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η decay into three pions and the on-mass-shell current algebra

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We use the on-mass-shell current algebra based on the SU(3) $\otimes$ SU(3) σ model for calculating the G-parity-violating decay $\eta \to 3\pi$. The model we use consists of the triplet of quarks coupled to the SU(3) scalar and pseudoscalar fields, σ and ϕ, and the chiral-symmetry-breaking operators u_8 and u_3 (tadpole terms). Owing to the terms $\phi_i\phi_j$ with $i \neq j = 0$, 3, and 8 in the Lagrangian, the axial-vector current sources and fields associated with the π^0, η, and η' particles are formed by mixtures of terms with opposite G parity; the mixing coefficient $\epsilon = u_3/(4u_8)$. Using these current operators, and having all particles on the mass shell, we evaluate the decay amplitude by the reduction formalism from the $\langle 3\pi|\eta\rangle$ matrix element, making use of pole dominance. The result is proportional to the coefficient $r = 4\epsilon$, which is determined by the tadpole parts of the masses and decay constants of the mesons through a set of relations which are derived from our Lagrangian. However, since these masses are not known satisfactorily, we first choose r so as to obtain the observed $\eta \to \pi^0\pi^+\pi^-$ decay rate (which is proportional to r^2), and then we verify that such an r value is consistent with the above set of relations. We find that $r = 2.12 \times 10^{-2}$, and in turn the tadpole part of the K^0-K^+ mass difference $m_{K^0}{}^2 - m_{K^+}{}^2 = 0.0005$ GeV^2. We also find the branching ratio and the ratio of the rates of the η and η decays in agreement with experiment. Comparing the decay amplitude obtained here with the one derived by the tree-graph method based on the U(3) $\otimes$ U(3) symmetry scheme, we realize the important role of the mixing of the ϕ_0, ϕ_8, and ϕ_3 field components, and the advantage of using the on-mass-shell current algebra for this process.

I. INTRODUCTION

It has been recognized that the rate of the η decay into three pions cannot be obtained simply by electromagnetic perturbation, and that this rate has not been satisfactorily calculated by the phenomenological Lagrangian method involving soft-pion formalism.[1] We see that in general the previous treatments contain some extrapolation, and give the decay amplitude in terms of certain parameters which are determined by certain assumptions and by fitting the observed decay rate. For instance, one such result is obtained from the SU(3) linear-σ-model Lagrangian and the other from the parametrization of the off-shell decay amplitude derived from the nonlinear σ model in which the tadpole term, u_3, is taken into account.[2,3] A different approach is based on the quark-gluon picture and the U(1) symmetry scheme in which the singlet component of the hadron axial-vector current is not conserved. Using this model, the decay amplitude is evaluated in the U(3)$\otimes$U(3) scheme by the tree-graph method in terms of a physically nonexistent "light boson," whose unknown mass is estimated to be less than 1.7 times the pion mass.[4,5] The contribution of this boson to the decay is seen to be appreciable and thus to invalidate the smoothness assumption upon which the usual partially conserved axial-vector current, PCAC, is based. By eliminating the contribution of this boson, as is done in Ref. 4, one gets the same result that was previously obtained by the SU(3)$\otimes$SU(3) theory, with the u_3 term, and the soft-pion technique. The best estimated rate from this treatment is less than $\frac{1}{3}$ of that observed.

Considering these points, it appears that this decay problem cannot be solved entirely by the choice of the symmetry and symmetry-breaking schemes alone. With the isospin-invariance-breaking term included in the Lagrangian, all the previous approaches yield more or less the same result. Thus, further improvement of the calculation seems to require a departure from the conventional PCAC and soft-pion formalism.[6]

In this paper, therefore, we evaluate the present decay by the current algebra of the on-mass-shell pion, based on the SU(3)$\otimes$SU(3) σ-model Lagrangian scheme.[7] In Sec. II we consider the Lagrangian of Gell-Mann and Lévy which consists of the triplets of quark field, q, and the nonets of scalar and pseudoscalar fields σ and ϕ.[7,8] Also, we define the quark mass matrix as

$$\mathfrak{M} = g\sum_i \lambda_i u_i\,, \quad i = 0, 3, 8 \tag{1}$$

where g is the quark-meson coupling constant, λ_i is the component of the Gell-Mann nonet λ matrix, u_0 is the quark-degenerate-mass parameter, while u_8 and u_3 parameters give the quark-mass-splitting terms. Owing to the u_8 and u_3 parameters, the Lagrangian contains terms proportional to $\phi_i \phi_j$ and $\sigma_i \sigma_j$ with i and $j=0$, 3, and 8. Consequently the axial-vector current source, J^μ_a, which is defined through the equation of motion and the chiral gauge technique as[9]

$$c_a(\partial^2+m^2)\phi_a=\partial_\mu J^\mu_a, \quad a=0 \text{ to } 8 \tag{2a}$$

