Possibility of a static scalar field in the Schwarzschild geometry*

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Some effects of a nonlinear coupling, $\mathcal{L}_I = -\lambda\varphi^4$, on a massless scalar field in a Schwarzschild geometry are studied. There exists a classical time-dependent solution (probably unique) which blows up as $4^{-1}(2MG)^{-1/2}\lambda^{-1/2}(r-2MG)^{-1/2}$ near the event horizon and goes to zero at infinity faster than r^{-1}. It is argued that (a) such behavior is admissible up to some very short distance from the horizon, at which point small couplings to other fields should come into play, and (b) a field which at some initial time has an arbitrarily small singular amplitude will develop in time into this classical solution. The effective potential for the propagation of scalar waves is drastically modified in the presence of the classical background field. This should lead to significant changes in the rate of emission of scalar quanta from a black hole.

I. INTRODUCTION

A well-known result in the theory of gravitational collapse is that the final state cannot support a static scalar field.[1] This is a consequence of the fact that there exists no static solution of the scalar wave equation, in the Schwarzschild geometry, which is regular at infinity and at the event horizon.

In the present paper we raise a question about the physical basis of the assumption that a solution cannot be singular, or nearly singular, at the event horizon, and we present some consequences of dropping the assumption. By a "nearly singular" field we mean a field which grows, as the event horizon is approached, up to some very large value at a point close to the horizon, at which point the classical field equations are assumed no longer to describe the system.

If we are dealing with a field which describes one of the particles of nature, this failure of the classical approximation must occur at some finite field strength because of the nonlinear couplings to other fields. For example, the electromagnetic field is coupled to the electron field; the neutrino field is coupled to all the leptons and hadrons through the weak interaction. For large values of the fields these couplings, however weak, must become important.

Suppose that a nearly singular configuration could exist, with the growth of the field near the horizon being resolved in complicated quantum physics very near the horizon. Since this physics would have to provide the hooks on which the exterior classical fields are hung, we shall not be able to establish the correctness of our speculations. However, the present work, in which we examine the simplest nonlinear model, gives some indications that such a field would be set up, in the course of gravitational collapse, or subsequent to gravitational collapse.

We shall consider the case of a massless scalar field theory with nonlinear self-coupling terms in the wave equation; this self-coupling is the additional element in our system beyond those contained in Ref. 1, or in recent discussions of black-hole radiance.[2-4] We take a $\lambda\varphi^4$ term in the Lagrangian as the only coupling of importance, so that the wave equation is

$$\varphi^{\mu}{}_{;\mu} = -4\lambda\varphi^3 . \tag{1}$$

We shall assume that the dimensionless coupling parameter, λ, is very small.[5] If λ were zero, the static, spherically symmetrical solution to (1), in Schwarzschild coordinates, which goes like r^{-1} as r approaches infinity, would have a logarithmic singularity at the event horizon, $r=r_0=2MG$. Clearly the nature of this singularity will be altered by the term $4\lambda\varphi^3$ in the field equation, since this term must become important at large field strengths.

Our original motivation in taking up this question was to see whether the effects of a very weak nonlinear coupling could itself regulate the behavior of φ near the event horizon, without departing from the classical field description as suggested above and as discussed further below. This can indeed happen for some forms of nonlinear coupling. It happens, for example, in the case of a $-\lambda\varphi^5$ interaction Lagrangian,[6] a field theory which, as a quantum theory, is probably unacceptable in flat space, because there is no lowest-energy state.

However, in the model with pure $\lambda\varphi^4$ coupling the singularity at the horizon is not eliminated by the nonlinear term; it is exacerbated instead, the behavior now being as $(r-r_0)^{-1/2}$ near the horizon. We argue nonetheless for the possible admissibility of this solution, or rather of a solution which grows as $(r-r_0)^{-1/2}$ up to some point very close to the horizon and then gradually disappears into a state involving a whole complex of elementary particles, in which the classical field φ is no longer defined.

