1}{2}m^2 g^{\mu\nu}, \qquad (1.33)$$

and where

$$\langle T^{\mu\nu}\rangle_{\mathrm{ren}}^{(n=2)} = \tfrac{1}{2}g^{\mu\nu}\mathcal{A}_2 - \frac{1}{4\pi}\int_0^\infty i\,ds(\ln\kappa^2 is)\frac{\partial}{\partial is}\times\left[e^{-m^2 is}T^{\mu\nu}(x;is;2)\right] \qquad (1.34)$$

Note that the infinite counterterm which appears in the stress-tensor renormalization in Eq. (1.32) is precisely twice the metric variational derivative of the action counterterm in Eq. (1.22). This relationship must, of course, hold in a consistent scheme of renormalization. It ensures that the covariant divergence of the counterterm vanishes. In four dimensions we find that

$$\langle T^{\mu\nu}\rangle^{(n=4)} = 2\left(\frac{1}{4-n}+L_4\right)\frac{1}{(4\pi)^2}\left\{\frac{\partial}{\partial is}T^{\mu\nu}(x;is;4)\bigg|_{s=0} - m^2T^{\mu\nu}(x;0;4)\right\} + \langle T^{\mu\nu}\rangle_{\text{ren}}^{(n=4)}, \tag{1.35}$$

where

$$\frac{\partial}{\partial is}T^{\mu\nu}(x;is;4)\bigg|_{s=0} = \tfrac{1}{180}(R^{,\mu;\nu} - 3R^{\mu\nu}{}_{;\sigma}{}^{;\sigma} + \tfrac{1}{2}g^{\mu\nu}R_{,\sigma}{}^{;\sigma} + 4R^{\mu\sigma}R^{\nu}_{\sigma} - 2R^{\mu\lambda\nu\kappa}R_{\lambda\kappa} - \tfrac{1}{2}g^{\mu\nu}R^{\lambda\sigma}R_{\lambda\sigma}$$
$$+\tfrac{1}{2}g^{\mu\nu}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\mu\alpha\beta\gamma}R^{\nu}_{\alpha\beta\gamma}) \tag{1.36}$$

and

$$m^2T^{\mu\nu}(x;0;4) = \tfrac{1}{4}m^4g^{\mu\nu}, \tag{1.37}$$

and where

$$\langle T^{\mu\nu}\rangle_{\text{ren}}^{(n=4)} = \tfrac{1}{4}g^{\mu\nu}\mathfrak{A}_4 + \frac{1}{(4\pi)^2}\frac{m^2}{18}G^{\mu\nu} - \frac{1}{(4\pi)^2}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2is}T^{\mu\nu}(x;is;4)]. \tag{1.38}$$

The expression in Eq. (1.36) is manifestly traceless. It is simplified by the use of the identity

$$C^{\mu\alpha\beta\gamma}C^{\nu}_{\alpha\beta\gamma} = \tfrac{1}{4}g^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \tag{1.39}$$

and we find that

$$180\frac{\partial}{\partial is}T^{\mu\nu}(x;is;4)\bigg|_{s=0} = R^{,\mu;\nu} - 3R^{\mu\nu}{}_{;\sigma}{}^{;\sigma} + \tfrac{1}{2}g^{\mu\nu}R_{,\sigma}{}^{;\sigma} + \tfrac{3}{2}g^{\mu\nu}R^{\lambda\kappa}R_{\lambda\kappa} - 6R^{\mu\lambda\nu\kappa}R_{\lambda\kappa} + 2R^{\mu\nu}R - \tfrac{1}{2}g^{\mu\nu}R^2$$
$$= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int (d^4x)\sqrt{-g}\,(3R_{\lambda\kappa}R^{\lambda\kappa} - R^2). \tag{1.40}$$

This establishes that the stress-tensor counterterm is indeed the correct metric variational derivative of the action counterterm as exhibited in Eq. (1.31b).

The weight $T^{\mu\nu}(x;is;n)$ is related to the biscalar $F(x,x';is;n)$ weight that appears in the proper-time representation of the Green's function. We prove in Sec. III that

$$T^{\mu\nu}(x;is;n)_{;\nu} = \frac{1}{n}(is)^{n/2}e^{m^2is}\frac{\partial}{\partial is}[(is)^{-n/2}e^{-m^2is}F(x,x;is;n)^{,\mu}], \tag{3.29}$$

with $(is)^{1-n/2}F(x,x;is;n)^{,\mu}$ vanishing at $s=0$. This establishes that the renormalized stress tensor is conserved at $n=2$ and at $n=4$. Since at $n=2$ or 4

$$\mathfrak{A}_n = \frac{2}{n}\frac{1}{(4\pi)^{n/2}}\left(\frac{\partial}{\partial is}\right)^{n/2}[e^{-m^2is}F(x,x;is;n)]\bigg|_{s=0}, \tag{1.41}$$

we can integrate by parts to prove this conservation law,

$$\langle T^{\mu\nu}\rangle_{\text{ren}}^{(n)}{}_{;\nu} = \frac{1}{n}\mathfrak{A}_n{}^{,\mu} - \frac{1}{n}\frac{1}{(4\pi)^{n/2}}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^{n/2}\left\{(is)^{n/2}\frac{\partial}{\partial is}[(is)^{-n/2}e^{-m^2is}F(x,x;is;n)^{,\mu}]\right\}$$
$$=0. \tag{1.42}$$

Note that the term $1/n\,g^{\mu\nu}\mathfrak{A}_n$ must appear in the renormalized stress tensor if this conservation law is to be obeyed.

The renormalized vacuum expectation value of the square of the scalar field is easily obtained by the dimensionally continued, proper-time technique. We find that in two dimensions

$$\langle\phi^2\rangle^{(n=2)} = 2\frac{1}{4\pi}\left(\frac{1}{2-n}+L_2\right) + \langle\phi^2\rangle_{\text{ren}}^{(n=2)}, \tag{1.43}$$

with

$$\langle \phi^2 \rangle_{\text{ren}}^{(n=2)} = -\frac{1}{4\pi}\int_0^\infty i\,ds(\ln\kappa^2 is)\,\frac{\partial}{\partial is}\,[e^{-m^2 is}F(x,x;is;2)]\,, \tag{1.44}$$

while in four dimensions

$$\langle \phi^2 \rangle^{(n=4)} = 2\,\frac{1}{(4\pi)^2}\left(\frac{1}{4-n}+L_4\right)+\langle \phi^2\rangle_{\text{ren}}^{(n=4)}\,, \tag{1.45}$$

with

$$\langle \phi^2 \rangle_{\text{ren}}^{(n=4)} = \frac{1}{(4\pi)^2}\left\{\tfrac{1}{18}R - \int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is;4)]\right\}. \tag{1.46}$$

Now we show in Sec. III that the trace of the weight appearing in the proper-time representations of the renormalized stress tensor, Eqs. (1.34) and (1.38), is given by

$$g_{\mu\nu}T^{\mu\nu}(x;is;n) = -m^2F(x,x;is;n). \tag{3.33}$$

Accordingly, for both $n=2$ and $n=4$, we find that

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle_{\text{ren}}^{(n)} = \mathfrak{A}_n - m^2\langle \phi^2\rangle_{\text{ren}}^{(n)}. \tag{1.47}$$

This is the stress-tensor trace anomaly: The naive identity (2.9) is violated by the occurrence of the anomalous term $\mathfrak{A}_n$.

