Gravitational and rotational effects in quantum interference

J. Anandan*
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
(Received 30 September 1976)

The phase shift due to gravitation and rotation in the quantum interference of two coherent beams is obtained relativistically and compared with the recent experiment of Colella, Overhauser, and Werner. A general expression relating the quantum phase shift to the transverse acceleration of a classical particle in the plane of interference for an arbitrary interaction with any external field is given. This can serve as a correspondence principle between quantum physics and classical physics. The phase shift due to the coupling of spin to curvature of space-time is deduced and written explicitly for the special case of a Schwarzschild field. The last result implies that a massless spinning particle can have at most two helicity states and its world line in a gravitational field is a null geodesic. Finally, new experiments are proposed to test the effect of rotation on quantum interference and to obtain direct evidence of the equivalence principle in quantum mechanics.

I. INTRODUCTION

A fundamental difference between classical and quantum physics is the phenomenon of interference. If a particle could arrive at a point B from a point A by two different paths, then according to classical physics the probability of its arrival from A to B is the sum of the probabilities of it taking each individual path. According to quantum theory, however, a probability amplitude (a complex number) must be associated with each possible path, the square of the modulus of this amplitude representing the probability. The probability amplitude for the particle to go from A to B is the sum of the probability amplitudes associated with each possible path from A to B. In general, the probability amplitude for a quantum-mechanical process is the sum of the amplitudes associated with each possible way in which it could occur, and this phenomenon is known as interference.

Quantum interference may alternatively be thought of as the interference of the de Broglie wave associated with an ensemble of particles with itself. The importance of the nature of such an interference lies in the fact that it enables us to predict, via the Huygens principle, the motion of the de Broglie wave. The modification of this motion in the presence of an external field can also be predicted if one knows how the external field causes a shift in the interference fringes. An experiment that provides this information for electrons moving in a magnetic field was suggested by Aharonov and Bohm[1] and subsequently performed by Chambers.[2] An equally valuable experiment that provides information on the influence of gravity on quantum interference was suggested by Overhauser and Colella[3] and performed subsequently by Colella, Overhauser, and Werner[4] (referred to from now on as the COW experiment).

The result of the experiment suggested by Aharonov and Bohm is consistent with both relativistic and nonrelativistic quantum mechanics. This is because in both relativistic and nonrelativistic wave equations, the presence of the electromagnetic field is realized by the substitution $\partial/\partial x^a \to \partial/\partial x^a - i(e/\hbar)A_a$, where A_a is the electromagnetic potential. This results in the same predicted *phase shift* $(e/\hbar)\oint A_a\,dx^a$ between the interfering beams due to the electromagnetic field, in both the relativistic and nonrelativistic theories. This, however, is not the case with the COW experiment. The purpose of the present paper is first to calculate relativistically the result of the latter type of experiment while taking into account not only the effect of gravity, but also the rotation of the apparatus which is always present since the earth rotates. The motion of a particle with respect to a rotating frame is different from its motion with respect to an inertial frame; it follows that there must be a phase shift between interfering beams due to the rotation of the apparatus so as to account for this modified motion.

The result of our calculation, though different from the nonrelativistic gravitational effect tested in the COW experiment, is in agreement with the result of this experiment within the limits of its accuracy. Experiments are then proposed which, if performed, will for the first time (i) test the effect of the earth's rotation on neutron interference (quantum-mechanical analog of the Michelson-Gale-Pearson experiment[5]), (ii) test the effect of rotation on neutron interference by means of a rotating table (quantum-mechanical analog of the Sagnac experiment[6]), and (iii) provide direct evidence of the equivalence principle at the quantum-mechanical level.

In Sec. II the general formalism for the motion of approximate locally plane waves in an arbitrary space-time will be stated. The phase difference between such waves interfering in a gravitational field will be obtained and thereby we shall generalize a result of Ashtekar and Magnon for light.[7] In Sec. III this general result is applied to the specific case of a Bonse-Hart type of interferometer[8] used in the COW experiment and needed for the proposed experiments mentioned above. In Sec. IV the connection between the quantum phase shift and the classical equation of motion is obtained using intuitive arguments for an arbitrary interaction between a particle and an external field. The preceding results, the Aharonov-Bohm effect, and the phase shift due to the coupling of spin to curvature of space-time are obtained as special cases of this general relationship. The last result, as expected, is very small, but turns out to have some interesting theoretical consequences. New experiments are proposed in Sec. V.

II. FORMALISM AND METHOD

A de Broglie wave in an arbitrary space-time can be uniquely specified by giving its amplitude α and phase ϕ as functions of space-time. ϕ is a real-valued scalar function; α may be a tensor-valued or spinor-valued function, depending on whether the wave describes bosons or fermions. At all points where ϕ is differentiable, define the wave vector k_a by

$$k_a = -\nabla_a \phi \ . \tag{1}$$

It follows that if γ is a circuit on which ϕ is piecewise continuous, then

$$\oint_\gamma k_a dx^a = \Delta\phi \ , \tag{2}$$

where $\Delta\phi$ is the algebraic sum of the discontinuities of ϕ in going around the closed curve γ.