These events, involving huge energy densities, would probably be confined to such a small volume as not to perturb significantly the large-scale gravitational structure, so that we can continue to neglect the back reaction on the metric. As an example let us consider the singular solution for the field φ, which is developed in the next section,

$$\varphi \underset{r\to r_0}{\sim} [r_0(r-r_0)\lambda]^{-1/2}4^{-1} . \quad (2)$$

Suppose that the description in terms of a single self-coupled field breaks down when the energy density ϵ_φ associated with the field φ reaches some value, say $\epsilon_\varphi c^{-2}=10^{15}$ g/cm^{-3}. The order of the thickness of the nonclassical layer, d, is then determined by

$$\epsilon_\varphi \sim \hbar c \left(\frac{\partial\varphi}{\partial r}\right)^2 \left(1-\frac{r_0}{r}\right)\Bigg|_{r=r_0+d}$$

$$\sim \hbar c\, r_0^{-2}\lambda^{-1}d^{-2} . \quad (3)$$

The total energy in this layer, E_L, and in the exterior classical field φ_c will be of the order of

$$E_L \sim r_0^2\epsilon_\varphi d \sim (\hbar c)^{1/2} r_0^{-1}\lambda^{-1/2}\epsilon_\varphi^{1/2} . \quad (4)$$

The fraction this represents of the mass energy of the star, $Mc^2 \sim r_0 c^4 G^{-1}$, is

$$R = E_L/Mc^2 \sim G\lambda^{-1/2}c^{-7/2}\hbar^{1/2}\epsilon_\varphi^{1/2} . \quad (5)$$

Setting, e.g., $\epsilon_\varphi c^{-2}=10^{15}$ g/cm^3 gives

$$R \sim 3.4\lambda^{-1/2}\times 10^{-40} . \quad (6)$$

According to this estimate the energy of the surface layer and the outlying scalar field is such a small fraction of the black hole's energy, for any reasonable conditions, that there will be very little perturbation of the metric.

Since the nonlinear coupling term, $-4\lambda\varphi^3$, in (1) did not, of itself, regulate the behavior at the horizon, and we are invoking the idea of a boundary layer instead, one might ask at this point whether the nonlinear coupling serves a purpose. The answer is that, as we shall see in the next two sections, the nonlinear term not only determines the way in which the static solution grows large near the event horizon, but it provides a mechanism for the singular behavior to grow in time. In the case of a free equation of motion, $\lambda=0$, the logarithmic singularity of the classical solution will not develop in time in the same manner.

The plan of the remainder of this paper is first to demonstrate the existence of the "nearly singular" static solution (Sec. II), and then to give some considerations, based on the time-dependent equation of motion, which indicate that the singular static configuration should be the final state of a system, given a variety of initial conditions (Secs. III and IV). However, these initial conditions all have some sort of singularity at the event horizon at the outset; they therefore beg the question of how singular (or nearly singular) behavior was established in the beginning. In Sec. V we mention three possible (perhaps not exclusive) ways in which such behavior might be established, and we do a numerical estimate in the case of one such way (trapped Hawking radiation). In Sec. VI we consider the modification in the effective potential for propagating waves (or quanta) of the scalar field, in the presence of the static classical field. This provides a dynamical basis for the trapping of Hawking radiation.

II. A STATIC SOLUTION

In Schwarzschild coordinates a spherically symmetical solution of (1) obeys

$$\frac{\partial^2}{\partial t^2}\varphi - r^{-2}\left(1-\frac{r_0}{r}\right)\frac{\partial}{\partial r}\left[r^2\left(1-\frac{r_0}{r}\right)\frac{\partial\varphi}{\partial r}\right] = -4\lambda\varphi^3\left(1-\frac{r_0}{r}\right) . \quad (7)$$

By direct substitution we can verify that time-independent solutions to (7) may behave near the event horizon as

$$\varphi(r) \underset{r\to r_0}{\sim} 4^{-1}(\lambda r_0)^{-1/2}(r-r_0)^{-1/2} + \text{less singular terms}. \quad (8)$$

We have shown that there exists a time-independent solution of (7) which has the behavior of (8), and which approaches zero at infinity. The demonstration was computational, through the following steps:

(a) At an arbitrarily chosen point $r_1=1.2r_0$ between r_0 and infinity a range of initial values $\varphi(r_1)$, $\partial\varphi/\partial r_1$ was chosen:

$$0.4 < 4\lambda^{1/2}(r_1-r_0)^{1/2}r_0^{1/2}\varphi(r_1) < 1.4 ,$$
$$-4 < 4\lambda^{1/2}r_0^{3/2}\frac{\partial}{\partial r}(r-r_0)^{1/2}\varphi(r)\Big|_{r_1} < 0 . \quad (9)$$

It was found that for each $\varphi(r_1)$ in this range, there is one initial value of $\partial\varphi/\partial r_1$ for which the solution extends to infinity without developing a singularity at some intermediate point.[7] This leads to a locus of acceptable initial conditions, for connecting to infinity, shown as one of the curves of Fig. 1. Next, the same procedure was applied, integrating inwards [after making appropriate changes of variables to remove the $(r-r_0)^{-1}$ factor from the field variable and to spread out the region between r_0 and r_1 into an infinite region]. Again, for each $\varphi(r_1)$ there is only one value of $\partial\varphi/\partial r_1$ which leads to a solution extending to the horizon. This solution has the behavior of (8) for r near r_0. The locus of the initial values of φ and $\partial\varphi/\partial r$ is plotted as the second curve of Fig. 1.

