Effective strong-interaction Lagrangian*

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We discuss the current-algebra treatment of the basic quark-model Lagrangian, which contains in addition the simplest (from a group-theoretical point of view) effective term that can solve the U(1) problem. Reasonable experimental agreement is found for the relevant pseudoscalar masses. It is shown how the σ-model solution for the masses is mapped into a whole family of solutions in this generalized quark-model case. A possible way of consistently getting Gell-Mann–Okubo–type mass relations while not going to the approximation of an exactly symmetric vacuum is pointed out. This leads to a simple relation among the pseudoscalar-meson decay constants.

I. EFFECTIVE STRONG-INTERACTION LAGRANGIAN

One expects that the fundamental structure of the strong interactions should be most clearly exposed at low energies, before unitarity corrections become important. This has led to the development, primarily inspired by Gell-Mann,[1] of a fundamental quark Lagrangian whose parameters could be partially determined by study of the low-lying spin-zero meson spectrum. In a schematic form this Lagrangian is

$$\mathcal{L} = -\sum_{a=1}^{3}\bar{q}_a\gamma_\mu\partial_\mu q_a + [\text{renormalizable } \mathrm{SU}(3)\times\mathrm{SU}(3)\text{-invariant interaction}] - \sum_{a=1}^{3} m_a\bar{q}_a q_a, \tag{1.1}$$

where the q_a are the quark fields and the m_a are their masses. Equation (1.1) may evidently be extended to the case of N quarks by changing 3 to N. For simplicity, however, we shall work first with the three-quark model. For the treatment of (1.1) it has been recognized that in addition to the explicit symmetry breaking represented by the last term, there should be spontaneous symmetry breaking (an unsymmetric vacuum) which in fact determines the main features of the low-lying multiplet structure. Furthermore, the requirement of getting the correct statistics for the ground-state baryons strongly suggests that each quark field should have an additional trivalent "color" label. The physical particles are taken to be color singlets, so we shall not write the color indices explicitly here.

The motivation for (1.1) originally was maximum symmetry of the interaction and maximum simplicity of the symmetry-breaking term. It has since been recognized[2] that the symmetry-breaking term could naturally originate from a weak-electromagnetic gauge theory as a term of the form $\bar{q}q\langle\phi\rangle_0$, where ϕ is a Higgs boson, possibly modified by radiative corrections. When one investigates the structure of the invariant interaction in more detail, on the other hand, a problem develops. Note that the interaction term was considered to be $\mathrm{SU}(3)\times\mathrm{SU}(3)$ symmetric. However, the first term in (1.1) actually is invariant under the larger symmetry group $\mathrm{U}(3)\times\mathrm{U}(3)$. Thus, on the grounds of maximum symmetry one should have considered the interaction to be $\mathrm{U}(3)\times\mathrm{U}(3)$ symmetric. In fact, the most reasonable choice for the strong interaction—a set of 8 Yang-Mills gauge bosons coupled to the color degree of freedom—does possess this $\mathrm{U}(3)\times\mathrm{U}(3)$ invariance. In such a case, the application of standard current-algebra techniques leads to a mass spectrum for the isoscalar 0^- particles, in violent disagreement with experiment. This has become known as the U(1) problem.[3]

Here we would like to suggest and investigate a possible effective strong-interaction Lagrangian which may be the simplest one to overcome the U(1) problem. This Lagrangian is

$$\mathcal{L} = -\sum_{a=1}^{3}\bar{q}_a\gamma_\mu\partial_\mu q_a + [\mathrm{U}(3)\times\mathrm{U}(3)\text{-invariant color gauge interaction}] - \sum_{a=1}^{3} m_a\bar{q}_a q_a - \mathrm{U}[\det\bar{q}(1+\gamma_5)q + \det\bar{q}(1-\gamma_5)q]. \tag{1.2}$$