In this section only de Broglie waves associated with spinless particles will be considered. The effect of spin on the results obtained is considered in Sec. IV and turns out to be negligible as far as the earth's gravitational field is concerned. Also the wave function $\psi = \alpha e^{i\phi}$ of the de Broglie wave associated with an ensemble of free (spinless) particles in space-time is assumed to satisfy the generalized Klein-Gordon equation:

$$\nabla^a \nabla_a \psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{3}$$

Since the de Broglie wave here represents spinless particles, α is a scalar. Without loss of generality α can and will be chosen to be real. In flat space-time, plane-wave solutions of (3) for which α is a constant can be obtained. In curved space-time this cannot be done in general, as will be readily seen later. But even in curved space-time approximate locally plane-wave solutions of (3) can be obtained by requiring that α is slowly varying in comparison with ϕ. On substituting $\psi = \alpha e^{i\phi}$ in (3) and requiring that $\nabla^a\nabla_a \alpha/\alpha$ is zero or negligible, one obtains for the real and imaginary parts of this equation

$$k_a k^a = \frac{m^2c^2}{\hbar^2} \tag{4}$$

and

$$k^a \nabla_a \alpha = -\tfrac{1}{2}(\nabla^a k_a)\alpha \ , \tag{5}$$

which govern the propagation of phase and amplitude, respectively.[9]

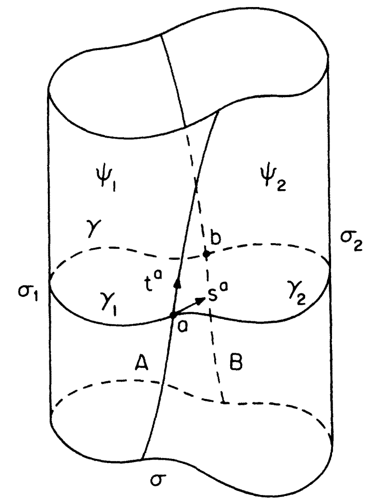

FIG. 1. Space-time representation of a general two-beam interference experiment. The original beam is split along the world line A and the two beams interfere along the world line B.

Consider now a beam of identical particles which is constantly being split into two beams, by means of say constant reflection and transmission by a small half-reflecting mirror attached to an apparatus. In space-time, we shall idealize the mirror by a world line A. The two beams travel along different paths made possible by suitable reflections on mirrors attached to the apparatus and interfere along a world line B of the apparatus. The paths along which the two beams travel will be idealized by two-dimensional timelike surfaces σ_1 and σ_2 which meet along A and B to form a tube-like submanifold σ in space-time (Fig. 1). The de Broglie waves ψ_1, ψ_2 associated with the two beams will be assumed to be approximately locally plane as described in the preceding paragraph so that all preceding equations are assumed to be valid on the submanifold σ.

Let t^a denote the four-velocity vector field of the particles constituting the apparatus. At each point on σ define a unit vector s^a orthogonal to t^a and tangent to σ such that the direction of s^a is always in the same sense around σ. Then

$$t^a t_a = 1, \quad s^a s_a = -1, \quad t^a s_a = 0 \ . \tag{6}$$

The wave vector k^a is tangent to σ and can be written as

$$k^a = \begin{cases} \omega t^a - \kappa s^a & \text{for } \psi_1 \ , \\ \omega t^a + \kappa s^a & \text{for } \psi_2 \ . \end{cases} \tag{7}$$

Since, for the present, the wave ψ_1 exists on σ_1 and ψ_2 on σ_2, (7) defines $\omega = k_a t^a$ and $\kappa = |k_a s^a|$ as non-negative scalar functions on σ whose values at each point are respectively the frequency and wave number as measured by a local observer at rest with respect to the apparatus. ω and κ are not independent: From (4), (7), and (6),

$$\omega^2 - \kappa^2 = \frac{m^2 c^2}{\hbar^2} \ . \tag{8}$$

Since at each point on A, ϕ is the same for both waves and t^a is tangent to A, ω is the same for both waves on A.

We wish to obtain the difference $\Delta\phi$ between the phases of the two waves at an arbitrary event b on B. Let γ be a closed curve on σ going around σ and containing b. Then the phase ϕ is continuous at all points on γ expect at b where its discontinuity is equal to $\Delta\phi$. Hence $\Delta\phi$ is obtained from (2) if k_a is known on σ. In general, if ω is given along A, then k^a can be constructed on σ_1 and σ_2 in the following manner. Note first that since by (1) k_a is curl-free, $k^a \nabla_a k_b = k^a \nabla_b k_a = 0$ by (4). Hence the integral curves of k^a are affinely parametrized geodesics on the submanifold σ. If ω is given at each point on A, then κ given by (8) determines at this point via (7) the wave vectors of the two waves ψ_1 and ψ_2. By parallel propagating each of these vectors along its direction on σ, and by doing this for each pair of wave vectors on A, k_a can be constructed on σ_1 and σ_2. Then the required phase difference is obtained from (2).

In general, however, the geodesics constructed in the above manner, if they are non-null, could cross each other, thus contradicting the assumption that k^a is well defined on σ_1 and σ_2. In such a situation if k^a is to be well defined, it follows that the integral curves of k^a cannot all be geodesics on σ. This is because Eq. (4), which was used in the preceding paragraph to show that the integral curves of the curl-free vector field k^a are geodesics, can in general be satisfied only approximately. Equations (1) and (4) can both be satisfied exactly only under special conditions, and the above prescription for obtaining $\Delta\phi$ is valid under these conditions.[10] Such a special set of conditions occurs if (i) the submanifold σ admits a timelike Killing vector field ξ^a such that the world lines of particles constituting the apparatus lie along the trajectories of this Killing field, and (ii) ω is constant along A. (i) corresponds to the physical conditions experienced by observers at rest with respect to the apparatus being independent of time. This includes for instance the case when the apparatus is fixed to the earth which has a constant angular velocity and a "time-independent" gravitational field. Condition (ii) physically corresponds to the frequency with respect to the half-reflecting mirror, of the incident wave being independent of time. (i) and (ii) include all cases of practical importance, and under these conditions an explicit formula for $\Delta\phi$ can be obtained as will be shown now.