Since the curves intersect at only one point, there is, within the range of initial values in which we searched, one solution which extends to infinity and has the behavior of (8) near the horizon. This is the classical solution, φ_c, the significance of which we investigate below. It is plotted in Fig. 2. For large r the function $\varphi_c(r)$ falls off faster than r^{-1}. Note that when the parameter λ is varied the scale of the whole solution varies as $\lambda^{-1/2}$; φ_c becomes very large for small λ.

III. TIME EVOLUTION OF THE SINGULARITY

Ideally we would solve the time-dependent wave equation (7) for various initial conditions in order to study the possible growth of the singularity. Or, better, we would solve the problem in a collapsing geometry. These would be difficult tasks. However, if we assume a limiting form for the solution approaching the event horizon,

$$\varphi(r) \underset{r\to r_0}{\sim} b(t)(r-r_0)^{-1/2}(r_0)^{-1/2} + \text{less singular terms}, \qquad (10)$$

then we can obtain consistency with the equation of motion (7) if the coefficient $b(t)$ obeys the ordinary differential equation,

$$r_0^2\frac{d^2b}{dt^2} - \tfrac{1}{4}b = -4\lambda b^3 . \qquad (11)$$

This can be seen by looking at the most singular terms in (7), that is, those terms which behave as $(r-r_0)^{-1/2}$ when (11) is substituted into (7). As can be anticipated by the form of the conservation law,

$$\frac{d}{dt}(r_0^2\dot{b}^2 - \tfrac{1}{4}b^2 + 2\lambda b^4) = 0 , \qquad (12)$$

most initial conditions $b(0), \dot{b}(0)$ will give solutions which oscillate around the minima of the potential term, $2\lambda b^4 - \frac{1}{4}b^2$, at $b = \pm 4\lambda^{-1/2}$.

Now we suppose that there is some energy dissipation mechanism, such that the dissipation is proportional to $(\partial\varphi/\partial t)^2$. This should certainly be the case in the actual physical system; if we have a field rapidly varying in time, quanta of all kinds will be emitted. In the equation for b we simply assume an extra term $-\gamma\dot{b}^2$ on the right-hand side of the conservation law, (12). The equation of motion for $b(t)$ is now

$$r_0^2\frac{d^2b}{dt^2} - \tfrac{1}{4}b + 4\lambda b^3 = -\frac{\gamma}{2}\frac{db}{dt} . \qquad (13)$$

The solutions to (13) have been investigated numerically and are found, for any initial conditions and any positive value of γ, to approach a constant value,

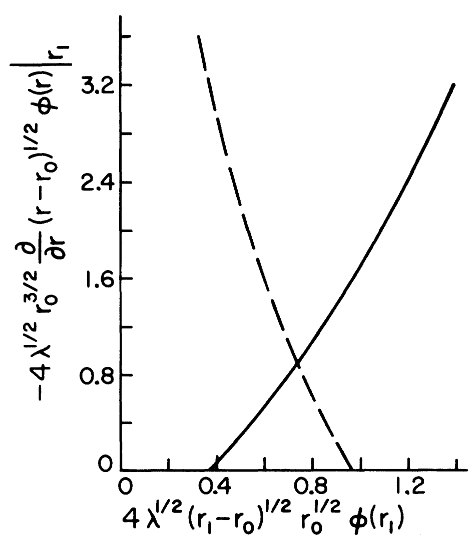

FIG. 1. Initial values at $r_1 = 1.2r_0$ for the function $\varphi(r)$ and its derivative. The solid line is the locus of initial values which lead to a solution extending to infinity. The dashed line is the locus of initial values which lead to a solution extending to the horizon and obeying the limiting condition (8). The intersection gives the initial conditions for the solution φ_c.