Here U is a new real parameter and

$$\det\bar{q}(1\pm\gamma_5)q = \frac{1}{3!}\sum \epsilon_{abc}\epsilon_{a'b'c'}\bar{q}_a(1\pm\gamma_5)q_{a'}\bar{q}_b(1\pm\gamma_5)q_{b'}\bar{q}_c(1\pm\gamma_5)q_{c'}.$$

(Again, color indices are suppressed, and the extension to four quarks is trivial.) The new term breaks U(3) $\times$ U(3) down to SU(3) $\times$ SU(3) $\times$ (baryon number) and is the simplest object which does so. It is clear that this term must be an *effective* rather than a fundamental term since it is not renormalizable in the usual way. One other motivation, beyond simplicity, is that such an extra term does occur in the SU(3) σ model.[4] This model, as we shall explicitly note again here, gives results very similar to the present one and does not have a U(1) problem. Thirdly, perhaps the most fascinating motivation for such a term is based on some recent work[5] of 't Hooft. He has pointed out that the vacuum of a gauge theory may be a rather complicated object corresponding to an infinite number of local minima with tunneling between them. When he calculates the transition amplitudes in such a (color gauge) theory he finds an effective term of the same *general form* as the new one in (1.2). Actually, his form is somewhat more complicated and may also include *CP* violations. Thus, we can only say that our term may turn out to be a reasonable approximation to what emerges from a more detailed study of the gauge-theory vacuum. We should also mention that 't Hooft's work is based on that of Belavin, Polyakov, Schwartz, and Tyupin[6] and that additional discussion of the underlying physics has been given by Jackiw and Rebbi[7] and Callan, Dashen, and Gross.[8] Furthermore, Pagels[9] has expressed a similar idea.

The rest of this paper is arranged as follows. In Sec. II the mass formulas for the low-lying spin-zero mesons are found, and the ranges of the parameters which ensure the positivity of these masses is given. In Sec. III we display the U(1) problem explicitly and give our proposed solution which leads to some fairly reasonable numerical predictions. The connection with the σ model is explored in Sec. IV. There it is shown that the mass formulas for the quark model can be mapped into those of the σ model. The σ-model solution gives rise to a whole family of quark-model solutions. We also mention a possible way in which Gell-Mann–Okubo-type mass formulas can be obtained from the current-algebra approach, nevertheless allowing a *nonsymmetric* vacuum. This leads to an amusing formula connecting the decay constants of the pseudoscalars. Finally, in Sec. V we briefly discuss the $\eta \to 3\pi$ decay and some extensions of this work.

II. MASS FORMULAS AND PARAMETER RANGES

By standard current-algebra arguments one can get a formula for the pseudoscalar-meson masses based on the idea that they have zero mass (i.e., are Nambu-Goldstone bosons) to zeroth order and pick up mass as a result of the non-U(3) $\times$ U(3)-invariant terms in (1.2) considered as a first-order perturbation. This formula[10] is

$$\sum_{A,B} F_A^a F_B^b (M^2)_{AB} = -\tfrac{1}{2}\int d^3x\, d^3y \langle 0 \| [P_4^a(\vec{x},0),[P_4^b(\vec{y},0),\mathcal{L}]] | 0\rangle + (a \leftrightarrow b). \tag{2.1}$$

Here $(M^2)_{AB}$ is the mass-squared matrix of the pseudoscalar fields expressed in some basis. $P_\mu^a(\vec{x},t)$ is the ath pseudovector current, while F_A^a is a number (decay constant) obtained by sandwiching the P_μ^a between the vacuum and state A. Note that state A may in general correspond to some linear combination of the usual fields. Substituting (1.2) into (2.1) gives the squared masses of the pion and kaon:

$$m_\pi^{\,2} = \frac{-1}{F_\pi^{\,2}}(m_1+m_2)(\langle \bar{q}_1 q_1\rangle_0 + \langle \bar{q}_2 q_2\rangle_0), \tag{2.2}$$

$$m_K^{\,2} = \frac{-1}{F_K^{\,2}}(m_1+m_3)(\langle \bar{q}_1 q_1\rangle_0 + \langle \bar{q}_3 q_3\rangle_0). \tag{2.3}$$