Let ξ^a be the Killing vector field (nonvanishing on σ) described in condition (i) above. Then we can write

$$\xi^a = \lambda^{1/2} t^a \ . \tag{9}$$

$\xi^a \xi_a = \lambda$ is constant along each Killing trajectory; choose ξ^a such that $\lambda = 1$ along A. Let ω_A be the constant value of ω on the world line A. The existence of a unique field k^a satisfying (1) and (4) and this boundary condition can be easily shown by the following constructive procedure. At any point on A there exist two tangent vectors to the submanifold σ, which are solutions for k^a in $k^a \xi_a = \omega_A$ and $k^a k_a = m^2 c^2/\hbar^2$. By Lie-propagating the unique affinely parametrized geodesics C_1, C_2 through these vectors along ξ^a, a congruence of curves is obtained on each of σ_1 and σ_2. The tangent vector field k^a of these two congruences of curves is well defined on σ and satisfies $k^a \nabla_a k_b = 0$ on C_1, C_2 and (the Lie derivative) $L_\xi k^b = 0$ on σ. It follows that $k^a \nabla_a (k^b k_b) = 0$ on C_1, C_2, $L_\xi (k^b k_b) = 0$, $k^a \nabla_a (k^b \xi_b) = 0$ on C_1, C_2, and $L_\xi (k^b \xi_b) = 0$ since ξ_a is a Killing field. Hence $k^b k_b = m^2 c^2/\hbar^2$ and $k^b \xi_b = \omega_A$ everywhere on σ Also it can be easily verified that k^a is curl-free[11] and k^a is therefore the wave vector field on σ for the waves ψ_1 and ψ_2 which have the common frequency ω_A on A.[12] Hence from (9), ω defined by (7) is given everywhere on σ by

$$k^a \xi_a = \lambda^{1/2} \omega = \omega_A \ . \tag{10}$$

The phase difference $\Delta\phi$ at b on B is given by (2) as mentioned in the paragraph following (8), and can be written, using (7) and (10), as

$$\Delta\phi = \omega_A \oint_\gamma \lambda^{-1/2} t_a dx^a + \int_{a\,(\gamma_1)}^{b} \kappa s_a dx^a + \int_{a\,(\gamma_2)}^{b} \kappa s_a dx^a \ , \tag{11}$$

where a is the point at which γ intersects A, and γ_1, γ_2 are the portions of γ that lie in σ_1 and σ_2, respectively (Fig. 1). Equation (11) can be evaluated if κ is expressed in terms of known constants and purely geometrical quantities. From

(8) and (10) the required expression is

$$\kappa=\left(\lambda^{-1}\omega_A{}^2-\frac{m^2c^2}{\hbar^2}\right)^{1/2} . \tag{12}$$

Consider now the more general case when either or both waves ψ_1 and ψ_2 travel one or more times around σ before experiencing interference. The convention adopted in (7) and also (10) and (12) ensures that in regions where the two waves overlap ω and κ for both waves are the same. Thus ω and κ are again well-defined functions on σ. Hence (11) can now be replaced by the more general expression

$$\Delta\phi=n\omega_A\oint_\gamma\lambda^{-1/2}t_a\,dx^a+\int_{\gamma_1 a}^{b}\kappa s_a dx^a+\int_{\gamma_2 a}^{b}\kappa s_a dx^a , \tag{13}$$

where n is a non-negative integer equal to the sum of the number of times the waves have gone around σ before interfering along B. In this general case of course γ_1 and γ_2 must also wind around σ as many times as the corresponding waves travel around σ. Also since γ can be displaced along the Killing trajectories of ξ^a without changing $\Delta\phi$, the latter must be independent of the chosen event b on B.

The last two terms of (13), roughly speaking, give the phase shift due to the path difference of the two beams and gravitational effects. The nonvanishing of the first term, which is experimentally testable, provides a criterion for rotation in general relativity.[7] In the special case when both waves travel once around σ before interfering along A itself, as in the experiments of Sagnac[6] and Michelson, Gale, and Pearson,[5] it is clear that $n=2$ and the last two terms in (13) make no contribution. But for the COW experiment[4] that we shall be concerned with in the next section, $n=1$ and the last two terms make the main contribution.

III. RELATIVISTIC PHASE SHIFT FOR THE COLELLA-OVERHAUSER-WERNER EXPERIMENT

Equation (13) combined with (12) is sufficient for the evaluation of the phase shift $\Delta\phi$ as soon as the gravitational field and the geometry of the apparatus is given. We shall demonstrate this now by calculating $\Delta\phi$ for the Bonse-Hart type of interferometer that was used in the COW experiment.[4]

In this apparatus, the two spatial paths taken by the two waves to go from A to B form a parallelogram $ADBC$. Let $AC=s$ and the distance between AC and DB be r. Let $\vec{G}$ be the resultant of the gravitational field and the centrifugal field due to earth's rotation. The direction of $\vec{G}$, which is the same as the direction of a plumb line at rest with respect to the earth, will be called the vertical direction. The apparatus is assumed to be oriented so that AC and DB are horizontal. Let g denote the component of $\vec{G}$ in the plane $ACBD$ at A. The apparatus is assumed to rotate with angular velocity $\vec{\Omega}$ about an axis through 0, the center of the parallelogram $ABCD$.

Assume also that the world lines of the apparatus are along the trajectories of a Killing vector field ξ^a. This condition is valid if the apparatus is fixed to the earth in which case $\vec{\Omega}$ equals the angular velocity of the earth, or if the apparatus is small and rotates with constant angular velocity with respect to the earth in a horizontal plane. Let t^a and T^a respectively be the four-velocity fields of the apparatus and a local nonrotating frame at rest with respect to 0. Then we can write

$$t^a=\left(1-\frac{\Omega^2\rho^2}{c^2}\right)^{-1/2}\left(T^a+\frac{\Omega\rho}{c}R^a\right) , \tag{14}$$

where R^a is a normalized rotational vector field such that each integral curve of R^a is a circle with radius ρ, lying entirely on a hypersurface element orthogonal to T^a with its center on the (instantaneous) axis of rotation and its plane normal to this axis in this hypersurface element.