[where the mass m_a and decay constant c_a are associated with the SU(3) field ϕ_a], contains terms with opposite G parity, for $a=0$, 3, and 8. Hence, using the physical particle operators, Eq. (2a) gives

$$J_\alpha=J^+_\alpha+J^-_\alpha, \tag{2b}$$

where the superscript denotes G parity.[10] This current spectrum stipulates that each physical field ϕ_α also consist of two parts with opposite G parity,

$$\phi_\alpha=\phi^+_\alpha+\phi^-_\alpha, \tag{2c}$$

where $\phi^\pm_\alpha$ are formed by the combination of the ϕ_0, ϕ_3, and ϕ_8 components of the nonet field ϕ. We will see how this mechanism allows us to evaluate the above decay directly by the reduction formalism from the $\langle 3\pi|\eta\rangle$ matrix element. By doing this in Sec. III, we see that the decay rates of the η and η' are produced, respectively, by the product of the current components (J^μ_8, J^μ_3) and (J^μ_0, J^μ_3). The matrix elements of these products are calculated, as in the previous work,[11] via a set of intermediate states and pole dominance. Making use of the conserved quantities we find the η and π^0 to be the only intermediate states which contribute appreciably to this decay amplitude.[18] The η or η' decay rate is found to be proportional to the square of the ratio $r=u_3/u_8$ which is also proportional to the tadpole part of the K^0-K^+ mass difference and represents a correction to the Gell-Mann–Okubo mass formula. All the parameters in the final results are among those parameters which appear in our Lagrangian, and so they are expressed in terms of the mesons masses and decay constants.[12] However, it is seen that the ratio r cannot satisfactorily be determined because of the inaccuracies in the tadpole parts of the meson masses and in the η decay constant c_η. We therefore determine first the r value from the observed η decay rate inserted in our decay rate formula. We then find that $r=2.12\times10^{-2}$, which satisfies all the relations we have among the meson masses and decay constants, with $c_\eta=1.28c_\pi$, which is close to $c_\eta=1.37c_\pi$, which is deduced from the SU(3)⊗SU(3) algebra developed by Gell-Mann, Oakes, and Renner, and others.[13] With the same r value we find the tadpole mass difference of the K^0 and K^+ mesons, $m_{K^0}{}^2-m_{K^+}{}^2=(5 \text{ to } 5.46)\times10^{-3}$ GeV^2,[16] and the branching ratio of the η and η' decaying into $3\pi^0$ and $\pi^0\pi^+\pi^-$, in agreement with experiment. Our results are consistent with other results previously produced from our Lagrangian model.[14]

A by-product of the present work is the information which we obtain on Weinberg's "light-boson" concept in Sec. IV. Comparing our result with that obtained by the tree-graph method in Ref. 4, we find that the effect of this boson field, ϕ_L, is mimicked by the mixing of the ϕ_0, ϕ_8, and ϕ_3 field components, which is our basic mechanism for explaining this G-parity-violating decay. We show in fact that the ϕ_L field is the ϕ^+_L part of the pion-field spectrum defined in Eq. (2c), and thus it will not directly be detectable. We discuss this matter further in Sec. V.

II. LAGRANGIAN FORMALISM

For evaluating the η or η' decay rate into three pions we use the on-mass-shell current algebra based on the σ-model Lagrangian of Gell-Mann and Lêvy[7] which is slightly modified and includes the u_3 "tadpole" symmetry-breaking term.[14] In this section we derive from this Lagrangian some relationships among the meson masses and decay constants, thereby determining those parameters relevant to the present work. We also obtain the expressions of the current source operators assocaited with the π, η, and η' mesons, as defined in Eq. (2). This Lagrangian may conveniently be written as