F_π and F_K are conventional; our axial-vector current is normalized so that numerically $F_\pi \simeq m(\pi^0)$. Corresponding to the nonconserved vector currents one can derive a formula similar to (2.1) for some of the possible scalar masses. This gives a formula for the mass squared of a particle with the quantum numbers of the κ meson:

$$m_\kappa^{\,2} = \frac{-1}{F_\kappa^{\,2}}(m_1-m_3)(\langle \bar{q}_1 q_1\rangle_0 - \langle \bar{q}_3 q_3\rangle_0). \tag{2.4}$$

Finally, setting

$$P_\mu^a(x) = i\bar{q}_a(x)\gamma_\mu\gamma_5 q_a(x), \tag{2.5}$$

we find for the matrix on the right-hand side of (2.1) specialized to the subspace of neutral nonstrange pseudoscalars;

$$C(\tilde{U}) \equiv -4\begin{pmatrix} \tilde{m}_1+\tilde{U} & \tilde{U} & \tilde{U} \\ \tilde{U} & \tilde{m}_2+\tilde{U} & \tilde{U} \\ \tilde{U} & \tilde{U} & \tilde{m}_3+\tilde{U}\end{pmatrix},$$

$$\tilde{m}_a = m_a\langle \bar{q}_a q_a\rangle_0, \qquad \tilde{U} = U\langle [\det\bar{q}(1+\gamma_5)q + \det\bar{q}(1-\gamma_5)q\,]\rangle_0. \tag{2.6}$$

We see that the new quantity U makes its first appearance in (2.6).

One interesting question is: What are the allowed ranges of the various parameters in the theory? The study of this question was begun by Mathur and Okubo.[11] We will give the restrictions which are imposed by requiring the squared masses above to be positive. Here we go to the limit of isotopic-spin invariance by setting

$$m_1 = m_2, \quad \langle \bar{q}_1 q_1 \rangle_0 = \langle \bar{q}_2 q_2 \rangle_0. \tag{2.7}$$

For convenience we define

$$R = \frac{m_3}{m_1}, \quad W = \frac{\langle \bar{q}_3 q_3 \rangle_0}{\langle \bar{q}_1 q_1 \rangle_0}. \tag{2.8}$$

The relevant parameters for discussing particle masses are then (apart from the decay constants) four in number:

R, W, $\tilde{m}_1$, $\tilde{U}$.

The condition on $\tilde{m}_1$ is trivial; the positivity of m_π^2 in (2.2) requires

$$\tilde{m}_1 = m_1 \langle \bar{q}_1 q_1 \rangle_0 < 0. \tag{2.9}$$

From (2.3) and (2.4) we see that positivity implies both

$$(R+1)(W+1) > 0 \quad \text{and} \quad (R-1)(W-1) > 0. \tag{2.10}$$

This limits R and W to the three regions shown in Fig. 1.

To proceed with the discussion of (2.6) we first make the simplifying assumption that there are just three low-lying neutral nonstrange pseudoscalars which are quark-antiquark composites involving only the first three quarks. We have in mind the π^0, η, and η'. The η' will be identified as the $X(960)$; recent work by Ueda[12] suggests that $E(1420)$ is not the proper choice. With this assumption the quantities F_A^a in (2.1), with A restricted to this subspace, form a square matrix, and we may write in matrix notation

$$FM^2F^T = C, \tag{2.11}$$

C being given by (2.6). Then if $\det C \neq 0$ (the special case[13] $\det C = 0$ may be handled separately), F must be a nonsingular matrix since $(\det F)^2(\det M^2) = \det C$. Then it may be seen that the positivity of M^2 is equivalent to the positivity of C. Thus it is sufficient to examine the secular equation which results from (2.6). Immediately, using (2.2) and (2.7), we find that one eigenvalue is m_π^2. The positivity of the remaining two eigenvalues results in both the following equations:

$$\tilde{U} < \tfrac{1}{12} F_\pi^2 m_\pi^2 (1+RW),$$
$$(1+2RW)\tilde{U} < \tfrac{1}{4} F_\pi^2 m_\pi^2 RW. \tag{2.12}$$