Let the world lines of $A, B, C, D, 0$ meet a hypersurface element orthogonal to T^a at a, b, c, d, o respectively. The first term of (13) is

$$\begin{aligned}\Delta\phi_1&=\omega_A\oint_\gamma \lambda^{-1/2}t_a dx^a\\&=\omega_A\int_\Sigma[\nabla_a(\lambda^{-1/2}t_b)-\nabla_b(\lambda^{-1/2}t_a)]ds^{ab}\end{aligned} \tag{15}$$

using Stokes's theorem, where Σ is a two-dimensional surface with γ as boundary.[7] Choose the parallelogram $abcd$ to be the closed curve γ and Σ to be the plane $abcd$. On this plane,

$$\xi^a\xi_a=\lambda=1+\frac{2\Phi}{c^2} ,\quad \frac{\partial\Phi}{\partial x}(a)=0,\quad \frac{\partial\Phi}{\partial y}(a)=g,\quad \Phi(a)=0 , \tag{16}$$

where x is the distance measured along ac and y is the distance measured on this plane perpendicular to ac, and ξ^a has been chosen so that $\lambda=1$ along the world line of A.

Hence from (15), (14), and (16), neglecting terms involving second and higher order in g, Ω, r, and s,

$$\Delta\phi_1\simeq\frac{2\omega_A\Omega_n A}{c} , \tag{17}$$

where A is the area of the apparatus, Ω_n is the component of the angular velocity normal to the apparatus. Substituting for ω_A from (12), for a massive particle ($m\neq 0$),

$$\Delta\phi_1 \simeq \frac{2\Omega_n Am}{\hbar} + \frac{\hbar\Omega_n A\kappa^2}{mc^2} . \qquad (18)$$

The remaining contribution to the phase shift according to (13) is

$$\Delta\phi_2 = \int_{adb} \kappa s_a dx^a + \int_{acb} \kappa s_a dx^a . \qquad (19)$$

To first order in Ω,

$$\int_d^b \kappa s_a dx^a + \int_a^c \kappa s_a dx^a \simeq (\delta\kappa)s , \qquad (20)$$

where $\delta\kappa$ is the difference between the values of κ along db and ac, respectively. To evaluate $\delta\kappa$ use (12) and (16):

$$\delta\kappa \simeq \frac{-\omega_A{}^2\delta\lambda}{2\kappa\lambda^2} \simeq \frac{-\omega_A{}^2 gr}{\kappa\lambda^2 c^2} , \qquad (21)$$

neglecting $(\partial^2\phi/\partial y^2)(r^2/c^2)$ and higher orders. Also it is easily seen from (12) and (16) that

$$\int_a^d \kappa s_a dx^a + \int_c^b \kappa s_a dx^a = 0 , \qquad (22)$$

neglecting terms in $(\partial^2\phi/\partial x^2)(s^2/c^2)$ and higher orders. It follows from (19)–(22) that

$$\Delta\phi_2 \simeq \delta\kappa s \simeq -\frac{\omega_A{}^2 grs}{\kappa c^2} = -\frac{\omega_A{}^2 gA}{\kappa c^2} = -\frac{gAm^2}{\hbar^2\kappa} - \frac{gA\kappa}{c^2} , \qquad (23)$$

using (12) and (16), to first order in g, r, and s.

The total phase shift for spinless particles is $\Delta\phi_1 + \Delta\phi_2$, where $\Delta\phi_1$ and $\Delta\phi_2$ represent respectively the effects due to rotation and gravitation. To first order in Ω, g, r, and s they are given by (17) and (23), respectively. The first term in (23) is the result derived nonrelativistically by Overhauser and Colella.[3] The first term in (18) is the nonrelativistic result given by Page.[13] The COW experiment[4] was accurate enough to measure only the first term in (23), and so our results are in agreement with this experiment within the limits of its accuracy. In Sec. V new experiments will be proposed to test the first term of (18). There appears to be no hope of testing experimentally the relativistic term in (23). But the relativistic term in (18) can be tested if significant advances are made in the production of monochromatic neutron beams, which will be discussed in Sec. V.

IV. EFFECT DUE TO THE COUPLING OF SPIN TO SPACE-TIME CURVATURE

The phase shift $\Delta\phi$ given by (13) applies only to spinless particles. It is known that the world lines of free particles with nonzero spin, according to the theory of relativity, are in general not geodesics. In particular, a modification arises due to the coupling of spin to the space-time curvature.[14] It follows that there must be a corresponding contribution to the phase shift in the two-beam interference to account for this modified motion.

We shall obtain the contribution to $\Delta\phi$ due to the coupling of spin to curvature by first obtaining a general correspondence between the quantum phase shift and the classical equation of motion. The results of the preceding section as well as the new result in the present section will follow as special cases of this general relation. The argument used to obtain this general relation is intuitive and not rigorous. But it has the advantage of being so general that it applies to any spin, which will not be the case if a particular wave equation such as Dirac's equation in curved space-time is used to obtain the result. Also, it provides support for and physical insight into the results obtained in the preceding section.

Let A be a point in three-dimensional space on a wave front of a de Broglie wave at a given instant. Consider two portions of the secondary wavelet originating from A and passing through points D and C respectively such that $AD = AC$ and AD, AC make the same small angle with the normal to the wave front at A. The secondary wavelets originating from D and C will interfere; let B be the central maximum of this interference pattern at the same distance l from DC as A (Fig. 2). In the absence of any external field $BD = BC$. The external field causes a phase shift between the interfering wavelets so that B is displaced. If $\delta\theta$ is the angular displacement as viewed from 0, the midpoint of CD, and κ is the wave number for the wave, then $\Delta\phi - \kappa d \sin\delta\theta = 0$, where $\Delta\phi$ is the phase shift between the wavelets ADB and ACB due to the external field, and $-\kappa d\sin\delta\theta$ is the phase shift at B due to the path difference. Since $\delta\theta$ is small, $\sin\delta\theta \simeq \delta\theta \simeq \delta v/v$, where v is the speed of the classical particle and δv is the change in transverse velocity in the plane of interference. Hence

$$\Delta\phi = \kappa d \frac{\delta v}{v} = \frac{\kappa d}{v}\frac{dv_\perp}{dt}\frac{l}{v} = \frac{pA}{\hbar v^2}\frac{dv_\perp}{dt} , \qquad (24)$$