It is easy to verify, using the specific forms (1.36) and (1.37) of the stress-tansor counterterm, that the anomaly in four dimensions, $\mathfrak{A}_4$, is precisely that given by the mechanism described in Eq. (1.8) above. The two-dimensional anomaly, $\mathfrak{A}_2$, cannot be derived by this mechanism because a potential counterterm that produces part of this anomaly, the Einstein tensor $G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$, vanishes identically in two dimensions.

In the development that we have described, we have chosen the factor of $R\phi^2$ in the scalar-field Lagrange function

$$\xi = \frac{n-2}{4(n-1)} \tag{2.5}$$

to be a continuous parameter. This is the conceptually clearest choice, for it ensures that the formal identity

$$g_{\mu\nu}T^{\mu\nu} = -m^2\phi^2 \tag{2.9}$$

is obeyed for arbitrary n. The functional dependence of ξ on n gives a contribution to the dimensionally continued, proper-time representation because the counterterm pole produces terms involving $\partial\xi/\partial n$. One could, however, fix $\xi=0$ or $\xi=\frac{1}{6}$ appropriate to the dimensions $n=2$ or $n=4$, and then perform the dimensional continuation in the proper-time representation to the dimension $n=2$ or $n=4$. This latter choice is, in fact, the technically simpler one. The only change resulting from holding ξ fixed is a change in a finite counterterm involving the scalar curvature R or its metric derivative, the Einstein tensor $G^{\mu\nu}$. In two dimensions, the only effect of holding $\xi=0$ fixed is to delete the term $(1/16\pi)R$ in the renormalized effective Lagrangian $\mathfrak{L}_{1\,\text{ren}}^{(n=2)}$ displayed in Eq. (1.19). In four dimensions, the effect of holding $\xi=\frac{1}{6}$ fixed is to delete the term

$$\frac{-1}{(4\pi)^2}\left(\frac{m^2}{36}R - \tfrac{1}{216}R_{,\sigma}{}^{;\sigma}\right)$$

in $\mathfrak{L}_{1\,\text{ren}}^{(n=4)}$ [Eq. (1.25)], the term

$$\frac{1}{(4\pi)^2}\,\frac{m^2}{18}\,G^{\mu\nu}$$

in $\langle T^{\mu\nu}\rangle_{\text{ren}}^{(n=4)}$ [Eq. (1.38)], and the term

$$\frac{1}{(4\pi)^2}\,\tfrac{1}{18}R$$

in $\langle \phi^2\rangle_{\text{ren}}^{(n=4)}$ [Eq. (1.46)]. These finite counterterm changes do not alter the stress-tensor trace anomaly, Eq. (1.47). Some of the technical details involved in the proper-time description are collected in the Appendix.

II. REVIEW

We turn now to review the techniques which we shall employ. First, we derive a Lagrange function for a scalar field in a space-time of arbitrary dimension which, in the absence of a mass, formally produces a stress tensor with a vanishing trace. This is achieved if the corresponding action is invariant under a conformal transformation of the metric tensor. The invariance will be ensured if $\sqrt{-g}\,\mathfrak{L}(\phi;g)$ is left *algebraically* invariant by the substitutions

$$g_{\mu\nu}(x)\to\lambda(x)^2g_{\mu\nu}(x),\quad \phi(x)\to\lambda(x)^p\phi(x), \tag{2.1}$$

for, by virtue of a field equation obeyed by ϕ, the action is not altered by a field variation $\phi(x)\delta\lambda(x)$. (This theorem is completely analogous to that which states that a Lagrange function which is algebraically invariant under a general coordinate transformation yields a conserved stress-energy tensor.) Such an algebraically invariant Lagrange function can be constructed from the quantity $\phi^{-n/p}\sqrt{-g}R(g_{\alpha\beta}\phi^{-2/p})$, where R is the curvature scalar. This quantity is obviously left invariant by the substitutions (2.1). If ϕ were constant, it would contain ϕ to the power $(2-n)/p$ since

$$R(\lambda^2 g_{\alpha\beta})=\lambda^{-2}R(g_{\alpha\beta}) \qquad (2.2)$$

if λ is constant. The scalar-field Lagrange function is quadratic in ϕ. Hence, we must have $(2-n)/p=2$. With this choice one finds, after some calculation, that

$$\phi^{-n/p}R(g_{\alpha\beta}\phi^{-2/p})=\phi^2 R(g_{\alpha\beta}) + 4\frac{n-1}{n-2}[\phi_{,\mu}\phi^{,\mu}-(\phi\phi_{,\mu})^{;\mu}]. \qquad (2.3)$$

On discarding the total divergence, normalizing the kinetic energy term in the conventional manner, and adding a mass term, we secure

$$\mathcal{L}=-\tfrac{1}{2}\phi_{,\mu}\phi^{,\mu}-\tfrac{1}{2}\xi R\phi^2-\tfrac{1}{2}m^2\phi^2, \qquad (2.4)$$

where

$$\xi=\frac{n-2}{4(n-1)}. \qquad (2.5)$$

The scalar-field Lagrange function (2.4) yields the field equation

$$-\phi_{,\mu}{}^{;\mu}+(\xi R+m^2)\phi=0, \qquad (2.6)$$

and the stress-energy tensor

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int(d^n x)\sqrt{-g}\,\mathcal{L} \\ &= \phi^{,\mu}\phi^{,\nu}-\tfrac{1}{2}g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}-\tfrac{1}{2}g^{\mu\nu}m^2\phi^2 \\ &\quad +\xi[G^{\mu\nu}\phi^2+g^{\mu\nu}(\phi^2)_{,\sigma}{}^{;\sigma}-(\phi^2)^{,\mu;\nu}]. \end{aligned} \qquad (2.7)$$

Since the conformal transformations (2.1) change the Lagrange function into

$$\sqrt{-g}\,\mathcal{L}\to\sqrt{-g}\,\mathcal{L}-(\lambda^2-1)\sqrt{-g}\tfrac{1}{2}m^2\phi^2, \qquad (2.8)$$

we conclude that

$$g_{\mu\nu}T^{\mu\nu}=-m^2\phi^2, \qquad (2.9)$$

which can also be directly verified from Eq. (2.7) by using the field equation (2.6).