For the old case where $\tilde{U} = 0$ these simply lead to

$$RW > 0, \tag{2.13}$$

so that the allowed regions in the second and fourth quadrants in the RW plane are eliminated. In the present case we may see that (2.12) implies, for any value of $\tilde{U}$, the weaker requirement

$$RW > -\tfrac{1}{2}. \tag{2.14}$$

This chops out the cross-hatched region in Fig. 1. Of course, for a particular $\tilde{U}$, (2.12) is more restrictive than (2.14). In fact one can transform the inequalities (2.12) to the following:

$$RW > -\frac{1}{2} + \frac{F_\pi^2 m_\pi^2}{2F_\pi^2 m_\pi^2 - 16\tilde{U}} > -\frac{1}{2},$$
$$\tfrac{1}{8} F_\pi^2 m_\pi^2 > \tilde{U}, \tag{2.15}$$

from which (2.13) and (2.14) both follow by inspection.

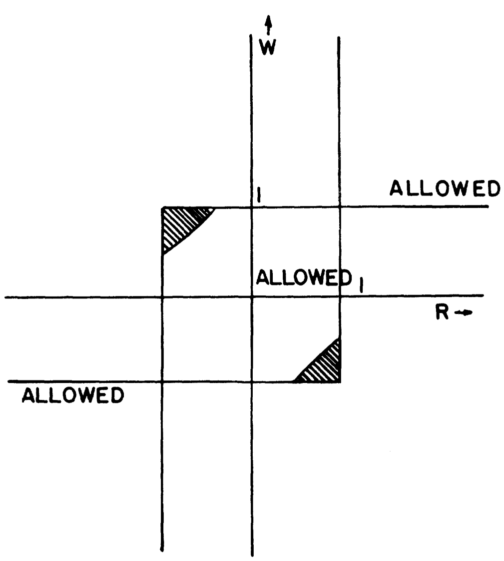

FIG. 1. The allowed regions in the R-W plane.

III. U(1) PROBLEM AND ITS POSSIBLE RESOLUTION

In the old case where $U = 0$, a bad mass spectrum results when one makes a "reasonable" choice for the decay constants which are not known from experiment. The precise "bad result" one gets depends on one's assumption. The simplest reasonable assumption is

$$F_A^a \simeq F_\pi \delta_A^a, \tag{3.1}$$

where the state labels A are considered to be in the same "group direction" as the pseudovector currents in (2.5). Then (2.6) becomes directly the following matrix of the three squared masses:

$$\frac{1}{F_\pi^2} C(0) = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & m_\pi^2 RW \end{pmatrix}. \tag{3.2}$$

Thus, one other isoscalar particle is actually degenerate with the π^0, in contradiction to experiment. If one wishes to temporarily ignore this problem, the results of Gell-Mann, Oakes, and Renner[14] may be derived using (3.1) and (3.2). In our present choice of basis the mathematical fields π^0, η, and η' are obtained by multiplication with the matrix

$$B=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}. \tag{3.3}$$

Then ignoring the fact that the mathematical η is not the physical η, we may write

$$m_\eta{}^2=\frac{1}{F_\pi{}^2}[BC(0)B^T]_{22}=\tfrac{1}{3}m_\pi{}^2(1+2R), \tag{3.4}$$

where we have set $W=1$ to make the vacuum SU(3) invariant. Taking this together with (2.2) and (2.3) in the $W=1$ limit:

$$\begin{aligned} m_\pi{}^2&=-4\tilde{m}_1/F_\pi{}^2,\\ m_K{}^2&=\tfrac{1}{2}(1+R)m_\pi{}^2, \end{aligned} \tag{3.5}$$

we recover the Gell-Mann–Okubo formula $4m_K{}^2=3m_\eta{}^2+m_\pi{}^2$ as well as $R=m_3/m\simeq 2m_K{}^2/m_\pi{}^2$. Of course, the derivation just given is wrong for the reason mentioned.