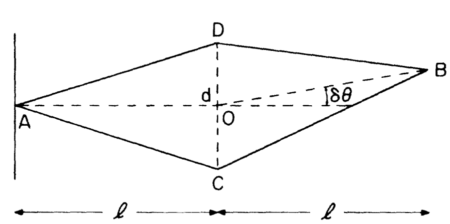

FIG. 2. The connection between the classical transverse acceleration and the phase shift in the two-beam interference experiment.

where $dv_\perp/dl$ is the component of acceleration in the direction CD in Fig. 2, $A = ld$ is the area enclosed by the two paths (assumed small), and the de Broglie relation $p = \hbar\kappa$ has been used. For massless particles $v = c$ in (24). For massive particles (24) may be written as

$$\Delta\phi = \frac{mA\,dv_\perp/dt}{\hbar(1 - v^2/c^2)^{1/2} v} \;. \tag{24'}$$

From (24) or (24′) the quantum phase shift $\Delta\phi$ can be obtained from the transverse acceleration $dv_\perp/dt$ in the plane of interference of a classical particle and vice versa, for any given external field. Consider now the following cases: (i) A particle with charge e moving in a magnetic field B. Then $(1 - v^2/c^2)^{-1/2} dv_\perp/dt = eBv/m$. Hence from (24′), $\Delta\phi = eBA/\hbar$, which is the Aharonov-Bohm effect. (ii) A spinless particle of mass m in a gravitational field, $dv_\perp/dt = -g$ (independent of m). From (24′) $\Delta\phi = -m^2Ag/\hbar(1 - v^2/c^2)p$. Using now the identity $(1 - v^2/c^2)^{-1} = 1 + p^2/m^2c^2$ and the de Broglie relation $p = \hbar\kappa$, $\Delta\phi = -gAm^2/\hbar^2\kappa - gA\kappa/c^2$, which is the same as (23). (iii) A particle moving in a Coriolis field. To first order in Ω, $dv_\perp/dt = 2|\vec{\Omega}\times\vec{v}| = 2\Omega_n v$. From (24′), $\Delta\phi = 2mA\Omega_n/\hbar(1 - v^2/c^2)^{1/2}$. Using now the Planck-Einstein law $E/c = \hbar\omega = mc/(1 - v^2/c^2)^{1/2}$, $\Delta\phi = 2\omega\Omega_n A/c$, which is the same as (17). It may be noted that even though (24′) was obtained using intuitive, arguments, it is evidently an exact relativistic expression valid to all orders of v^2/c^2. Also in each case $\Delta\phi$, which was obtained from (24′), yields the correct result for the massless case in the limit as $m \to 0$. Finally consider (iv) the classical force $f^a = Dp^a/Ds$ due to the coupling of spin to curvature given by Mathisson and Papapetrou[14] which can be written as

$$f^a = \tfrac{1}{2}\epsilon^{cdef} R^{ab}{}_{cd} S_e v_b v_f \;, \tag{25}$$

where $R^{ab}{}_{cd}$ is the Riemann tensor, v_a is the four-velocity of the particle, and S_a is the spin vector. From (25), the transverse acceleration due to the coupling of spin to curvature can be computed. Equation (24′) then yields the corresponding phase shift for the two-beam interference.

We shall now compute the transverse acceleration due to (25) in a Schwarzschild field, which includes the earth's gravitational field as a special case (if the rotation of the earth is neglected). The Schwarzschild field has a unique Killing vector field τ^a which is timelike everywhere (including infinity). Let ϕ^a be a rotational Killing vector field and ρ^a be the "radial vector field" such that the integral curves of ρ^a pass through the "center" of the Schwarzschild field and τ^a, ϕ^a, and ρ^a are mutually orthogonal at every point. Choose τ^a, ϕ^a, ρ^a such that they are normalized at a given point a in the worldline of the test particle. Let θ^a be the normalized four-vector at a orthogonal to τ^a, ρ^a, ϕ^a such that ρ^a, θ^a, ϕ^a form a right-handed system. For simplicity assume that the four-velocity v^a at a has no radial component and write

$$v^a = \gamma\left(\tau^a + \frac{v}{c}\theta^a\right) \;. \tag{26}$$

Since $v^a v_a = 1$, $\gamma = (1 - v^2/c^2)^{-1/2}$. On substituting (26) in (25), we obtain for the Schwarzschild field[15]

$$f^a = \gamma^2 \frac{v}{c}\left[(R^{a\hat{\theta}}{}_{\hat{\phi}\hat{\theta}} - R^{a\hat{\tau}}{}_{\hat{\phi}\hat{\tau}})S_{\hat{\rho}} + (R^{a\hat{\tau}}{}_{\hat{\rho}\hat{\tau}} - R^{a\hat{\theta}}{}_{\hat{\rho}\hat{\theta}})S_{\hat{\phi}}\right], \tag{27}$$

where $S_{\hat{\rho}} = S^a\rho_a$, $R^{a\hat{\tau}}{}_{\hat{\phi}\hat{\tau}} = R^{ab}{}_{cd}\tau_b\phi^c\tau^d$, etc. Hence[15]

$$\begin{aligned} f^{\hat{\rho}} &\equiv f^a\rho_a = -\frac{\gamma^2 GM}{c^3R^3} vS_{\hat{\phi}} \;, \\ f^{\hat{\phi}} &\equiv f^a\phi_a = \frac{\gamma^2 GM}{c^3R^3} vS_{\hat{\rho}} \;, \\ f^{\hat{\tau}} &\equiv f^a\tau_a = 0, \quad f^{\hat{\theta}} \equiv f^a\theta_a = 0 \;, \end{aligned} \tag{28}$$

where G is the gravitational constant, M is the mass of the source of the Schwarzschild field, and R is the distance from the center.