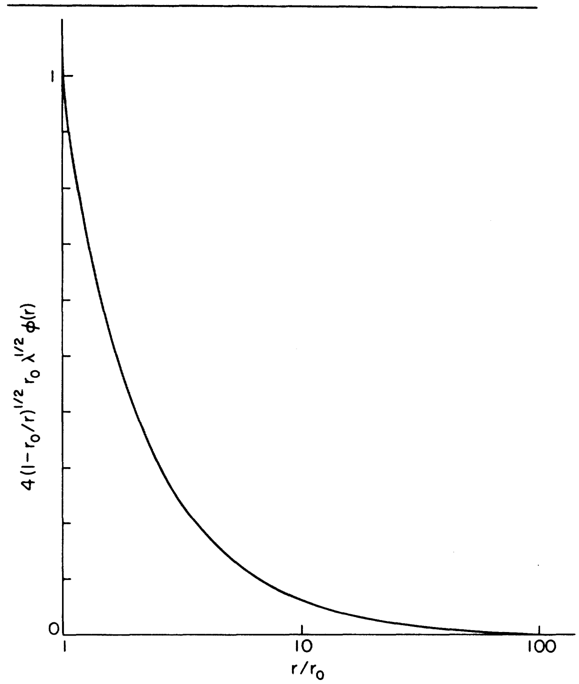

FIG. 2. The function $\varphi_c(r)$ plotted out to 100 times the event horizon.

$$b(t) \xrightarrow[t \to \infty]{} \pm 4\lambda^{-1/2}. \tag{14}$$

Thus if we begin with an arbitrarily small value of b, it will ultimately grow to the value characteristic of the time-independent solution, $\varphi_c(r)$.

It is this fact which encourages us to investigate further the possibility of the classical field $\varphi_c(r)$ being set up during or after gravitational collapse. It also provides an important distinction between a solution which is singular at the horizon, and a solution which has a singularity at some other value of r, say r_2. The static field equation admits a singularity at any point, $r_0 < r_2 < \infty$, of the form $\varphi \sim \text{const}(r - r_2)^{-1}$. Once again we could have argued that in the region of very large fields other physics could regulate the energy density. However, in contrast to the earlier case, this singularity does not change in time, as can be seen by substituting the singular term into (7). Thus it would have no way of developing from a small initial fluctuation.

It should be emphasized that the arguments in this section are merely suggestive, since we have not proved that there exists a time-dependent solution to the field equation (7) which has factorized time dependence near the event horizon, as assumed in (10). One result of the following section will be to establish the existence of such a solution in the neighborhood of the event horizon. However, this will not establish the existence of a solution everywhere in space, and at all times, which is singular only at the horizon.

IV. EVOLUTION OF THE FIELD IN SPACE AND TIME NEAR THE EVENT HORIZON

Do the results of the last section mean that if we begin with an initial condition which has some amount of singular field $\varphi \sim (r - r_0)^{-1/2}$, then the entire static field $\varphi_c(r)$ of Sec. II will inevitably develop? We think not, because of some inconsistencies in our assumptions. Recall that the physical arguments presented in Sec. I led to the conclusion that the field could be at most "nearly singular," in the sense of growing as $(r - r_0)^{-1/2}$ until, at some small value of $r - r_0 = d$, the state becomes one not describable in terms of a classical field. Thus the discussion of the time growth of the singularity is not entirely convincing, since it only describes the limit $r \to r_0$ of the solution of the time-dependent equation, and this limit takes us out of the domain of applicability of the equation.

Can we then discuss the growth of nearly singular behavior? Unfortunately, a complete treatment of this question would depend on exact knowledge of what goes on in the layer surrounding the event horizon, and we have, for now, no hope of detailed understanding of this layer. However, in this section and in the next, we present two fragmentary considerations which point to the possibility of the growth of "nearly singular" behavior, without going into the mechanics of the transitional layer.

To discuss the time and space evolution for r very near r_0, it is convenient to introduce a new space coordinate, y,

$$r = \frac{r_0}{1 - \exp(y)}, \tag{15}$$

such that the region near the event horizon is given by large negative values of y. We also introduce a new field variable, ψ,

$$\varphi(y, t) = \exp\left(-\frac{y}{2}\right)\psi(y, t), \tag{16}$$

such that the expected singular behavior $\varphi \sim (r - r_0)^{-1/2}$ as $r \to r_0$ is replaced by $\psi(y) \to \text{const}$ as $y \to \infty$. Equation (7) becomes

$$\psi_{tt} + r_0^{-2}(-\psi_{yy} + \psi_y - \tfrac{1}{4}\psi)[1 - \exp(y)]^4 = -4\lambda\psi^3. \tag{17}$$

For large negative y we can discard the $\exp(y)$ term in (17). The resulting equation is

$$\psi_{tt} + r_0^{-2}(-\psi_{yy} + \psi_y - \tfrac{1}{4}\psi) = -4\lambda\psi^3. \tag{18}$$

We note that the dependence of the field φ near the event horizon assumed in Sec. III, with factorized space and time dependence for the function φ, is characteristic of solutions for ψ which are independent of the variable y. In this case $\psi(t)$ is simply the function $r_0^{-1}b(t)$, where $b(t)$ is as defined in (10). However, since the coefficients of ψ and its derivatives in (18) are independent of the variables y and t, we can also find special solutions of the form

$$\psi(y, t) = f(\xi), \tag{19}$$

where $\xi = y + at$.