This U(1) problem may be avoided if the new term is present so that $U\neq 0$. First, let us attempt a fit to the η and η' masses when the approximations (3.1) and (3.5) as well as $W=1$ are made. This corresponds to the approximations of a U(3)-invariant vacuum and U(3)-invariant decay constants. In this case the matrix of squared masses of the π^0, η, and η' is given by

$$M^2=\frac{1}{F_\pi{}^2}\,C(\tilde{U}), \tag{3.6}$$

$C(\tilde{U})$ being defined by (2.6). Diagonalizing this matrix and using (3.5) results in the following formula for the squared masses of the η and η':

$$\begin{aligned} m^2(\eta,\eta')&=m_K{}^2+3b\pm[(m_K{}^2+3b)^2-2b(4m_K{}^2-m_\pi{}^2)\\ &\qquad -m_\pi{}^2(2m_K{}^2-m_\pi{}^2)]^{1/2}, \\ b&=-2\tilde{U}/F_\pi{}^2. \end{aligned} \tag{3.7}$$

We interpret the plus sign as giving the η' squared mass and the minus sign giving the η squared mass. A plot of these as a function of b is shown in Fig. 2. Interestingly enough, despite the simplicity of our approximation, both masses come out pretty well for $b\simeq 7m^2(\pi^0)$. The η-η' mixing angle θ may also be calculated to be[15]

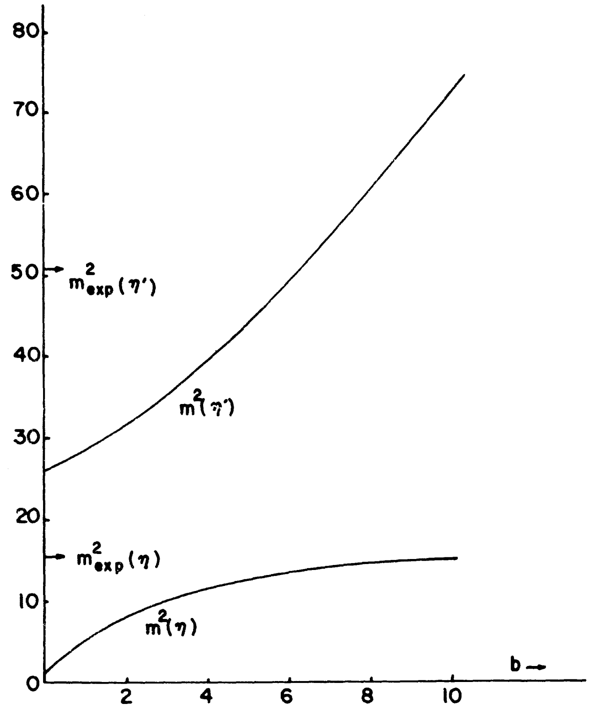

FIG. 2. $m^2(\eta)$ and $m^2(\eta')$ plotted against b, all in units of $m^2(\pi^0)$.

$$\tan 2\theta=\frac{2\sqrt{2}(\tilde{m}_3-\tilde{m}_1)}{\tilde{m}_3-\tilde{m}_1-9\tilde{U}}. \tag{3.8}$$

For $b\simeq 7m^2(\pi^0)$ one gets $\theta\simeq -18°$, which is reasonable but a little too large in magnitude.[16] Thus, three quantities can be correlated in terms of the single parameter $\tilde{U}$, which may turn out to have some fundamental significance. A slightly different fitting of the mass spectrum will be discussed in the next section.