We need the Green's function[8]

$$G(x,x')=\langle iT(\phi(x)\phi(x'))\rangle \qquad (2.10)$$

to compute the one-loop action and the stress-tensor expectation value. It obeys

$$\left[-\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu+\xi R+m^2\right]G(x,x') = \frac{1}{\sqrt{-g}}\delta(x-x'). \qquad (2.11)$$

We shall construct a dimensionally continued, proper-time representation for the Green's function, following the work of Schwinger[6] as extended by DeWitt.[3,7] It is convenient to define

$$\tilde{G}(x,x')=[-g(x)]^{1/4}G(x,x')[-g(x')]^{1/4}, \qquad (2.12)$$

and to consider this function as the coordinate matrix element of an operator,

$$\tilde{G}(x,x')=\langle x|\tilde{G}|x'\rangle. \qquad (2.13)$$

Then, on introducing an operator with the coordinate representation

$$H=-(-g)^{-1/4}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu(-g)^{-1/4}+\xi R+m^2, \qquad (2.14)$$

the Green's function differential equation (2.11) appears in the operator form

$$H\tilde{G}=1. \qquad (2.15)$$

On writing

$$\tilde{G}=H^{-1}=\int_0^\infty i\,ds\,e^{-isH}, \qquad (2.16)$$

we obtain the proper-time representation[6]

$$\tilde{G}(x,x')=\int_0^\infty i\,ds\langle x,s|x',0\rangle, \qquad (2.17)$$

where

$$\langle x,s|x',0\rangle=\langle x|e^{-isH}|x'\rangle. \qquad (2.18)$$

In writing this representation we are tacitly assuming that it is taken as the limit of a complex continuation of the mass, $m^2\to m^2(1-i\epsilon)$, $\epsilon\to0^+$. If the metric tensor can be expanded about the Minkowski metric with space-time asymptotically flat, this will yield the time-ordered, vacuum-state Green's function characterized by positive-frequency boundary conditions at large times. Although the proper-time representation may not give the correct boundary conditions for all geometrical configurations, it should give the correct counterterms and trace anomalies since these are quantities that involve only short-distance limits.

The transformation function (2.18) is defined by the "Schrödinger equation"

$$-\frac{\partial}{\partial is}\langle x,s|x',0\rangle=H\langle x,s|x',0\rangle, \qquad (2.19)$$

with the boundary condition

$$s\to0:\ \langle x,s|x',0\rangle\to\langle x|x'\rangle=\delta(x-x'). \qquad (2.20)$$

Since we need only the short-distance limit of this transformation function, we shall write it in a "WKB" form,[3,7]

$$\langle x, s| x', 0\rangle = \frac{i}{(4\pi is)^{n/2}}[-g(x)]^{1/4}\Delta^{1/2}(x,x')[-g(x')]^{1/4} \times F(x,x';is;n)\exp\left[-\frac{\sigma(x,x')}{2is} - m^2 is\right]. \tag{2.21}$$

Here the biscalar $\sigma(x,x')$, the "world function,"[12] is equal to one-half of the square of the distance along the geodesic between x' and x. It is a symmetrical function,

$$\sigma(x,x') = \sigma(x',x), \tag{2.22}$$

which satisfies an equation of the Hamilton-Jacobi form

$$\sigma_{,\mu}\sigma^{,\mu} = 2\sigma, \tag{2.23}$$

and it, along with its first derivative, vanishes with coincident coordinates,

$$x = x': \quad \sigma = 0 = \sigma_{,\mu} = \sigma_{,\mu'}, \tag{2.24}$$

while

$$x = x': \quad \sigma_{,\mu;\nu} = g_{\mu\nu} = -\sigma_{,\mu;\nu'}. \tag{2.25}$$

Here we use a suffix to denote a derivative with respect to the variable x, and a primed suffix to denote a derivative with respect to x'. The biscalar $\Delta^{1/2}(x,x')$ is a symmetrical function,

$$\Delta^{1/2}(x,x') = \Delta^{1/2}(x',x), \tag{2.26}$$

defined by

$$n\Delta^{1/2} = \Delta^{1/2}\sigma_{,\mu}{}^{;\mu} + 2\Delta^{1/2}{}_{,\mu}\sigma^{,\mu} \tag{2.27}$$

and the coincident limit

$$x = x': \quad \Delta^{1/2} = 1. \tag{2.28}$$

Substituting the WKB structure (2.21) into the Schrödinger equation (2.19), and using the differential equations (2.23) and (2.27) obeyed by σ and $\Delta^{1/2}$ gives the weight function equation

$$-\frac{\partial F}{\partial is} = \xi RF + \frac{1}{is}\sigma^{,\mu}F_{,\mu} - \Delta^{-1/2}(\Delta^{1/2}F)_{,\mu}{}^{;\mu}. \tag{2.29}$$

The weight function F is regular at $s=0$. Hence, the differential equation (2.29) implies that $\sigma^{,\mu}F_{,\mu}$ must vanish at $s=0$, which requires that $F(x,x';0;n)$ be a constant. The overall normalization is determined by the boundary condition (2.20) or, equivalently, by the short-distance limit in a locally flat frame

$$\tilde{G}(x,x') \to \frac{i}{(4\pi)^{n/2}}\int_0^\infty \frac{ids}{(is)^{n/2}} e^{-(1/4)is(x-x')^2}. \tag{2.30}$$

Since $\sigma(x,x') \to \frac{1}{2}(x-x')^2$ in the locally flat frame, we infer that

$$F(x,x';0;n) = 1. \tag{2.31}$$

Note that $F(x,x';is;n)$ is a symmetrical function

$$F(x',x;is;n) = F(x,x';is;n). \tag{2.32}$$

In the work that follows we shall make use of the power-series development

$$F = 1 + isf_1 + (is)^2 f_2 + \cdots. \tag{2.33}$$

Inserting this series into Eq. (2.29) yields

$$-f_1 = \xi R + \sigma^{,\mu}f_{1,\mu} - \Delta^{-1/2}\Delta^{1/2}{}_{,\mu}{}^{;\mu}, \tag{2.34a}$$

$$-2f_2 = \xi R f_1 + \sigma^{,\mu}f_{2,\mu} - \Delta^{-1/2}(\Delta^{1/2}f_1)_{,\mu}{}^{;\mu}, \tag{2.34b}$$

and so forth. Various short-distance limits of the biscalars σ, Δ, f_1, and f_2 are needed for our work. The derivation of these limits is reviewed in the Appendix.

III. ACTION, STRESS TENSOR

The one-loop action functional $W_1[g_{\alpha\beta}]$ has the formal (divergent) definition

$$W_1 = \tfrac{1}{2}i \ln \mathrm{Det}G^{-1}, \tag{3.1}$$

where G^{-1} is the inverse of the operator corresponding to the Green's function (2.10). Since

$$\int (d^n x)\sqrt{-g}\; \mathcal{L} = -\tfrac{1}{2}(\phi, G^{-1}\phi), \tag{3.2}$$

the metric variation of the one-loop action functional produces the vacuum expectation value of the stress-energy tensor,

$$\begin{aligned}\delta W_1 &= \tfrac{1}{2}i\,\mathrm{Tr}G\delta G^{-1}\\ &= \langle -\tfrac{1}{2}(\phi, \delta G^{-1}\phi)\rangle\\ &= \left\langle \int (d^n x)\delta(\sqrt{-g}\;\mathcal{L})\right\rangle\\ &= \int (d^n x)\sqrt{-g}\langle T^{\mu\nu}\rangle \tfrac{1}{2}\delta g_{\mu\nu}.\end{aligned} \tag{3.3}$$

Recalling the definitions (2.12) and (2.15) of $\tilde{G}$ and its inverse H, we have

$$\delta W_1 = \tfrac{1}{2}i\,\mathrm{Tr}\tilde{G}\delta H. \tag{3.4}$$

[Here a partial renormalization has been performed by deleting a term involving $\mathrm{Tr}GHg^{-1}\delta g \sim \delta^{(n)}(0)\int \delta \ln g$.] We now follow Schwinger[6] and DeWitt[3,7] and introduce the proper-time representation to write

$$\delta W_1 = -\delta\tfrac{1}{2}i\,\mathrm{Tr}\int_0^\infty \frac{ids}{is} e^{-isH}. \tag{3.5}$$