$$\mathcal{L}=\mathcal{L}_q+\mathcal{L}^M_M+\mathcal{L}^I_M, \tag{3a}$$

with

$$\mathcal{L}_q=\bar{q}[-i\gamma_\mu\partial^\mu+\mathfrak{M}+g\lambda_a(\sigma_a+i\gamma_5\phi_a)]q, \tag{3b}$$

$$\mathcal{L}^M_M=\tfrac{1}{2}\{\partial_\mu\Phi\partial^\mu\Phi\}-\tfrac{1}{2}\sum_{a,b}(m_{ab}{}^2\phi_a\phi_b+m'_{ab}{}^2\sigma_a\sigma_b), \tag{3c}$$

$$\begin{aligned}\mathcal{L}^I_M=&-\rho\{\Phi\Phi^\dagger\}-\lambda\{\Phi\Phi^\dagger\Phi\Phi^\dagger\}-\nu(\det\Phi+\det\Phi^\dagger)\\&+4\rho' u_i\sigma_i\{\Phi\Phi^\dagger\}\\&+2\lambda_i d_{iam}d_{mbc}[\sigma_a\sigma_b\sigma_c+2\phi_c(\sigma_a\phi_b-\phi_a\sigma_b)],\end{aligned} \tag{3d}$$

with a and $b=0$ to 8, but $i=0$, 3, and 8. Here $\mathcal{L}_q$ is the part of the Lagrangian which is due to a triplet of quarks, while $\mathcal{L}^M_M$ consists of the meson-mass terms, and $\mathcal{L}^I_M$ represents the mesonic interactions. Also $\{\cdots\}$ denotes the trace,

$$\Phi=\sigma+i\phi.$$

$\mathfrak{M}$ is the quark mass given by Eq. (1), and the meson masses are obtained, in terms of the quantities $X=\sqrt{2}u_8u_0{}^{-1}$ and $Y=\sqrt{2}u_3u_0{}^{-1}$, as

$$m_{ab}{}^2 = \mu^2\delta_{ab} + h_1[\delta_{ab} + \sqrt{3}\,(Xd_{8ab} + Yd_{3ab})] + h_2[-\delta_{ab} + 3\delta_{0a}\delta_{0b} + \sqrt{3}\,(Xd_{8ab} + Yd_{3ab}) - 3\sqrt{2}\,(X\delta_{8a} + Y\delta_{3a})\delta_{ab}] \tag{4a}$$

for pseudoscalar mesons, and

$$m'_{ab}{}^2 = \mu^2\delta_{ab} + 2h_3\delta_{0a}\delta_{0b} + 3h_1\left[1 + \sqrt{3}\,(Xd_{8ab} + Yd_{3ab}) + \frac{\sqrt{2}}{2}(X+Y)\delta_{ab}\right] + h_2[\delta_{ab} - 3\delta_{0a}\delta_{0b} - \sqrt{3}\,(Xd_{8ab} + Yd_{3ab}) + 3\sqrt{2}\,(X\delta_{8a} + Y\delta_{3a})\delta_{ab}] \tag{4b}$$

for scalar mesons. In writing Eqs. (4a) and (4b) we have neglected the terms involving X^2, Y^2, and XY. We have also defined μ as the meson "unbroken" mass, and

$$h_1 = \tfrac{4}{3}u_0{}^2\lambda,\quad h_2 = 2\sqrt{3}\,u_0\nu,\quad h_3 = 4u_0{}^2\rho, \tag{4c}$$

where ρ, λ, ν are the parameters of the chiral-invariance terms, I_ρ, I_λ, I_ν, in Eqs. (3c). The mixing-mass terms needed in the present work are obtained from Eqs. (4):

$$m_{08}{}^2 = \sqrt{2}\,(h_1 - h_2)X,\quad m_{03}{}^2 = \sqrt{2}\,(h_1 - h_2)Y,\quad m_{38}{}^2 = \tfrac{1}{2}(h_1 + h_2)Y. \tag{4d}$$

By shifting the σ field by $\lambda_i(\sigma - \sigma')_i = \mathfrak{M}/g$ in the Lagrangian (3) and again making use of the chiral formalism we obtain the familiar relation[13]

$$c_a m_a{}^2\phi_a = \partial_\mu A_a^\mu,\quad a = 0 \text{ to } 8,\qquad A_a^\mu = J_a^\mu - c_a\partial^\mu\phi_a. \tag{5a}$$