The function $f(\xi)$ is then determined by solving the ordinary differential equation

$$f_{\xi\xi}(a^2 - r_0^{-2}) + (f_\xi - \tfrac{1}{4}f)r_0^{-2} + 4\lambda f^3 = 0 \tag{20}$$

or

$$\frac{d}{d\xi}[2^{-1/2}(a^2 - r_0^{-2})f_\xi^2 - \tfrac{1}{8}r_0^{-2}f^2 + \lambda f^4] = -f_\xi^2 r_0^{-2}. \tag{21}$$

The second form, (21), looked at as a conservation law with a dissipation term on the right-hand side, suggests that all solutions approach one of two possible limits at $\xi = \infty$,

$$f(\xi) \xrightarrow[\xi \to \infty]{} \pm 4^{-1}\lambda^{-1/2}r_0^{-1}. \tag{22}$$

These are the values which minimize the analog of the potential energy in the conservation law, (21). Numerical computations show this to be the case. A typical example of f is plotted in Fig. 3.

If the parameter a is taken as positive, so that the solution moves to the left (toward the event horizon), and the parameter $a > r_0^{-1}$, we see that at a fixed value of y, no matter how negative, and at sufficiently large times, the field will approach the value (22), which characterizes the time-independent solution $\varphi_c(r)$, near the horizon

$$\varphi_c(r) \underset{r\to r_0}{\sim} (4\lambda)^{-1/2}(r-r_0)^{-1/2} r_0^{-1/2}. \tag{23}$$

We now suppose that at some large value of $-y$, such as $y = \ln 10^{-10}$, for the case of an $r_0 = 1$-cm black hole with a surface layer $d \approx 10^{-10}$ cm thick, we encounter the unknown physics of the boundary layer. In the present case we have only to assume that this layer absorbs the wave moving into it from the right; then the solution will be given by

$$\begin{aligned}\varphi(r,t) &= \exp\left(\frac{-y}{2}\right) f(y+at)\\ &= \left(1-\frac{r_0}{r}\right)^{-1/2} f\left(\ln\left(\frac{r-r_0}{r_0}\right)+at\right)\end{aligned} \tag{24}$$

for all times, at all points $r > r_0 + d$. Since for

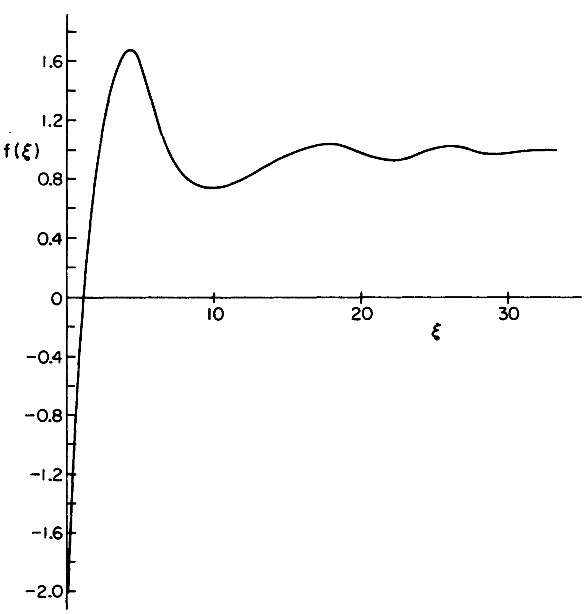

FIG. 3. The function $4\lambda^{1/2} r_0 f(\xi)$, where $f(\xi)$ is a solution to (20). We chose $a = (2r_0^{-1})^{1/2}$, $4\lambda^{1/2} r_0 f(\xi = 0) = -2$, $4\lambda^{1/2} r_0 f'(0) = 2$ in this solution. Note that for large ξ the quantity $4\lambda^{1/2} r_0 f(\xi)$ approaches the value 1, which characterizes the time-independent solution φ_c near the horizon. Under the above conditions ($a > 0$) this configuration moves in time to the left, toward the horizon, leaving behind the solution φ_c.

large enough times, at any $r > r_0$ the function f approaches one of the values given by (22), the field eventually settles into the static configuration, φ_c.