Thus

$$W_1 = \int (d^n x)\sqrt{-g}\,\mathcal{L}_1, \tag{3.6}$$

where the one-loop effective Lagrangian is given by

$$\mathcal{L}_1 = -\frac{i}{2\sqrt{-g}}\int_0^\infty \frac{ids}{is}\langle x|\,e^{-isH}|\,x\rangle$$
$$= \frac{1}{2}\frac{1}{(4\pi)^{n/2}}\int_0^\infty \frac{ids}{(is)^{1+n/2}}\,e^{-m^2 is}F(x,x;is;n). \tag{3.7}$$

This effective Lagrangian is divergent, and it must be renormalized. We shall perform this renormalization by the method of dimensional continuation discussed in the Introduction. The continuation is effected through sufficiently small values of n so that integration by parts with no end-point contributions can be performed until potential logarithms of the proper time appear. Accordingly,

$$\mathcal{L}_1 = \frac{1}{(4\pi)^{n/2}}\frac{1}{n}\int_0^\infty \frac{ids}{(is)^{n/2}}\frac{\partial}{\partial is}[e^{-m^2 is}F(x,x;is;n)]. \tag{3.8}$$

Let us consider first the limit in two-dimensional space-time. As discussed in the Introduction, we introduce an auxiliary scale mass κ to keep the integrand at a fixed scale dimension. Then we have an integral identical in form to that evaluated in Eqs. (1.2) and (1.3):

$$\mathcal{L}_1^{(n=2)} = \frac{1}{1-n/2}\frac{1}{4\pi}\frac{1}{2}\frac{\partial}{\partial is}[e^{-m^2 is}F(x,x;is;2)]\Big|_{s=0} - 2\frac{\partial}{\partial n}\left\{\frac{1}{(4\pi)^{n/2}}\frac{1}{n}\frac{\partial}{\partial is}[e^{-m^2 is}F(x,x;is;n)]\Big|_{s=0}\right\}\Big|_{n=2}$$
$$-\frac{1}{8\pi}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is;2)]. \tag{3.9}$$

Here

$$F(x,x;0;n) = 1 \tag{3.10}$$

and

$$\frac{\partial}{\partial is}F(x,x;is;n)\Big|_{s=0} = f_1(x,x;n), \tag{3.11}$$

and we define

$$L_2 = \frac{\partial}{\partial n}\ln(4\pi)^{n/2}\Big|_{n=2}. \tag{3.12}$$

Thus

$$\mathcal{L}_1^{(n=2)} = \left(\frac{1}{2-n}+L_2\right)\frac{1}{4\pi}[f_1(x,x;2)-m^2] + \frac{1}{2}\frac{1}{4\pi}[f_1(x,x;2)-m^2]$$
$$-\frac{1}{8\pi}\left\{2\frac{\partial}{\partial n}f_1(x,x;n)\Big|_{n=2} + \int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)[e^{-m^2 is}F(x,x;is;2)]\right\}. \tag{3.13}$$

We have separately displayed the term $\frac{1}{2}(1/4\pi)(f_1 - m^2)$ which arises from the dimensional derivative of $1/n$ in Eq. (3.9). We do not incorporate this into the infinite counterterm because we want the counterterm to correspond precisely to that appearing in the stress-energy tensor. Now, using the result of the Appendix,

$$f_1(x,x;n) = (\tfrac{1}{6}-\xi)R, \tag{A20}$$

and remembering that

$$\xi = \frac{n-2}{4(n-1)}, \tag{2.5}$$

we obtain the results [Eqs. (1.17)–(1.19)] quoted in the Introduction. (Note that if ξ is held fixed at $\xi = 0$, then the quantity $\partial f_1/\partial n$ is deleted from $\mathcal{L}_1^{(n=2)}$.) To perform the dimensional continuation for four-dimensional space-time, we first integrate by parts in Eq. (3.8),

$$\mathcal{L}_1 = \frac{1}{(4\pi)^{n/2}}\frac{1}{n}\frac{1}{n/2-1}\int_0^\infty \frac{ids}{(is)^{n/2-1}}\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is;n)]. \tag{3.14}$$

Then, using the process which must now be familiar, we get

$$\mathcal{L}_1^{(n=4)}=\left(\frac{1}{4-n}+L_4\right)\frac{1}{2}\,\frac{1}{(4\pi)^2}\{m^4-2m^2f_1(x,x;4)+2f_2(x,x;4)\}$$
$$+\frac{1}{4}\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\{m^4-2m^2f_1(x,x;4)+2f_2(x,x;4)\}$$
$$-\frac{1}{2(4\pi)^2}\left\{\frac{\partial}{\partial n}[-2m^2f_1(x,x;n)+2f_2(x,x;n)]\bigg|_{n=4}+\tfrac{1}{2}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3[e^{-m^2 is}F(x,x;is;4)]\right\},\tag{3.15}$$

where

$$L_4=\tfrac{1}{2}+\frac{\partial}{\partial n}\ln(4\pi)^{n/2}\bigg|_{n=4}.\tag{3.16}$$

Again we have displayed separately a piece that could have been incorporated into the counterterm so as to make the counterterm correspond precisely to that appearing in the stress tensor. Using Eqs. (A20), (2.5), and

$$f_2(x,x;n)=\tfrac{1}{2}[(\tfrac{1}{6}-\xi)R]^2+\tfrac{1}{6}(\tfrac{1}{5}-\xi)R_{,\sigma}{}^{;\sigma}+\tfrac{1}{180}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-R_{\alpha\beta}R^{\alpha\beta}),\tag{A24}$$

gives the results [Eqs. (1.23)–(1.25)] quoted in the Introduction. [Note that if ξ is held fixed at $\xi=\frac{1}{6}$, then the quantity $(\partial/\partial n)(-2m^2f_1+2f_2)$ is deleted from $\mathcal{L}_1^{(n=4)}$.]

We turn now to derive the dimensionally continued, proper-time representation for the vacuum expectation value of the stress-energy tensor (2.7),

$$\langle T^{\mu\nu}\rangle=\langle\phi^{,\mu}\phi^{,\nu}-\tfrac{1}{2}g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}-\tfrac{1}{2}g^{\mu\nu}m^2\phi^2+\xi[G^{\mu\nu}\phi^2+g^{\mu\nu}(\phi^2)_{,\sigma}{}^{;\sigma}-(\phi^2)^{,\mu;\nu}]\rangle.\tag{3.17}$$

We use Eqs. (2.10), (2.12), (2.17), and (2.21) to express

$$\langle T(\phi(x)\phi(x'))\rangle=\frac{1}{(4\pi)^{n/2}}\Delta^{1/2}(x,x')\int_0^\infty\frac{ids}{(is)^{n/2}}F(x,x';is;n)\exp\left[-\frac{\sigma(x,x')}{2is}-m^2is\right].\tag{3.18}$$