It is clear from (28) that the force is normal to the plane containing the directions of velocity and spin, and its magnitude is independent of the orientation of this plane about the direction of velocity. The magnitude of the additional transverse acceleration due to this force is given for a massive particle using (28) by[16]

$$m\frac{\gamma^2}{c^2}\frac{dv_\perp}{dt} = -\frac{\gamma^2 GMvS_n}{c^3R^3} \;, \tag{29}$$

where S_n is the component of the spin normal to the velocity in the plane containing the velocity and spin. If the interference plane is chosen so as to contain the directions of velocity and acceleration, then $dv_\perp/dt$ that occurs in (24′) and (29) is the same and eliminating it gives

$$\Delta\phi_3 = -\frac{GMS_n\gamma A}{\hbar cR^3} = -\frac{GMS_n\omega A}{mc^2R^3} \tag{30}$$

using the Planck-Einstein law $mc\gamma = \hbar\omega$. The subscript in $\Delta\phi_3$ denotes the fact that (30), which is the contribution due to the coupling of spin to curvature, arises in addition to the contributions $\Delta\phi_1$, $\Delta\phi_2$ due to the rotation of the apparatus and the usual gravity effect, respectively, which were obtained in Sec. III for the COW experiment. In the latter experiment if the neutron beam is polarized normal to the plane of interference, then $S_n = \frac{1}{2}(\hbar/c)$ and (30) reads

$$\Delta\phi_3 = -\frac{1}{2}\frac{\hbar GMA\omega}{mc^3R^3} \;. \tag{31}$$

But (31) appears to be far too small to be detected by any terrestrial experiment.

A glance at (30) shows that as $m \to 0$ with S_n and ω held fixed, $\Delta\phi_3 \to \infty$. This is unlike any of the phase shifts obtained earlier and suggests that in the limit when $m=0$, if $\omega \neq 0$, then S_n must be zero. This means that if the particle is massless and has nonzero energy, its spin must be either parallel or antiparallel to its velocity. This will now be shown to be the case in general, in an arbitrary space-time, so long as (25) is nonvanishing for nonvanishing S_n. We can always write S^a in the form $S^a = \alpha v^a + \beta w^a$, where v^a and w^a are linearly independent. It is the second term in this expression that contributes to the nonzero value of f^a in (25). It follows then from the above argument by considering the limit as $m \to 0$ that when $m=0$, $\beta = 0$. This implies that for a massless particle (with nonzero energy) S^a is either parallel or antiparallel to v^a, so that it can have at most two helicity states.[17] It follows then from (25) that $f^a = 0$, which means that the world lines of massless particles are geodesics and the corresponding phase shift $\Delta\phi_3$ is zero. Finally, the subsidiary condition $S^a v_a = 0$ implies that if S^a $(=\alpha v^a)$ is to be nonzero then v^a must be null. This means that the world lines of massless spinning particles are null geodesics.

Equation (24′) may be regarded as a correspondence principle between classical and quantum mechanics that places a constraint on the possible relativistic quantum theories that could conceivably be created: The "correct" quantum theory must predict for the two-beam interference the same phase shift $\Delta\phi$ as that given by (24′) from the known classical law of motion. We have shown in the present paper that relativistic quantum mechanics for spinless particles as formulated by (3) is in agreement with this correspondence principle. Equation (30) then places a restriction on the possible quantum theories on curved space-time for particles with nonzero spin. If, on the other hand, the result predicted by (24) or (24′) from the classical equation of motion is violated by experiment, then this may call for radical modifications of our present conceptions of physical geometry.

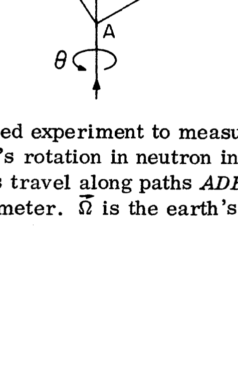

FIG. 3. Proposed experiment to measure the phase shift due to earth's rotation in neutron interference. The interfering beams travel along paths ADB and ACB in a neutron interferometer. $\vec{\Omega}$ is the earth's angular velocity.

V. PROPOSED EXPERIMENTS

We shall now explore the possibility of experimentally verifying the results obtained in Sec. IV. Experiments will be proposed which make use of the Bonse-Hart type of interferometer.[8,4]

The effect of the earth's rotation on neutron interference can be verified by using a vertical beam of neutrons[18] as the incident beam for the interferometer (Fig. 3). If the interferometer is rotated about AD (the direction of the incident beam), the direction of $\vec{G}$ (defined in Sec. III) with respect to the interferometer will not change. Hence the phase shift due to $\vec{G}$ will remain the same for the different positions of the interferometer obtained in this way. But since the component of the earth's angular velocity Ω_n normal to the apparatus changes when it is rotated about AD, the phase shift due to the earth's rotation will change [see Eq. (18)]. Ω_n in (18) can in this case be written as

$$\Omega_n = \Omega \sin\psi \sin\theta ,$$

where ψ is the constant angle between AD and the earth's angular velocity $\vec{\Omega}$, and θ is the angle of rotation of the interferometer about the vertical axis AD, with the initial position $(\theta=0)$ of the interferometer being parallel to $\vec{\Omega}$.

The variation of the phase shift as θ varies from $0°$ to $360°$ is maximum at the equator $(\psi = 90°)$. This maximum value for the apparatus used in the COW experiment is 0.63 fringes. This variation can be increased and made even more easily detectable if the apparatus is constructed with greater area than the original apparatus. A phase shift of this magnitude would have been too small to be detected in the COW experiment. However, in this proposed experiment the apparatus is to be rotated about a vertical axis, unlike in the COW experiment where the axis of rotation was horizontal. This gives the proposed experiment the following advantages over the latter: The contribution to the phase shift due to (i) the bending of the interferometer under its own weight and (ii) the deviation of the neutron beams due to gravitational acceleration is the same for all orientations of the apparatus. Hence these errors are completely eliminated since they make no con-