IV. CONNECTION WITH THE σ MODEL

The linear σ model is constructed out of scalar fields S_a^b transforming like $\bar{q}_b q_a$ and pseudoscalar fields ϕ_a^b transforming like $\bar{q}_b\gamma_5 q_a$. The Lagrangian density may be written[4]

$$\begin{aligned} \mathcal{L}=&-\tfrac{1}{2}\sum_{a,b}(\partial_\mu\phi_a^b\partial_\mu\phi_b^a+\partial_\mu S_a^b\partial_\mu S_b^a)+[\text{U(3)}\times\text{U(3)-invariant nonderivative interaction}]\\ &+2\sum_a A_a S_a^a-6V_4[\det(S+i\phi)+\det(S-i\phi)]. \end{aligned} \tag{4.1}$$

Each term is analogous to and has the same chiral transformation properties as the corresponding term in (1.2). The A_a are analogous to the m_a and the quantity V_4 to the new parameter U. In order to compute masses from the σ model, no more information than that given above is needed. Actually, the Lagrangian may even contain an arbitrary function of the last term, but (4.1) is sufficient for our present purposes. In this model the symmetry breaking in the vacuum is measured by the quantities

$$\langle S_b^a \rangle_0 = \alpha_a \delta_b^a. \tag{4.2}$$

In addition, the decay constants are given by

$$\begin{aligned} F_\pi &= \alpha_1 + \alpha_2, \\ F_K &= \alpha_1 + \alpha_3, \\ F_\kappa &= \alpha_3 - \alpha_1. \end{aligned} \tag{4.3}$$

To find the spin-zero masses we may either use (2.1) or the Ward-like identities given in Ref. 4. The results are[4]

$$\begin{aligned} m_\pi^2 &= \frac{2}{F_\pi^2}(A_1 + A_2)(\alpha_1 + \alpha_2), \\ m_K^2 &= \frac{2}{F_K^2}(A_1 + A_3)(\alpha_1 + \alpha_3), \\ m_\kappa^2 &= \frac{2}{F_\kappa^2}(A_1 - A_3)(\alpha_1 - \alpha_3), \end{aligned} \tag{4.4}$$

and for the 3×3 mass-squared matrix M^2 of the π^0, η, and η' we have

$$\begin{aligned} FM^2F^T &= 4\begin{pmatrix} 2A_1\alpha_1 - 6\tilde{V}_4 & -6\tilde{V}_4 & -6\tilde{V}_4 \\ -6\tilde{V}_4 & 2A_2\alpha_2 - 6\tilde{V}_4 & -6\tilde{V}_4 \\ -6\tilde{V}_4 & -6\tilde{V}_4 & 2A_3\alpha_3 - 6\tilde{V}_4 \end{pmatrix}, \\ F &= 2\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \\ \tilde{V}_4 &= 2V_4\alpha_1\alpha_2\alpha_3. \end{aligned} \tag{4.5}$$

As might be expected these are remarkably similar to (2.2), (2.3), (2.4), and (2.6). In fact if we were to make the substitutions

$$\begin{aligned} m_a &= \frac{1}{K}A_a, \\ \langle \bar{q}_a q_a \rangle_0 &= -2K\alpha_a, \\ \tilde{U} &= 6\tilde{V}_4, \end{aligned} \tag{4.6}$$

where K is an arbitrary constant, the mass formulas for the quark model and the σ model actually coincide. [This, of course, also requires that the matrix F be chosen as in (4.5).]

One immediate consequence of this correspondence is that if we also set $R = A_3/A_1$ and $W = \alpha_3/\alpha_1$, the allowed regions in the R-W plane are exactly the same for the two models. This was previously noted in Ref. 4.