The coincident limits

$$x=x':\quad \sigma=0=\sigma_{,\mu}=\sigma_{,\mu'},\tag{2.24}$$
$$\sigma_{,\mu;\nu}=-\sigma_{,\mu;\nu'}=g_{\mu\nu},\tag{2.25}$$
$$\Delta^{1/2}=1,\tag{2.28}$$
$$\Delta^{1/2}{}_{,\mu}=\Delta^{1/2}{}_{,\mu'}=0,\tag{A11}$$

and

$$\Delta^{1/2}{}_{,\mu;\nu}=-\Delta^{1/2}{}_{,\mu,\nu'}=\tfrac{1}{6}R_{\mu\nu}\tag{A13}$$

are needed for this evaluation of the stress tensor Eq. (3.17). Moreover, we integrate by parts in that contribution which arises when the derivatives act upon the world function σ,

$$\int_0^\infty\frac{ids}{(is)^{n/2}}e^{-m^2is}F(g^{\mu\lambda}g^{\nu\kappa}-\tfrac{1}{2}g^{\mu\nu}g^{\lambda\kappa})\partial_\lambda\partial'_\kappa e^{-\sigma/2is}\bigg|_{x=x'}=\left(\frac{2-n}{2n}\right)g^{\mu\nu}\int_0^\infty\frac{ids}{(is)^{n/2}}\frac{\partial}{\partial is}(e^{-m^2is}F).\tag{3.19}$$

Thus we obtain the proper-time representation

$$\langle T^{\mu\nu}(x)\rangle=\frac{1}{(4\pi)^{n/2}}\int_0^\infty\frac{ids}{(is)^{n/2}}e^{-m^2is}T^{\mu\nu}(x;is;n),\tag{3.20}$$

with

$$T^{\mu\nu}(x;is;n)=-m^2\frac{1}{n}g^{\mu\nu}F(x,x;is;n)+\frac{2-n}{2n}g^{\mu\nu}\frac{\partial}{\partial is}F(x,x;is;n)$$
$$+(g^{\mu\lambda}g^{\nu\kappa}-\tfrac{1}{2}g^{\mu\nu}g^{\lambda\kappa})F_{,\lambda,\kappa'}(x,x';is;n)\big|_{x=x'}$$
$$-(\tfrac{1}{6}-\xi)G^{\mu\nu}F(x,x;is;n)+\xi(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\lambda}g^{\nu\kappa})[F(x,x;is;n)]_{,\lambda;\kappa}.\tag{3.21}$$

The proper-time representation (3.20) yields a formally conserved stress tensor. To establish this, we first note that $F(x, x; is; n)$ is a symmetrical function of x and x' [Eq. (2.32)]. Hence

$$F_{,\lambda,\kappa'}(x, x'; is; n) = F_{,\kappa,\lambda'}(x', x; is; n), \tag{3.22}$$

$$\{F_{,\lambda}{}^{,\lambda'}(x, x'; is; n)|_{x=x'}\}^{,\mu} = 2F_{,\lambda}{}^{,\lambda';\mu'}(x, x'; is; n)|_{x=x'}, \tag{3.23}$$

and

$$\{(g^{\mu\lambda}g^{\nu\kappa} - \tfrac{1}{2}g^{\mu\nu}g^{\lambda\kappa})F_{,\lambda,\kappa'}(x, x'; is; n)|_{x=x'}\}_{;\nu} = F_{,\nu}{}^{;\nu,\mu'}(x, x'; is; n)|_{x=x'}. \tag{3.24}$$

The divergence of the last term in Eq. (3.21) is of the form

$$f_{,\nu}{}^{;\nu;\mu} - f^{,\mu}{}_{;\nu}{}^{;\nu} = -R^{\mu\nu}f_{,\nu}. \tag{3.25}$$

These results, together with the conservation of the Einstein tensor $G^{\mu\nu}$, imply that

$$\begin{aligned} T^{\mu\nu}(x; is; n)_{;\nu} &= \left(-\frac{m^2}{n} + \frac{2-n}{2n}\frac{\partial}{\partial is}\right)F(x, x; is; n)^{,\mu} \\ &\quad + F_{,\nu}{}^{;\nu,\mu'}(x, x'; is; n)|_{x=x'} - [(\tfrac{1}{6} - \xi)G^{\mu\nu} + \xi R^{\mu\nu}]F(x, x; is; n)_{,\nu}. \end{aligned} \tag{3.26}$$

To put this expression in a useful form, we differentiate

$$-\frac{\partial F}{\partial is} = \xi RF + \frac{1}{is}\sigma^{,\nu}F_{,\nu} - \frac{1}{\Delta^{1/2}}(\Delta^{1/2}F)_{,\nu}{}^{;\nu} \tag{2.29}$$

with respect to x' and then set $x' = x$. Using the coincident limits Eqs. (2.24), (2.25), (2.28), (A11), and (A13) listed in the previous paragraph and

$$x = x': \quad \Delta^{1/2}{}_{,\nu}{}^{;\nu,\mu'} = 0, \tag{A16}$$

we get

$$\begin{aligned} -\frac{\partial}{\partial is}F^{,\mu'}(x, x'; is; n)\Big|_{x=x'} &= (\xi - \tfrac{1}{6})RF^{,\mu'}(x, x'; is; n)|_{x=x'} \\ &\quad - \left(\frac{1}{is}g^{\mu\nu} - \tfrac{1}{3}R^{\mu\nu}\right)F_{,\nu}(x, x'; is; n)\Big|_{x=x'} - F_{,\nu}{}^{;\nu;\mu'}(x, x'; is; n)|_{x=x'}, \end{aligned} \tag{3.27}$$

and, taking account of the symmetrical nature of $F(x, x'; is; n)$,

$$F_{,\nu}{}^{;\nu,\mu'}(x, x'; is; n)|_{x=x'} - [\tfrac{1}{6}R^{\mu\nu} + (\xi - \tfrac{1}{6})\tfrac{1}{2}g^{\mu\nu}R]F(x, x; is; n)_{,\nu} = \tfrac{1}{2}\left(\frac{\partial}{\partial is} - \frac{1}{is}\right)F(x, x; is; n)^{,\mu}. \tag{3.28}$$

Hence

$$\begin{aligned} T^{\mu\nu}(x; is; n)_{;\nu} &= \left(-\frac{m^2}{n} + \frac{1}{n}\frac{\partial}{\partial is} - \frac{1}{2is}\right)F(x, x; is, n)^{,\mu} \\ &= (is)^{n/2}e^{m^2 is}\frac{1}{n}\frac{\partial}{\partial is}[(is)^{-n/2}e^{-m^2 is}F(x, x; is; n)^{,\mu}]. \end{aligned} \tag{3.29}$$

This formula establishes that the stress tensor is formally conserved, for we now have

$$\langle T^{\mu\nu}\rangle_{;\nu} = \frac{1}{(4\pi)^{n/2}}\int_0^\infty i\,ds\,\frac{1}{n}\frac{\partial}{\partial is}[(is)^{-n/2}e^{-m^2 is}F(x, x; is; n)^{,\mu}] = 0. \tag{3.30}$$