It should be emphasized that this solution is valid only in the region of space

$$d < r - r_0 \ll r_0,$$

where the latter inequality is to ensure that $\exp(y) \ll 1$, in order to obtain (18). However, the present considerations establish that there do exist solutions in the region near the horizon which confirm the limiting behavior at large time derived in Sec. III for the case in which the solution was extended all the way to r_0. However, this special solution may not be appropriate for the boundary conditions of the problem, since it has an enormous flux of φ-field energy inwards at early times.

The particular solutions presented here do not require the introduction of a damping term, such as was introduced in Sec. III, in order to settle down at long times. However, a small damping term of the same type as introduced in Sec. III (energy loss proportional to $\dot{\varphi}^2$) does not qualitatively disturb the results of this section.

V. GROWTH FROM A VACUUM FLUCTUATION

The considerations on the time development presented in the last two sections presuppose that at some initial time the field is singular or nearly singular at the event horizon. Suppose we ask the question of whether, beginning at an initial time with a nonsingular field, a singular field would develop in time. We have already seen that this probably does not happen in the static Schwarzschild geometry, if φ is a classical field, and it is doubtful that it would happen in the geometry of a collapsing star.

However, the latter possibility is not ruled out, so that as the first possible way of investigating the development of nearly singular behavior we suggest:

(a) The solution of the collapse problem with the particles in the collapsing cloud coupled (as a source) to the scalar field, and the φ^4 interaction included in the field equations. Since this picture does not describe the physics of our boundary layer, the question which should be asked is probably whether the scalar field is becoming large near the point where the horizon is going to form, prior to its formation.

As other possible ways we suggest:

(b) The transition from $\langle\varphi\rangle = 0$ to $\langle\varphi\rangle \neq 0$ may be spontaneous. That is, it may require no source of field φ. Phase transitions of the vacuum of this kind are under intensive study for their possible applications to other problems in physics. However, it is obscure to us exactly how the problem can be attacked in the present context.

(c) The trapping of Hawking radiation. We look at a vacuum fluctuation as a seed for the development of the classical field and obtain the following interesting result: When we estimate the energy flux outward from some point slightly outside of the event horizon at a time shortly after the creation of the fluctuation (a time given by $\hbar$ divided by the energy of the fluctuation), we find a result of the order of Hawking's result for black-hole radiance.[2] This suggests a sequence of events in which, after a small singular component of the field has been established through a vacuum fluctuation, some of the energy which would have been radiated by the Hawking mechanism is trapped, by the new potential barrier discussed in the next section, and goes into the building of the classical field, φ_c. However, we do not know how to work out the details of the process.

The estimate of the initial energy flux goes as follows: Consider a field which behaves near the event horizon as

$$\begin{aligned}\varphi_{\rm fluct} &\sim \zeta(r-r_0)^{-1/2}, \quad r_0 \gg r-r_0 > d \\ &\sim \zeta d^{-1/2}, \quad r-r_0 < d.\end{aligned} \tag{25}$$

For small values of d the energy of this fluctuation will be governed by the gradient terms in the Hamiltonian.

$$\begin{aligned}E &= 2\pi\int r^2 dr\left(1-\frac{r_0}{r}\right)\left(\frac{\partial\varphi}{\partial r}\right)^2 \\ &\sim \zeta^2 r_0 d^{-1}.\end{aligned} \tag{26}$$

This fluctuation can endure for a time $\Delta t \approx \hbar\zeta^{-2}r_0^{-1}d$. Taking

$$\varphi(r,t=0)=\varphi_{\rm fluct}(r) \quad \text{and} \quad \frac{\partial\varphi}{\partial t}(r,t=0)=0$$

as initial values for the classical field equation, we can calculate $\partial\varphi/\partial t$ at small values of t, from the field equation, (7), or, if we want the dominant term near the horizon, from the equation of evolution of the singularity, (11). From the latter we find that

$$\frac{\partial\varphi(r,\Delta t)}{\partial t} \sim \hbar d\zeta^{-1}(r-r_0)^{-1/2}, \tag{27}$$

as long as the scale of the fluctuation is small enough to neglect the cubic term in (11), $\zeta \ll \lambda^{-1/2}r_0^{-1/2}$. The outgoing radial energy flux is given by

$$\text{flux} = -\left(1-\frac{r_0}{r}\right)\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial t}. \tag{28}$$