Another consequence is that if we know a numerical solution to the mass spectrum from the σ model we can generate a family of solutions to the quark model (one solution for each value of K). In Ref. 4 a fitting of the masses and decay constants was obtained with the following parameters:

$$\begin{aligned} &\tfrac{1}{2}(\alpha_1 + \alpha_2) \simeq 0.5m(\pi^0), \quad \alpha_3 = 0.86m(\pi^0), \\ &\tfrac{1}{2}(A_1 + A_2) = 0.25m^3(\pi^0), \quad A_3 = 9.05m^3(\pi^0), \\ &V_4 = -1.85m(\pi^0). \end{aligned} \tag{4.7}$$

These values were obtained by taking the π^0, K^0, η, and η' masses as well as F_π for input.[17] Output predictions included $F_K \simeq 1.37F_\pi$ and $\theta \simeq 0.3°$ as well as a variety of decay widths and scattering amplitudes. Note that (4.7) corresponds to a (slightly) SU(3)-broken vacuum since $\alpha_3 \neq \alpha_1$ and also $F_K \neq F_\pi$. This is the chief difference between this family of solutions and the SU(3)-symmetric vacuum solution (with $F_\pi = F_K$) given in Sec. III.

It is clear that the quark-model calculation gives less information than that of the σ model since the F's and the vacuum expectation values $\langle \bar{q}_a q_a \rangle_0$ are completely undetermined theoretically. In fact, we realize that the assumption of an SU(3)-invariant vacuum which we would appear to have the freedom to make in the quark model, can only be an approximation since it violates Coleman's dictum[18] in that the symmetry of the vacuum [SU(3)] is *greater* than the symmetry of the physical states [SU(2)]. The proper connection between the vacuum parameters α_a and the explicit symmetry-breaking parameters A_a is, on the other hand, automatically provided in the σ model by an extremum equation for the potential. (In principle, this may be done for the quark model too, but the calculation would appear difficult.)

Which then of the two solutions given here is a better approximation to nature? The advantage of the family of solutions (4.7) is that it corresponds to $F_K/F_\pi \neq 1$, more like the experimental $F_K/F_\pi \simeq 1.28$. On the other hand, the success of the Gell-Mann–Okubo-type mass formulas in general seems to indicate that the effective symmetry breaker behaves like an octet. From either (4.4) or (2.2) and (2.3) we see that, in general, unless the F's and vacuum expectation values are SU(3) symmetric, the symmetry breaking will have more complicated group-transformation properties. Thus, it would seem that nature may interpolate between the two solutions given.

We would like to speculate briefly here on a way in which this interpolation may be achieved. We shall work with the SU(4) version of the model given here, since in the SU(3) case the differences between the two kinds of solutions may not show up so drastically. Since a tensor notation was used, all the basic formulas here have obvious generalizations to higher SU(N) groups. In particular, for the D and F masses[19] we have

$$m^2(D) = -\frac{1}{F_D{}^2}(m_1+m_4)\,(\langle\bar{q}_1q_1\rangle_0+\langle\bar{q}_4q_4\rangle_0),$$
$$m^2(F) = -\frac{1}{F_F{}^2}(m_3+m_4)(\langle\bar{q}_3q_3\rangle_0+\langle\bar{q}_4q_4\rangle_0). \tag{4.8}$$

Comparing (4.8) with (2.2) and (2.3) shows that the effective symmetry breaker (at least for the π, K, D, and F) will transform as a member of the $\underline{15}$ provided that

$$\frac{e_1+e_2}{F_\pi{}^2}=\frac{e_1+e_3}{F_K{}^2}=\frac{e_1+e_4}{F_D{}^2}=\frac{e_3+e_4}{F_F{}^2}, \tag{4.9}$$
$$e_a \equiv \langle\bar{q}_aq_a\rangle_0.$$

Equation (4.9) ensures that the relevant squared pseudoscalar masses are all proportional to the sums of their constituent quark masses with the same constant of proportionality. It is also consistent with the approximation made in Sec. III. Note that the analogous formula for the σ-model-type solution is different:

$$\frac{\alpha_1+\alpha_2}{F_\pi}=\frac{\alpha_1+\alpha_3}{F_K}=\frac{\alpha_1+\alpha_4}{F_D}=\frac{\alpha_3+\alpha_4}{F_F}=1.$$

It would be interesting if a dynamical derivation of (4.9) could be given. Note that (4.9) has a direct consequence the formula

$$F_F{}^2=F_D{}^2+F_K{}^2-F_\pi{}^2. \tag{4.10}$$

This may be checked experimentally in the future. The analogous formula for the σ-model-type solutions is $F_F=F_D+F_K-F_\pi$. An additional consequence of (4.9) is the well-known relation[19] $m^2(F)-m^2(D)=m^2(K)-m^2(\pi)$.