Note that A_a^μ with $a = 1$ to 8 is the usual weak hadron current.[15] The decay constants $c_a = c_{ab}\delta_{ab}$, and

$$\begin{aligned}
c_\pi &= c_{11} = c_{22} = c_3 = (\tfrac{1}{6})^{1/2}(2 + X)\,u_0,\\
c_{K^\pm} &= c_{44} = c_{55} = (\tfrac{1}{6})^{1/2}\left(2 - \frac{X}{2} + \frac{\sqrt{3}}{2}Y\right)u_0,\\
c_{K^0} &= c_{66} = c_{77} = (\tfrac{1}{6})^{1/2}\left(2 - \frac{X}{2} - \frac{\sqrt{3}}{2}Y\right)u_0,\\
c_\eta &= c_{88} = (\tfrac{1}{6})^{1/2}[2 - (\sqrt{2} - 1)X]\,u_0,\\
c_{\eta'} &= c_{00} = (\tfrac{2}{3})^{1/2}u_0,
\end{aligned} \tag{5b}$$

where we have neglected Y^2, X^2, and XY terms. Experimentally we have

$$\frac{c_K}{c_\pi} = 1.22 - 1.28. \tag{5c}$$

Before going further, we present some relationships which will be used in the present work. First assigning $m = m_3 \equiv m_{\pi^0}$ the π^0 mass, $m_8 \equiv M$ the η mass, and $m_0 \equiv M'$ the η' mass, Eqs. (4) give

$$\begin{aligned}
h_2 &= \tfrac{1}{6}(2M'^2 - m^2 - M^2),\\
h_1 &= \frac{1}{2X}(m^2 - M^2) - h_2,\\
\sqrt{3}\,r &= (4m_K{}^2 - m^2 - 3M^2)(M^2 - m^2)^{-1},
\end{aligned} \tag{6a}$$

where $r = Y/X$ and the last relation represents a correction to the Gell-Mann–Okubo mass formula.

In order to calculate the η decay rate we will require the parameters h_1, h_2, and r in the next section. Yet we cannot accurately determine these parameters from Eqs. (6a), since these relations depend sensitively on the tadpole parts of the meson masses which are not well determined.[16] However, we assume that turning on the electromagnetic interaction does not appreciably alter the ratio c_K/c_π, or the ratios M/m and M'/m which involve the neutral-meson masses. Hence, choosing $M = 4m$ and $M' \doteq 7.09\,m$, and combining Eqs. (5), (6a), and (4c), we find that

$$u_0 = 1.1m,\quad h_1 = 9.35\,m^2,\quad h_2 = 13.8\,m^2,\qquad \lambda = 5.9,\quad X = -0.336,\quad r = 4.23\Delta\,m^2{}_{K,t}. \tag{6b}$$

Here $\Delta m^2{}_{K,t} = m_{K^0}{}^2 - m_{K^+}{}^2$ is the tadpole contribution to the K^0-K^+ mass splitting, and r satisfies

$$\left(\frac{c_\eta}{c_\pi} - 1\right) = \left(\frac{2}{3}\right)^{1/2}\frac{2}{\sqrt{3} - r}\left(\frac{c_K}{c_\pi} - 1\right), \tag{6c}$$

in which c_η/c_π is not known. If we compare expression (6c) with

$$\left(\frac{c_\eta}{c_\pi} - 1\right) \simeq \frac{4}{3}\left(\frac{c_K}{c_\pi} - 1\right), \tag{6d}$$

which is derived approximately from the familiar $SU(3)\otimes SU(3)$ algebra containing only the u_0 and u_8 symmetry-breaking terms, we find that $r = 0.5$.[13,17] This is too large a number to produce an acceptable value for $\Delta m^2{}_{K,t}$ from Eqs. (6b), or for the η decay rate from the formula that we will present below. On the other hand, it will be seen in the next section that $r = 0.021$ deduced from the observed η decay rate (which is proportional to r^2) provides a satisfactory $\Delta m^2{}_{K,t}$, and a c_η/c_π ratio which is a few percent different from what we get from Eqs. (6d) and (5c). Finally by writing the equation of motion with the Lagrangian (3), the field–current-source relation (2) becomes

$$c_\alpha(\partial^2+m_\alpha{}^2)\phi_\alpha=S_\alpha, \tag{7}$$

$$S_\alpha=\partial_\mu J'^\mu_\alpha+\tfrac{1}{2}c_\alpha(1-\delta_{i\alpha})G_i\,,\quad i=0,3,8$$