The proper-time representation (3.20) also obeys the formal trace identity (2.9). To establish this, we use the coincident limit of Eq. (2.29),

$$x = x': \quad \frac{\partial F}{\partial is} = (\tfrac{1}{6} - \xi)RF + F_{,\mu}{}^{;\mu}, \tag{3.31}$$

to write the trace of Eq. (3.21) in the form

$$\begin{aligned} g_{\mu\nu}T^{\mu\nu}(x; is; n) &= -m^2F(x, x; is; n) \\ &\quad + \frac{2-n}{2}[F_{,\mu}{}^{;\mu}(x, x'; is; n) + F_{,\mu}{}^{,\mu'}(x, x'; is; n)]\Big|_{x=x'} + \xi(n-1)F(x, x; is; n)_{,\nu}{}^{;\nu}. \end{aligned} \tag{3.32}$$

Thus, on taking account of the symmetry of $F(x, x'; is; n)$ in x and x',

$$g_{\mu\nu}T^{\mu\nu}(x;is;n) = -m^2F(x,x;is;n) + (n-1)\left[\xi - \frac{n-2}{4(n-1)}\right]F(x,x;is;n)_{,\nu}{}^{;\nu}. \tag{3.33}$$

The second expression on the right-hand side of this equation vanishes with $\xi = [n-2/4(n-1)]$ [Eq. (2.5)]. Recalling now the proper-time representation (3.18) of the vacuum expectation value of two field operators, we have

$$\langle\phi^2\rangle = \frac{1}{(4\pi)^{n/2}}\int_0^\infty \frac{ids}{(is)^{n/2}}e^{-m^2is}F(x,x;is;n), \tag{3.34}$$

and we see that Eq. (3.33) does imply the formal trace identity

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = -m^2\langle\phi^2\rangle. \tag{3.35}$$

We turn at last to the renormalization of the proper-time stress-tensor representation (3.20). In a two-dimensional space-time

$$\langle T^{\mu\nu}\rangle^{(n=2)} = 2\left(\frac{1}{2-n}+L_2\right)\frac{1}{4\pi}T^{\mu\nu}(x;0;2) - 2\frac{1}{4\pi}\frac{\partial}{\partial n}T^{\mu\nu}(x;0;n)\Big|_{n=2}$$
$$-\frac{1}{4\pi}\int_0^\infty i\,ds(\ln\kappa^2 is)\frac{\partial}{\partial is}[e^{-m^2is}T^{\mu\nu}(x,is;2)], \tag{3.36}$$

where, according to Eq. (3.21),

$$T^{\mu\nu}(x;0;n) = g^{\mu\nu}\left[-\frac{m^2}{n}+\frac{2-n}{2n}f_1(x,x;n)\right] - (\tfrac{1}{6}-\xi)G^{\mu\nu}. \tag{3.37}$$

Remembering that the Einstein tensor $G^{\mu\nu}$ vanishes in two dimensions, we obtain the results [Eqs. (1.33) and (1.34)] quoted in the Introduction.

In a four-dimensional space-time, we first integrate by parts,

$$\langle T^{\mu\nu}\rangle = \frac{2}{n-2}\frac{1}{(4\pi)^{n/2}}\int_0^\infty \frac{ids}{(is)^{n/2-1}}\frac{\partial}{\partial is}[e^{-m^2is}T^{\mu\nu}(x;is;n)], \tag{3.38}$$

and then take the limit $n\to 4$,

$$\langle T^{\mu\nu}\rangle^{(n=4)} = 2\left(\frac{1}{4-n}+L_4\right)\frac{1}{(4\pi)^2}\left[\frac{\partial}{\partial is}T^{\mu\nu}(x;is;4)\Big|_{s=0} - m^2T^{\mu\nu}(x;0;4)\right]$$
$$-2\frac{\partial}{\partial n}\frac{1}{(4\pi)^2}\left[\frac{\partial}{\partial is}T^{\mu\nu}(x;is;n)\Big|_{s=0} - m^2T^{\mu\nu}(x;0;n)\right]\Bigg|_{n=4}$$
$$-\frac{1}{(4\pi)^2}\int_0^\infty i\,ds(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2is}T^{\mu\nu}(x;is;4)]. \tag{3.39}$$

The weight $T^{\mu\nu}(x;is;n)$ depends upon n in two ways: There is the explicit n dependence displayed in Eq. (3.21) and the implicit n dependence that arises from $\xi(n)$. The explicit n dependence gives

$$\left(\frac{\partial}{\partial n}\right)_{\exp}T^{\mu\nu}(x;is;n) = \frac{1}{n^2}g^{\mu\nu}\left(m^2 - \frac{\partial}{\partial is}\right)F(x,x;is;n) \tag{3.40}$$

and

$$-2\left(\frac{\partial}{\partial n}\right)_{\exp}\frac{1}{(4\pi)^2}\left[\frac{\partial}{\partial is}T^{\mu\nu}(x;is;n)\Big|_{s=0} - m^2T^{\mu\nu}(x;0;n)\right]\Bigg|_{n=4} = \frac{1}{2}\frac{1}{4}g^{\mu\nu}\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2is}F(x,x;is;4)]\Big|_{s=0}$$
$$= \tfrac{1}{4}g^{\mu\nu}\mathcal{A}_4. \tag{3.41}$$

This is the anomaly contribution to the stress tensor displayed in Eq. (1.38) of the Introduction. We mentioned toward the end of the Introduction that we could fix $\xi = \frac{1}{6}$ appropriate to $n=4$. We see now that this is the simplest procedure, and that with this prescription Eq. (3.41) would give the only dimensional derivative contribution to the stress tensor. We have, however, for conceptual clarity, considered ξ to be a function of n so that the formal trace identity $g_{\mu\nu}T^{\mu\nu} = -m^2\phi$ is maintained for all n values. Let us observe that

$$f_{1,\lambda,\kappa'}(x,x';n)|_{x=x'} = \tfrac{1}{2}f_1(x,x;n)_{,\lambda;\kappa} - f_{1,\lambda;\kappa}(x,x';n)|_{x=x'} \tag{3.42}$$

and define

$$f_{1,\lambda;\kappa}(x,x';n)\big|_{x=x'}=\tfrac{1}{3}(\tfrac{1}{6}-\xi)R_{,\lambda;\kappa}+\bar{f}_{\lambda\kappa}. \tag{3.43}$$

Then, using Eq. (2.34b) in Eq. (3.21), and recalling that

$$f_1(x,x;n)=(\tfrac{1}{6}-\xi)R, \tag{A20}$$

we get

$$\frac{\partial}{\partial is}T^{\mu\nu}(x;is;n)\bigg|_{s=0}=\frac{1}{n}g^{\mu\nu}[-m^2+\tfrac{1}{2}(2-n)(\tfrac{1}{6}-\xi)R](\tfrac{1}{6}-\xi)R$$
$$+(\tfrac{1}{6}-\xi)^2(R^{;\mu;\nu}-g^{\mu\nu}R_{,\sigma}{}^{;\sigma}-G^{\mu\nu}R)+\tfrac{1}{3}(\tfrac{1}{6}-\xi)\left(\frac{1}{n}-\frac{1}{4}\right)g^{\mu\nu}R_{,\sigma}{}^{;\sigma}+\frac{1}{n}g^{\mu\nu}\bar{f}^{\sigma}_{\sigma}-\bar{f}^{\mu\nu}. \tag{3.44}$$