To end this section it may be of some interest to make a crude numerical estimate of the new parameter U. From the mass spectrum we have found $\tilde{U}$. We then make the following (σ-model inspired) semiclassical approximation:

$$\tilde{U}=U\langle[\det\bar{q}(1+\gamma_5)q+\det\bar{q}(1-\gamma_5)q]\rangle_0$$
$$\simeq 2U\langle\bar{q}_1q_1\rangle_0\langle\bar{q}_2q_2\rangle_0\langle\bar{q}_3q_3\rangle_0\,. \tag{4.11}$$

With (4.6) this may be written as

$$U\simeq -\frac{3}{4}V_4\left(\frac{m_3}{A_3}\right)^3=1.9\times10^{-3}(m_3)^3, \tag{4.12}$$

where (4.7) was used in the last step. All numerical quantities are here expressed in units where $m(\pi^0)=1$. If we were to take for the mass of the third quark $m_3\simeq 3m(\pi^0)$, (4.12) would predict $U\simeq 0.051m^{-5}(\pi^0)$.

V. DISCUSSION

A. $\eta\to 3\pi$

Another aspect of the U(1) problem[3] is that in a current-algebra approximation the amplitude for this process is related to the four-momentum-conserving matrix element

$$\int d^4x\langle\pi^+\pi^-|(\bar{q}_1\gamma_5q_1+\bar{q}_2\gamma_5q_2)|\eta\rangle. \tag{5.1}$$

Calculating the divergence of the pseudovector current $P^a_\mu(x)$ in (2.5) by using the equations of motion for (1.2) gives

$$\partial_\mu P^a_\mu = 2\,im_a\bar{q}_a\gamma_5q_a$$
$$+iU[\det\bar{q}(1+\gamma_5)q-\det\bar{q}(1-\gamma_5)q]. \tag{5.2}$$

Now for $U=0$ the operator in (5.1) may be written, using (5.2), as a four-divergence of a current. This then gives zero for (5.1). Obviously this bad result does not hold in the present model. In fact, using the σ model, which as we have seen in the last section corresponds to making a particular choice of parameters in the new quark model, gives a result in fair agreement with the latest experimental value. This was carried out in Ref. 20 where, with a conventional normalization, the amplitude for $\eta\to\pi^+\pi^-\pi^0$ was predicted to be

$$T_{\rm theor}\simeq 0.64\left(1-\frac{2\omega_0}{m_\eta}\right),$$

where ω_0 is the (relativistic) energy of the π^0. This is in reasonable agreement[21] with the new experimental result,[22] which may be presented as

$$T_{\rm exp}=(0.56\pm0.04)\left(1-\frac{2\omega_0}{m_\eta}\right).$$

Note that the experimental value has changed drastically since Ref. 20 was written.

B. Generalizations

A discussion of the four-quark case in more detail than the few sentences in Sec. IV is clearly called for. This will be given elsewhere. Actually, the associated σ model has already been discussed[23] with the present notation and point of view. From that work it is clear that the effect of the fourth quark on the world of the first three quarks is relatively small, so that the discussion given here should not be changed much. Two amusing features[22] of the σ-model case (when restricted to be renormalizable) were (i) the prediction $m(D) > m(F)$, and (ii) relatively large values of F_D/F_π and F_F/F_π. In the last section we have shown how (i) may be avoided in the more general quark model and how (ii) may conceivably be accommodated.

We should stress that we have here investigated the *simplest* Lagrangian with an effective term which overcomes the U(1) problem. It is clear that effective terms of similar forms can be handled by the same technique. The σ model would seem to be a valuable tool for guidance in understanding and interpreting the quark-model results. For the case of complicated decays like $\eta \to 3\pi$ it is especially convenient.

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