where $\partial\mu J'^\mu_a$ represents the old pion source given in Ref. 8, and G_i is an operator which carries a G parity opposite to that of the J'^μ current. The G_i operators for physical particles are found to be

$$\begin{pmatrix} G_{\eta'} \\ G_\eta \\ G_{\pi^0}\end{pmatrix}=\begin{pmatrix} 0 & 0 & m_{30}{}^2 \\ 0 & 0 & m_{38}{}^2 \\ m_{03}{}^2 & m_{08}{}^2 & 0\end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_8\\ \phi_3\end{pmatrix}, \tag{8}$$

with m_{ij} given by Eq. (4d). Such a S_a source with mixed G parity implies a mixture of the field for the physical fields of the π^0, η, and η' mesons, through three Euclidian angles θ, θ_1, and θ_2, in the isospin space. Knowing that these admixtures must be very small, and using Eqs. (7) and (6), we find that

$$\begin{pmatrix}\phi_{\eta'}\\ \phi_\eta\\ \phi_{\pi^0}\end{pmatrix}=\begin{pmatrix} 0 & 0 & \epsilon\sin\theta \\ 0 & 0 & \epsilon\cos\theta \\ \epsilon\sin\theta & -\epsilon\cos\theta & 0\end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_8\\ \phi_3\end{pmatrix}, \tag{9a}$$

where $\epsilon\equiv\sin\theta_1\ll 1$ and $\theta_2\approx 0$ are assumed. Comparing Eq. (9a) with Eq. (17b) and making use of Eqs. (4d) and (6c) we find that

$$\tan\theta=\sqrt{2}\,\frac{(M^2-m^2)(h_1-h_2)}{(M'^2-m^2)(h_1+h_2)}\,, \tag{9b}$$

$$\epsilon=\tfrac{1}{4}r\simeq 0.58e^2\,,$$

so the mixing coefficient, ϵ, between ϕ_3 and other fields is of the order of $e^2=4\pi\alpha$.

III. THE $\eta\to 3\pi$ DECAY

Now that the sources of the π^0, η, and η' fields contain terms with opposite G parity, we can evaluate the above decay by the direct reduction formalism in $\langle 3\pi|\eta\rangle$ as

$$T=i\int d^4x\,d^4y[\exp i(k\cdot x-p\cdot y)]K_{\pi x}K_{\eta y}\times\langle 2\pi,P\,|T\{\phi_\pi(x),\phi_\eta(y)\}|\,0\rangle, \tag{10a}$$

with $K_{\pi,x}=\partial_x{}^2+m^2$ and $K_{\eta y}=\partial_y{}^2+m^2$. Here p and k are the 4-momenta of the π^0 and η while $P=p_1+p_2$ is the sum of the 4-momenta of the remaining two pions. Using Eq. (7), we have

$$K_{\pi x}K_{\eta y}T\{\phi_\pi(x),\phi_\eta(y)\}=c_\pi{}^{-1}c_\eta{}^{-1}T\{S_\pi(x),S_\eta(y)\}+c_\pi{}^{-1}\delta(x_0-y)[S_\pi(x),\partial^0\phi_\eta(y)]. \tag{10b}$$

We note that the last term of the identity (10b), in Eq. (10a), vanishes for the physical pion and η mesons, making use of the usual canonical commutation rules. Thus for physical pions these relations give

$$T=\frac{i}{c_\pi c_\eta}\int d^4x\,d^4y[\exp i(k\cdot x-p\cdot y)]\times\langle 2\pi,P\,|\,\theta(x_0-y_0)[S_\pi(x),S_\eta(y)]|\,0\rangle. \tag{11}$$