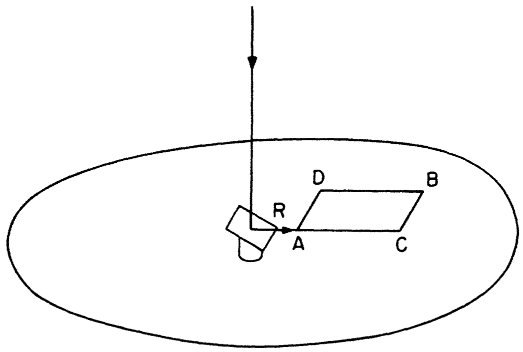

FIG. 4. Rotating-table arrangement to measure the phase shift due to rotation and to test the equivalence principle in quantum mechanics. The neutron beam travels along the axis of rotation and is Bragg-reflected by the crystal R which is rigidly attached to the table. The interferometer, which is also attached to the table, is schematically represented by $ADBC$.

tribution to the variation in phase shift when the angle θ is varied. The elimination of error due to (i), which was of the order of 5 fringes for the COW experiment, is very important and it should render the effect due to the earth's rotation detectable if this experiment is performed, even though it could not be detected in the COW experiment.

A much larger variation Ω than is allowed by the earth's rotation is possible if the experiment is performed with the apparatus fixed to a rotating horizontal table. The known neutron sources that can be fixed to the table, however, do not yield a neutron flux of sufficient intensity. But this problem can be overcome by the following arrangement: A vertical beam of neutrons obtained from a reactor[18] (which has the necessary intensity) passes along the axis of rotation of a horizontal rotating table (Fig. 4) and is Bragg-reflected at R by a suitable crystal rigidly attached to the table and oriented so that the reflected beam is horizontal. The interferometer and the detector are also rigidly attached to the table in such a way that the reflected beam, which has a constant direction with respect to the table, is the incident beam RA for the interferometer $ACBD$. It is possible to construct the reflector and the interferometer from the same cubic crystal by using the $(4,2,2)$ and $(\bar{2},2,0)$ planes to reflect the neutron beam at R and A.[19] This new interferometer, which will have greater stability than if the reflector and interferometer were made of two separate crystals, might also find other applications.

In addition to the contribution to the phase shift due to rotation given in (17), there are contributions that depend on higher orders of Ω. According to the principle of equivalence the centrifugal field locally has the same effect as an appropriate gravitational field and hence contributes to the phase shift. Also, owing to the centrifugal field, strains are set up in the interferometer resulting in a phase shift. Since the centrifugal field is proportional to Ω^2, the last two contributions depend on powers of Ω^2 (or even powers of Ω) and can be eliminated as follows. Measure the phase shifts $\Delta\phi_+$ and $\Delta\phi_-$ when the apparatus rotates with the same angular speed Ω but in opposite senses. Then $\Delta\phi_+ - \Delta\phi_-$ will depend on odd powers of Ω, whereas $\Delta\phi_+ + \Delta\phi_-$ will depend on even powers of Ω. So the effects of centrifugal fields contribute to $\Delta\phi_+ + \Delta\phi_-$ but make no contributions to $\Delta\phi_+ - \Delta\phi_-$.

The major contribution to $\Delta\phi_+ - \Delta\phi_-$ is twice the first term in (18) (with $\Omega_n = \Omega$). There is also a contribution due to Coriolis forces which cause a slight bending of the beam. It can be shown, using arguments similar to those of Overhauser and Colella for the bending of beams due to gravity,[3] that there is no contribution due to the Coriolis force to first order in Ω. There is another contribution due to the velocities of beams along opposite legs being slightly different as a result of the centrifugal acceleration, which is proportional to Ω^3: The difference in velocities is proportional to Ω^2; the differences in the Coriolis force $2\vec{\Omega}\times\vec{V}$ are then proportional to Ω^3. Also, in obtaining (18) we neglected relativistic contributions that will be proportional to Ω^3 and higher odd powers of Ω. For small Ω these contributions are all negligible compared to the main contribution to $\Delta\phi_+ - \Delta\phi_-$, which is proportional to Ω.

The major contribution to $\Delta\phi_+ + \Delta\phi_-$ is due to the centrifugal field which plays locally the role of a gravitational field according to the principle of equivalence. The confirmation of this effect will provide the first direct evidence of the equivalence principle on a quantum-mechanical level.[20] This evidence is crucial for the generalization of quantum mechanics to curved space-times, since present generalizations assume the equivalence principle—obtained from classical phenomena—to be valid in quantum phenomena as well. If the experiment is carried out with the interferometer at a distance r from the axis of rotation which is large compared to the dimensions of the interferometer, then the centrifugal field in the region of the apparatus is approximately uniform and the phase shift is given by (23) with $g = \Omega^2 r$.

The relativistic term in (18) depends not only on v^2/c^2, which is small for thermal neutrons, but also on Ω, which is controllable in the proposed experiment. This provides hope of detecting this term by this experiment. When the apparatus rotates at the rate of about 500 revolutions per second, the second term in (18) is of measurable magnitude. If the experiment is performed with neutron beams of different wavelengths (use higher-order Bragg reflections) then the differences in the phase shifts obtained will depend on the second term of (18) but not on the first term. Unfortunately, however, since the ratio of the first term to the second is of the order of 10^9 for thermal neutrons, when the second term is of measurable magnitude, the overall phase shift due to both terms is so large that the two neutron beams will be coherent at D only if each wave packet contains 10^9 waves. The uncertainty principle $\Delta x\, \Delta p \sim h$ implies that such a neutron beam must be highly monochromatic, and at present it is not possible to produce such a beam. One is confronted with the same problem if one attempts to measure the second term in (23) by using the effective g on a fast rotating table. Thus it appears that the measurement of relativistic terms in neutron interference must await the development of highly monochromatic neutron beams or the discovery of interferometric techniques for high-energy neutrons.