Using the result (A22) of the Appendix, the decomposition (3.43) gives

$$\bar{f}_{\mu\nu}=\tfrac{1}{180}(-R_{,\mu;\nu}+3R_{\mu\nu;\sigma}{}^{;\sigma}-4R_{\mu\sigma}R^{\sigma}_{\nu}+2R_{\mu\lambda\nu\kappa}R^{\lambda\kappa}+2R_{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}), \tag{3.45}$$

which is independent of the ξ parameter. With

$$\frac{\partial}{\partial n}\xi\bigg|_{n=4}=\tfrac{1}{36}, \tag{3.46}$$

we can now compute the implicit dependence

$$\left(\frac{\partial}{\partial n}\right)_{\text{imp}}\left[\frac{\partial}{\partial is}T^{\mu\nu}(x;is;n)\bigg|_{s=0}\right]\bigg|_{n=4}=\tfrac{1}{4}g^{\mu\nu}m^2\tfrac{1}{36}R. \tag{3.47}$$

The remaining implicit dimensional derivative term involves

$$T^{\mu\nu}(x;0;n)=\frac{1}{n}g^{\mu\nu}[-m^2+\tfrac{1}{2}(2-n)(\tfrac{1}{6}-\xi)R]-(\tfrac{1}{6}-\xi)G^{\mu\nu}, \tag{3.48}$$

with

$$\left(\frac{\partial}{\partial n}\right)_{\text{imp}}m^2T^{\mu\nu}(x;0;n)\bigg|_{n=4}=\tfrac{1}{4}g^{\mu\nu}m^2\tfrac{1}{36}R+m^2\tfrac{1}{36}G^{\mu\nu}. \tag{3.49}$$

Hence,

$$-2\left(\frac{\partial}{\partial n}\right)_{\text{imp}}\frac{1}{(4\pi)^2}\left[\frac{\partial}{\partial is}T^{\mu\nu}(x;is;n)\bigg|_{s=0}-m^2T^{\mu\nu}(x;0;n)\right]\bigg|_{n=4}=\frac{1}{(4\pi)^2}\tfrac{1}{18}m^2G^{\mu\nu}, \tag{3.50}$$

which is the other dimensional derivative contribution to the stress tensor displayed in Eq. (1.38) of the Introduction. It could be omitted from the stress tensor if the corresponding quantities are also omitted from the effective Lagrangian and from the ϕ^2 vacuum expectation value. The counterterms displayed in Eqs. (1.36) and (1.37) in the Introduction follow immediately from Eqs. (3.44), (3.45) and (3.48).

The dimensional continuation renormalization of the vacuum expectation value of the square of the scalar field displayed in Eqs. (1.43)–(1.46) in the Introduction follows rather directly from Eq. (3.34).

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APPENDIX

The coincident coordinate limits of various derivatives of the world function $\sigma(x,x')$ are needed for our work. Most of these appear in the books by Synge[12] and DeWitt.[7] We present here an outline of the derivation of these quantities for the convenience of the reader. The world function is defined as the symmetrical solution of

$$\sigma_{,\mu}\sigma^{,\mu}=2\sigma, \tag{2.23}$$

with the boundary condition

$$x=x':\quad \sigma=0=\sigma_{,\mu'}, \tag{2.24}$$

$$\sigma_{,\mu;\nu}=g_{\mu\nu}. \tag{2.25}$$

The third derivative of Eq. (2.23) gives the coincident limit

$$x=x':\quad \sigma_{,\alpha;\beta;\gamma}+\sigma_{,\gamma;\alpha;\beta}=0. \tag{A1}$$

Now $\sigma_{,\alpha;\beta;\gamma}$ is symmetrical in the indices α,β.

Hence, Eq. (A1) implies that

$$x=x':\quad \sigma_{,\gamma;\alpha;\beta}=\sigma_{,\gamma;\beta;\alpha}=\sigma_{,\beta;\gamma;\alpha}$$
$$=\sigma_{,\beta;\alpha;\gamma}=\sigma_{,\alpha;\beta;\gamma}=0. \qquad \text{(A2)}$$

Using these results, we can take four derivatives of Eq. (2.23) to get

$$x=x':\quad \sigma_{,\alpha;\beta;\gamma;\delta}+\sigma_{,\alpha;\delta;\beta;\gamma}+\sigma_{,\alpha;\gamma;\beta;\delta}=0. \qquad \text{(A3)}$$

The derivatives can now be put in the same order by commuting them with the use of the curvature tensor. This gives

$$x=x':\quad \sigma_{,\alpha;\beta;\gamma;\delta}=\tfrac{1}{3}(R_{\delta\alpha\beta\gamma}+R_{\gamma\alpha\beta\delta}). \qquad \text{(A4)}$$

Note, for example, that

$$\sigma_{;\alpha}{}^{;\alpha}{}_{,\mu;\nu}=-\tfrac{2}{3}R_{\mu\nu}. \qquad \text{(A5)}$$

Next we take five derivatives of Eq. (2.23) to get

$$x=x':\quad \sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu}+\sigma_{,\mu;\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}+2\sigma_{,\beta;\alpha}{}^{;\alpha;\beta}{}_{,\mu}=0. \qquad \text{(A6)}$$

Placing the indices in the same order with the aid of the curvature tensor gives

$$x=x':\quad \sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu}=-R_{,\mu}. \qquad \text{(A7)}$$

Finally, we take six derivatives of Eq. (2.23) to get

$$x=x':\quad \sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu;\nu}+\sigma_{,\mu;\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}{}_{;\nu}+\sigma_{,\nu;\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}{}_{;\mu}+2\sigma^{,\beta}{}_{;\alpha}{}^{;\alpha}{}_{;\beta,\mu;\nu}$$
$$+2\sigma_{,\lambda;\alpha}{}^{;\alpha}{}_{;\beta}\sigma^{,\lambda;\beta}{}_{;\mu;\nu}+2\sigma_{,\lambda}{}^{;\alpha}{}_{;\alpha;\mu}\sigma^{,\lambda}{}_{;\beta}{}^{;\beta}{}_{;\nu}+2\sigma_{,\lambda;\alpha;\beta}{}^{;\beta}\sigma^{,\lambda;\alpha}{}_{;\mu;\nu}+4\sigma_{,\lambda;\alpha;\beta;\mu}\sigma^{,\lambda;\alpha;\beta}{}_{;\nu}=0. \qquad \text{(A8)}$$

Putting the terms with six derivatives of σ in the same index order and using Eq. (A4) gives

$$x=x':\quad \sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu;\nu}=\tfrac{4}{15}R_{\mu\sigma}R_{\nu}{}^{\sigma}-\tfrac{2}{5}R_{\mu\nu;\sigma}{}^{;\sigma}-\tfrac{6}{5}R_{,\mu;\nu}-\tfrac{8}{45}R_{\mu\alpha\beta\nu}R^{\alpha\beta}-\tfrac{4}{15}R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}. \qquad \text{(A9)}$$

From this follows

$$x=x':\quad \sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\gamma}{}^{;\gamma}=-\tfrac{8}{5}R_{,\sigma}{}^{;\sigma}+\tfrac{4}{15}R_{\mu\nu}R^{\mu\nu}$$
$$-\tfrac{4}{15}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \qquad \text{(A10)}$$