Noticing by Eq. (7a) that $[S_\pi,S_\eta]$ contains commutators which will not contribute to this G-parity-violating case, and that on account of the smallness of X and Y involved in Eqs. (7) and (4d), we can neglect the $[G_\pi,G_\eta]$ contribution, Eq. (11a) gives in the rest frame of the η meson

$$T=T_1+T_2, \tag{12a}$$

$$T_1=\frac{i}{c_\pi c_\eta}\sum_n\frac{\langle 2\pi,P|\partial_\mu J'^\mu_{\pi^0}|n\rangle\langle n|G_\eta|0\rangle}{M-p_{\eta^0}}\,\delta^3(\vec{k}-\vec{p}_\eta), \tag{12b}$$

$$T_2=\frac{-i}{c_\pi c_\eta}\sum_n\frac{\langle 2\pi,P|\,\partial_\mu J'^\mu_\eta|\,n\rangle\langle n|G_{\pi^0}|0\rangle}{p_0+p_{\eta^0}}\,\delta^3(\vec{p}+\vec{p}_n), \tag{12c}$$

dropping $(2\pi)^4\delta^4(k-p-P)$. Here p_n is the 4-momentum of the set of intermediate states n which are introduced in Eq. (11a). We note that the J and G operators have opposite G parities, but that G-parity conservation must be respected in determining the $|\,n\rangle$ states. Considering parities and other conserved quantities, and taking all particles on the mass shell, we determine the states n in Eq. (12). Then assuming pole dominance up to and partly including the three-particle intermediate states, we find the contributing states n to be $|\,\pi^0\rangle$ in Eq. (12b) and $|\,\eta\rangle$ in Eq. (12c).[18] Finally making use of these states and the Appendix, Eqs. (12) yield

$$T=\epsilon A(2\pi)^4\delta^4(k-p-P)\left(1-\frac{2}{M}E-B\right), \tag{13a}$$

where ϵ is given by Eq. (9b), $E=p_0m^{-1}$, and

$$A=\frac{1}{4m^3}(M^2-m^2)(2Mg_{\pi^0}+mg_\eta), \tag{13b}$$

$$B=(2mg_{\pi^0}-mg_\eta)(2Mg_{\pi^0}+mg_\eta)^{-1}.$$

Using Eqs. (A7) and (A8) of the Appendix in Eq.

(13b),

$$A = \frac{1}{12m^3}(M^2 - m^2)(6M + m)\lambda,$$
$$B = 5m(6M + m)^{-1} \quad (14a)$$

for $\eta \to \pi^0 + \pi^+ + \pi^-$, and

$$A = \frac{1}{12m^3}(M^2 - m^2)[9m(2\rho + \lambda) + m\lambda],$$
$$B = [9m(2\rho + \lambda) - m\lambda][9m(2\rho + \lambda) + m\lambda]^{-1} \quad (14b)$$

for $\eta \to 3\pi^0$. With the usual formula of the transition probability we find from Eq. (13) the decay rate Γ,

$$\Gamma = \frac{m^2 A^2 I \epsilon^2}{64\pi^3 M s}, \quad (15a)$$

where $s = 1$ for $\eta \to \pi^0\pi^+\pi^-$, $s = 6$ for $\eta \to 3\pi^0$, and

$$I = \tfrac{1}{3}\int_1^M (1 - aE)^2[(R' - E)^3 - (R - E)^3 - 3]\,dE. \quad (15b)$$

Here $R = M/m - 2$, $R' = M/m - 1$, and

$$a = \frac{2}{M}(1 - B), \quad (15c)$$

with $M = 4m$, gives

$$a = 0.625 \text{ for } \eta \to \pi^0\pi^+\pi^-,$$
$$a = 0.664 \text{ for } \eta \to 3\pi^0. \quad (15d)$$

Using Eqs. (6), (9), and (15), we find for

$$\Gamma_{0+-} = 4.56 \times 10^5 r^2$$
$$= 8.2 \times 10^{-6}(\Delta m_{K,t}{}^2)^2. \quad (16)$$

If we take Dashen's sum rule for the chiral-symmetry limit,[19]

$$(m_{K^+}{}^2 - m_{K^0}{}^2)_{\text{em}} = (m_{\pi^+}{}^2 - m_{\pi^0}{}^2)_{\text{em}} \quad (17)$$

(where em denotes the contribution of the purely electromagnetic interaction) and make use of the observed $\Delta m_{K,t}{}^2 = 4.16 \times 10^{-3}\ \text{GeV}^2$, we find[19] that

$$\Delta m_{K,t}{}^2 = 5.46 \times 10^{-3}\ \text{GeV}^2,$$
$$r = 2.32 \times 10^{-2}. \quad (18a)$$