Using (25) to compute $\partial\varphi/\partial r$ and using (27) for $\partial\varphi/\partial t$ we find that the flux at $r-r_0=d$, or $r-r_0$ = (a few times d), is of the order of

$$\text{flux} \sim \frac{\hbar}{r_0^4}. \tag{29}$$

The significant results of this estimate are that (a) the order of magnitude of the flux is independent of the scale of the initial fluctuation, (b) the order of magnitude of the flux is independent of the thickness of the surface layer, d, and (c) the order of magnitude of the flux is the same as that calculated by Hawking as steady-state emission of massless scalar quanta from a black hole.[2]

Thus, although we do not know how to establish the development of the field φ_c during gravitational collapse, we have some indication that the same energy source that provides the radiation of φ quanta in the linear case can be drawn on to build φ_c.

VI. NEW EFFECTIVE POTENTIAL

The classical background field φ_c in conjunction with the nonlinear term in the Lagrangian density, $-\lambda\varphi^4$, makes an important modification in the effective potential for scalar waves propagating in the Schwarzschild metric. In terms of the coordinate r^*,

$$r^* = r + r_0\ln(r/r_0 - 1),$$

the linear radial wave equation in the absence of $\mathcal{L}_I$ and φ_c takes the form[8]

$$-\varphi^l_{tt} + \varphi^l_{r^*r^*} = \left(1-\frac{r_0}{r}\right)\left[\frac{r_0}{r^3} + \frac{l(l+1)}{r^2}\right]\varphi^l. \tag{30}$$

In the presence of φ_c and $\mathcal{L}_I$ we define a "quantized" field, for the wave, by

$$\varphi = \varphi_c + \varphi_Q. \tag{31}$$

Then $\mathcal{L}_I$ can be written as

$$\mathcal{L}_I = -\lambda(\varphi_c^4 + 4\varphi_c^3\varphi_Q + 6\varphi_c^2\varphi_Q^2 + 4\varphi_c\varphi_Q^3 + \varphi_Q^4). \tag{32}$$

The wave equation for φ_Q is

$$\varphi_{Q;\mu}^{\mu} = -12\lambda\varphi_c^2\varphi_Q - 12\lambda\varphi_c\varphi_Q^2 - 4\lambda\varphi_Q^3, \tag{33}$$

where the wave equation obeyed by φ_c has been used to remove a number of terms.

In the usual approximation of linearization we drop the last two terms on the right-hand side of (33). We note that, since $\varphi_c \sim \lambda^{-1/2}$, these terms are superficially small for small λ.

Having dropped the φ_Q^2 and φ_Q^3 terms in (33), we can now separate the angular dependence and write a radial equation in the form of (30),

$$-\varphi^l_{Q,tt}+\varphi^l_{Q,r_*r_*}=\left(1-\frac{r_0}{r}\right)\left[\frac{r_0}{r^3}+\frac{l(l+1)}{r^2}+12\lambda\varphi_c{}^2\right]\varphi^l_Q$$
$$\equiv V^l_{\rm eff}(r)\varphi^l_Q(r,t)\,. \qquad (34)$$

The function φ_c approaches $(r-r_0)^{-1/2}r_0{}^{-1/2}4^{-1}\lambda^{-1/2}$ near the horizon, so that the new term in the effective potential,

$$12\lambda\varphi_c{}^2(1-r_0/r)\underset{r\to r_0}{\sim}\tfrac{3}{4}r_0{}^{-1}\,,$$

makes a contribution to $V^l_{\rm eff}(r)$ which approaches a positive, constant, λ-independent value at $r=r_0$. This is in marked contrast to the case in which $\varphi_c=0$, in which $V_{\rm eff}$ vanishes at $r=r_0$. The new potential barrier extends all the way to the event horizon.

The barrier plays an important role in Hawking's description of black-hole radiance, where one factor in the rate for emission of quanta is the barrier penetration probability. In our modified problem this emission rate will vanish. It is not clear, however, that the problem of particle emission can be completely formulated and solved until our present assumptions are embedded in a theory of gravitational collapse.

VII. DISCUSSION

There is no known massless scalar particle in nature, and the massless φ^4 theory is pathological as a quantum theory because of infrared difficulties. (Since in the present work we did not discuss closed-loop quantum corrections, we did not encounter this pathology.) We have discussed this model because of its simplicity and because a massless scalar field has been the basis of the various treatments of black-hole radiance.