The coincident coordinate limits of various derivatives of the symmetrical function $\Delta^{1/2}(x,x')$ are also needed for our work. Most of these appear in the book by DeWitt,[7] but we again outline their derivation for the convenience of the reader. This function is defined by

$$n\Delta^{1/2}=\Delta^{1/2}\sigma_{,\mu}{}^{;\mu}+2\Delta^{1/2}{}_{,\mu}\sigma^{,\mu}, \qquad \text{(2.27)}$$

and the boundary condition

$$x=x':\quad \Delta^{1/2}=1. \qquad \text{(2.28)}$$

The first derivative of Eq. (2.27) gives the coincident limit

$$x=x':\quad \Delta^{1/2}{}_{,\mu}=0. \qquad \text{(A11)}$$

Two derivatives of Eq. (2.27) give

$$x=x':\quad 0=\sigma_{,\lambda}{}^{;\lambda}{}_{,\mu;\nu}+4\Delta^{1/2}{}_{,\mu;\nu}, \qquad \text{(A12)}$$

and, remembering Eq. (A5),

$$x=x':\quad \Delta^{1/2}{}_{,\mu;\nu}=\tfrac{1}{6}R_{\mu\nu}. \qquad \text{(A13)}$$

Three derivatives of Eq. (2.27) give

$$x=x':\quad 0=\sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu}+4\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu}+2\Delta^{1/2}{}_{,\mu;\alpha}{}^{;\alpha}. \qquad \text{(A14)}$$

Since the coincident coordinate limit of $\Delta^{1/2}{}_{,\mu}$ vanishes, the derivatives of $\Delta^{1/2}$ in Eq. (A14) can be freely commuted. Hence, using Eq. (A7), we have

$$x=x':\quad \Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu}=\tfrac{1}{6}R_{,\mu}. \qquad \text{(A15)}$$

This, together with Eq. (A13), implies that

$$\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu'}(x,x')|_{x=x'}=[\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}(x,x')|_{x=x'}]_{,\mu}$$
$$-\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu}(x,x')|_{x=x'}$$
$$=0. \qquad \text{(A16)}$$

Finally, we take four derivatives of Eq. (2.27) to get the coincident limit

$$x=x':\quad 0=\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}\sigma_{,\beta}{}^{;\beta}{}_{,\mu;\nu}+\Delta^{1/2}{}_{,\mu;\nu}+\Delta^{1/2}{}_{,\mu;\nu}\sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}$$
$$+2\Delta^{1/2}{}_{,\alpha;\mu}\sigma_{,\beta}{}^{;\beta,\alpha}{}_{;\nu}+2\Delta^{1/2}{}_{,\alpha;\nu}\sigma_{,\beta}{}^{;\beta,\alpha}{}_{;\mu}+\sigma_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}{}_{,\mu;\nu}+4\sigma_{,\alpha;\beta;\mu;\nu}\Delta^{1/2,\alpha;\beta}$$
$$+2\sigma^{,\alpha}{}_{;\beta}{}^{;\beta}{}_{;\mu}\Delta^{1/2}{}_{,\alpha;\nu}+2\sigma^{,\alpha}{}_{;\beta}{}^{;\beta}{}_{;\nu}\Delta^{1/2}{}_{,\alpha;\mu}+4\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu;\nu}+2\Delta^{1/2}{}_{,\mu;\alpha}{}^{;\alpha}{}_{;\nu}+2\Delta^{1/2}{}_{,\nu;\alpha}{}^{;\alpha}{}_{;\mu}. \qquad \text{(A17)}$$

Placing the derivative indices of $\Delta^{1/2}$ in the same order with the curvature tensor and using Eqs. (A4) and (A13) gives

$$x=x': \quad \Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu;\nu}=\tfrac{3}{20}R_{,\mu;\nu}+\tfrac{1}{20}R_{\mu\nu;\alpha}{}^{;\alpha} +\tfrac{1}{36}RR_{\mu\nu}-\tfrac{1}{15}R_{\mu\alpha}R_{\nu}{}^{\alpha} +\tfrac{1}{30}R_{\mu\alpha\nu\beta}R^{\alpha\beta}+\tfrac{1}{30}R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} \tag{A18}$$

and

$$x=x': \quad \Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\beta}{}^{;\beta}=\tfrac{1}{5}R_{,\alpha}{}^{;\alpha}+\tfrac{1}{36}R^2 -\tfrac{1}{30}R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{30}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \tag{A19}$$

The proper-time representation of the Green's function involves the weight function

$$F=1+isf_1+(is)^2f_2+\cdots, \tag{2.33}$$

with

$$-f_1=\xi R+\sigma^{;\mu}f_{1,\mu}-\frac{1}{\Delta^{1/2}}\Delta^{1/2}{}_{,\mu}{}^{;\mu} \tag{2.34a}$$

and

$$-2f_2=\xi Rf_1+\sigma^{;\mu}f_{2,\mu}-\frac{1}{\Delta^{1/2}}(\Delta^{1/2}f_1)_{,\mu}{}^{;\mu}. \tag{2.34b}$$

Using Eq. (A13), we get the coincident limit

$$f_1(x,x;n)=(\tfrac{1}{6}-\xi)R. \tag{A20}$$

We take two derivatives of Eq. (2.34a) to find

$$x=x': \quad -f_{1,\mu;\nu}=\xi R_{,\mu;\nu}+2f_{1,\mu;\nu}+\Delta^{1/2}{}_{,\mu;\nu}\Delta^{1/2}{}_{,\alpha}{}^{;\alpha} -\Delta^{1/2}{}_{,\alpha}{}^{;\alpha}{}_{,\mu;\nu}, \tag{A21}$$

or, using Eqs. (A13) and (A18),

$$f_{1,\mu;\nu}(x,x';n)|_{x=x'}=(\tfrac{1}{20}-\tfrac{1}{3}\xi)R_{,\mu;\nu} +\tfrac{1}{60}R_{\mu\nu;\alpha}{}^{;\alpha}-\tfrac{1}{45}R_{\mu\alpha}R_{\nu}{}^{\alpha} +\tfrac{1}{90}R_{\mu\alpha\nu\beta}R^{\alpha\beta}+\tfrac{1}{90}R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}. \tag{A22}$$

Hence

$$f_{1,\mu}{}^{;\mu}(x,x';n)|_{x=x'}=\tfrac{1}{3}(\tfrac{1}{5}-\xi)R_{,\mu}{}^{;\mu} -\tfrac{1}{90}R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{90}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \tag{A23}$$

Finally, we compute the coincident limit of Eq. (2.34b) using Eqs. (A20), (A11), (A13), and (A23):

$$f_2(x,x;n)=\tfrac{1}{2}[(\tfrac{1}{6}-\xi)R]^2+\tfrac{1}{6}(\tfrac{1}{5}-\xi)R_{,\alpha}{}^{;\alpha} -\tfrac{1}{180}R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \tag{A24}$$

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[8]To be more explicit, we note that our "vacuum expectation value of an operator X" refers to the normalized matrix element between vacuum states in the remote past and in the remote future,

$$\langle X\rangle=\frac{\langle \text{vac, out}|X|\text{vac, in}\rangle}{\langle \text{vac, out}|\text{vac, in}\rangle}.$$

If no particle production takes place, then the in and out vacuum states are identical (although their vectors may differ in phase).

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