ACKNOWLEDGMENTS

I am indebted to Professor P. Trucano for several stimulating conversations concerning the material in Sec. V of this paper. I wish to thank Professor P. Trucano and Professor R. Colella also for explaining to me many experimental aspects related to the present paper. Professor L. A. Page is thanked for a discussion concerning the nonrelativistic effect of rotation on quantum interference. Professor C. V. Vishveshwara and Dr. R. O. Hansen are thanked for useful comments. Finally I wish to thank Professor R. Roskies and Professor E. T. Newman for reading through the manuscript of this paper and for their useful comments concerning the presentation.

*Work supported in part by the A. W. Mellon Foundation.

[1]Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959). See also W. Ehrenberg and R. E. Siday, Proc. Phys. Soc. London B62, 8 (1949).

[2]R. G. Chambers, Phys. Rev. Lett. 5, 3 (1960).

[3]A. W. Overhauser and R. Colella, Phys. Rev. Lett. 33, 1237 (1974).

[4]R. Colella, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. 34, 1472 (1975).

[5]A. A. Michelson, Astrophys. J. 61, 137 (1925); A. A. Michelson, H. G. Gale, and F. Pearson, *ibid.* 61, 140 (1925).

[6]M. G. Sagnac, C. R. Acad. Sci. (Paris) 157, 708 (1913); 157, 1410 (1913).

[7]Abhay Ashtekar and Anne Magnon, J. Math. Phys. 16, 342 (1975).

[8]U. Bonse and M. Hart, Z. Phys. 194, 1 (1966). See also H. Rauch, W. Treimer, and U. Bonse, Phys. Lett. 47A, 369 (1974).

[9]The approximation being made here is very similar, but not identical, to the WKB approximation of Schrödinger's equation and the geometric optics approximation for light. See, for instance, A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1958), Chap. VI; M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1970), Chap. III; C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 22.5.

[10]There exist, of course, an infinite number of solutions for which (4) is approximately valid. The results of this paper based on (4) will then be valid for these waves to the corresponding degree of approximation. But it is necessary to show that the limiting case for which (4) is exactly valid exists, in order to do the constructions or calculations which assume the exact validity of (4). We show this to be the case under the conditions (i) and (ii) assumed in this paper.

[11]An elegant way of showing this is by noting that there is only one independent bivector at each point on the two-dimensional submanifold σ. So we can write $\nabla_{[a}k_{b]}=\rho t_{[a}k_{b]}$. Since $\nabla_b(k^a k_a)=0$, $\nabla_b(k^a \xi_a)=0$, and $\nabla_{(a}\xi_{b)}=0$, $\xi^b k^a \nabla_{[a}k_{b]}$. Hence $\rho=0$. $k^a\nabla_{[a}k_{b]}=0$. Hence $\rho=0$.

[12]Once k^a is constructed in the manner described, the amplitude α is determined by (5) if its value is given on A. However, the question arises whether α would then satisfy $\nabla^a\nabla_a\alpha=0$, which is the necessary and sufficient condition for the exact validity of (4) for a solution of (3). It is easy to show that for massless particles, $\nabla^a k_a=0$. Therefore all the above-mentioned equations can be satisfied exactly in this case by choosing the scalar function α to be a constant on σ. However, for massive particles $\nabla^a k_a$ is not zero in general. But here also calculation shows that under the conditions (i) and (ii) assumed in this paper, if α is a constant along A (i.e., the amplitude of the wave as seen by the half-reflecting mirror is independent of time) then $\nabla^a\nabla_a\alpha$ can be taken to be zero to a very high degree of approximation for an experiment that is done in a laboratory. Hence the results of this paper will also be valid for such a wave to the same high degree of approximation.

[13]Lorne A. Page, Phys. Rev. Lett. 35, 543 (1975).

[14]M. Mathisson, Acta Phys. Polon. 6, 163 (1937); A. Papapetrou, Proc. R. Soc. London A209, 258 (1951). The spin angular momentum tensor S^{ab} in the last paper is related to the spin vector S^a of the present paper by $S^a=\frac{1}{2}\epsilon^{abcd}v_b S_{cd}$; the subsidiary conditions $S^{ab}v_b=0$ are assumed. See also F. A. E. Pirani, Acta Phys. Polon. 15, 389 (1956).

[15]For the nonzero components of $R^a{}_{bcd}$ in a Schwarzschild metric that are used in (27) and (28), see for instance, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 31.2.

[16]Actually $f^a=Dp^a/Ds=mDv^a/Ds+\epsilon^{abcd}(D^2v_d/Ds^2)S_b v_c$, which reflects the fact that in relativistic physics

p^a and v^a for a spinning particle are in general not parallel. So even in flat space-time for which $f^a = 0$, it appears that there can be a phase shift due to the second term in the above expression which might be related to the Zitterbewegung of the Dirac electron. In this paper, however, we are concerned only with the gravitational effect and therefore have ignored the contribution due to the second term. If S^a is parallel to v^a, which we show to be the case when the particle is massless, then the second term is zero.

[17]Mashhoon has obtained this conclusion as well as the subsequent conclusion that paths of massless particles are geodesics, using only classical considerations on the basis of the Mathisson-Papapetrou equations. B. Mashhoon, Ann. Phys. (N.Y.) 89, 254 (1975). However, Mashhoon assumes that a massless particle follows a null path, in order to reach these conclusions. This assumption is not needed in the argument used in the present paper but is instead obtained as a consequence, upon using the supplementary condition $S^a v_a = 0$.

[18]The nuclear reactors that serve as the source of neutrons in these experiments are built so that the neutron beam produced is horizontal. A vertical beam can be produced by reflecting this beam by a suitable crystal. Also reactors can be constructed so that the produced beam is vertical.

[19]This possibility was realized in a conversation I had with Professor P. Trucano. I wish to thank him for supplying the key ideas in this discussion.

[20]The equivalence principle on a quantum-mechanical level has been verified by the COW experiment (Ref. 4). But this verification is somewhat indirect since the effect of acceleration on quantum interference was given in this paper by a Gedanken experiment. In the experiment proposed in the present paper, the interferometer is accelerating with respect to the distant stars and hence the evidence obtained will be more direct.