The massless particles in nature, besides the graviton, are the photon and the neutrino. However, most of the nonlinear couplings of these fields are not self-couplings, but couplings to other fields. Can we look for singular solutions of the coupled Dirac and Maxwell fields which give interesting behavior of, say, $\langle\bar\psi\psi\rangle$, $\langle\bar\psi\gamma_r\psi\rangle$, $\langle\bar\psi\gamma_0\psi\rangle$ near the horizon? Or, to pose a simpler problem, can we look at a self-coupled neutrino field with an interaction

$$\mathcal{L}_W=G_W[\bar\psi_\nu\gamma_\mu(1+\gamma_5)\psi_\nu][\bar\psi_\nu\gamma^\mu(1+\gamma_5)\psi_\nu]$$

and find behavior analogous to that which we found in the scalar case?

Even here the development cannot be absolutely parallel. There is no classical Fermi field ψ_ν. What we might anticipate is singular behavior of $\langle\psi_\nu^\dagger\psi_\nu\rangle$. This would be in disagreement with Hartle's result on the vanishing of neutrino pair fields of a black hole,[9] but it would be based on relaxing the demands on the behavior of the neutrino solutions at r_0, just as the present work has been based on relaxing the conditions for a scalar field.

In the self-coupled neutrino case, the only approach which suggests itself immediately is a self-consistent calculation of

$$\langle{\rm vac}|\bar\psi_\nu\gamma_\mu\psi_\nu|{\rm vac}\rangle$$

by making something roughly like the Hartree approximation,

$$(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)\approx(\bar\psi\gamma^\mu\psi)\langle{\rm vac}|\bar\psi\gamma_\mu\psi|{\rm vac}\rangle\,,$$

but this is not an adequate approach in our opinion. Thus we see no simple solution of the nonlinear neutrino force problem.

To sharpen our point on the inevitability of interactions which are usually negligible becoming significant in the presence of strong fields, we estimate the values of electromagnetic fields and neutrino pair fields above which couplings will have to be taken into account in discussing vacuum propagation of photons or neutrinos, respectively. In the case of electromagnetism an electric or magnetic field in excess of $m_e{}^2c^3e^{-1}\hbar^{-1}$ leads to significant effects on the propagation of light. The (energy density)/c^2 in this case is of the order of 10^6 g/cm^3. In the case of a static neutrino pair field

$$\langle\nu^\dagger(\vec{x})\nu(\vec{x})\rangle=\chi(\vec{x})\,. \qquad (35)$$

We find, assuming a neutral-current coupling of strength $G_W\approx10^{-5}\hbar^3c^{-1}M_p{}^{-2}$, that if

$$G_W\chi>r_0{}^{-1}\hbar c\,, \qquad (36)$$

then the propagation of neutrinos of wavelength $\geq r_0$ (which are those which predominate in the black-hole emission process) is significantly affected. The energy density ϵ_χ divided by c^2 of this pair field is estimated to be of the order of $\hbar c^{-1}\chi^{4/3}$, so that

$$\epsilon_\chi/c^2>(r_0G_W)^{-4/3}\hbar^{7/3}c^{1/3}$$
$$\sim r_0{}^{-4/3}\times10^5\ {\rm g/cm^3}\ \ (r_0\ {\rm in\ cm})\,. \qquad (37)$$

Thus in the photon and neutrino cases the nonlinear interactions become important at energy densities which are low compared to that assumed in making the estimate of (6).

These considerations involving photons and neutrinos are all a little vague. However, we have made, in our scalar field considerations, a beginning in treating the simplest model with a nonlinear coupling. The results indicate the possibility that particle interactions can alter both the long-accepted "no-hair" theorems proved at a classical level for free field theories and the properties of the recently discovered black-hole radiance, which results when the free field theory is quantized.

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[5]We choose units for φ of cm^{-1}, so that the coupling constant λ is dimensionless. The energy density is then of the form $\hbar c$ times a function of the fields which does not contain any factor of $\hbar$. This is why we find, below, factors of $\hbar$ in classical results. We do this because "weak coupling" is defined only at the quantum level.

[6]In this case there is a time-independent solution φ_c to the field equation $\varphi^{;\mu}_{;\mu} = 5\lambda\varphi^4$, which is $\varphi_c = (5\lambda)^{-1/3} r^{-1} \times r_0^{-1/3}$.

[7]Of course, in the actual computation, an improved choice of the initial derivative merely pushes the point of blowup farther out toward infinity. We stopped improving the initial conditions when the blowup was at more than $r = 500 r_0$. This solution dies off faster than r^{-1} up to a distance greater than $r = 300 r_0$.

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