Using these data in Eq. (16) we have

$$\Gamma_{0+-} = 245\ \text{eV}, \quad (18b)$$

which is close to the recently observed rate $\Gamma_{\text{ob}} = 204 \pm 22$ eV.[20] With a deviation in Eq. (17), which would reduce $(m_{K^+}{}^2 - m_{K^0}{}^2)_{\text{em}}$ by about 30%, the same calculation yields

$$\Delta m^2{}_{K,t} = 5 \times 10^{-3}\ \text{GeV}^2,$$
$$r = 2.12 \times 10^{-2}, \quad (19)$$
$$\Gamma_{0+-} = 205\ \text{eV}.$$

We note from Eq. (16) that the $\Delta m_{K,t}{}^2$ value is much less sensitive than the Γ value to the approximation involved in the evaluation of the amplitude (12). The $\Delta m_{K,t}{}^2$ value found here agrees with the one given by the authors in Refs. 16 and 19. It is also to be compared with the $\Delta m_{K,t}{}^2$ range obtained from a perturbation formalism combined with the phenomonological expression of the η decay amplitude by Langacker and Pagels.[21]

Finally making use of Eqs. (14) to (15) we find that

$$R = \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^0\pi^+\pi^-)} \simeq 1.56, \quad (20a)$$

in good agreement with the observed data.[22]

To obtain the rate of the $\eta' \to 3\pi$, we follow the same above treatment in which the source S_η in Eq. (11) is replaced by $S_{\eta'}$ given by Eq. (7), and M is changed to $M' = 7.09m$. In this way we find $m_{30}{}^2$ in place of $m_{38}{}^2$ and $I(\eta' \to 3\pi)$ in place of $I(\eta \to 3\pi)$ in Eq. (15). Hence

$$\frac{\Gamma(\eta' \to 3\pi)}{\Gamma(\eta \to 3\pi)} = \frac{M'}{M}\left(\frac{m_{03}}{m_{83}}\right)^4 \frac{I(\eta' \to 3\pi)}{I(\eta \to 3\pi)}. \quad (20b)$$

According to the Particle Data Group the upper limit of the η' total width is 1 MeV.[22] Taking this figure, $\Gamma_\eta = 204$ eV obtained from Eq. (16), and Eq. (20a), we find the upper limit on the branching ratio of the $\eta' \to \pi^0\pi^+\pi^-$ and $\eta' \to 3\pi^0$ to be 2.9% and 4.6%, respectively, consistent with the observed limit 5%.[22]

IV. COMPARISON WITH THE U(3)⊗U(3) APPROACH

We now consider the result of the decay amplitude which is derived in Ref. 4 by the tree-graph method in the U(3)⊗U(3) picture. This amplitude can be expressed in terms of two parameters F and m_L as

$$T = F\left(1 - \frac{2}{M}E + \frac{m_L{}^2}{m^2 - m_L{}^2}\right), \quad (21)$$

where m_L in Ref. 4 is considered to be the mass of an unobserved boson, to be called the "L" particle. The fields associated with the "L" and η particles are given in terms of the ϕ_0-ϕ_8 mixing as

$$\begin{pmatrix}\phi_L\\ \phi_\eta\end{pmatrix} = \begin{pmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_8\end{pmatrix}; \quad (22a)$$

β is the mixing angle.

Here, comparing first Eqs. (22a) and (9) and setting $\beta = \theta$ we find that

$$\phi_\pi = \phi_3 + \epsilon\phi_L. \quad (22b)$$

So the "L"- boson field ϕ_L is actually the ϕ_π^+ part of our pion-field spectrum defined by Eq.

(2c). Next we note that the two amplitudes (21) and (13) are equal, if we set

$$F = \epsilon A, \qquad m_L^2 = B(1-B)^{-1}m^2, \tag{23}$$

with A and B given by Eq. (13b). An inspection of Eqs. (9) and (13b) shows that the m_L is imaginary and the upper limit of $|m_L|$, for η and η' decays, is $1/\sqrt{3}\ m$.[23]

V. CONCLUDING REMARKS

We have seen that the Lagrangian based on the SU(3)⊗SU(3) linear σ model, combined with the on-mass-shell current treatment, is capable of giving the η and η' decay rates in agreement with experiment. This success is partially due to the isospin-breaking term u_3 in the quark mass matrix, and is partly due to the field–current–source relation (7) which provides a mechanism for handling this G-parity-violating decay through the usual reduction formalism. The term u_3 has a far-reaching consequence in the entire formalism. On the one hand, the quantity $r = u_3/u_8$ comes out proportional to $\Delta m^2_{K,t}$ which is the tadpole contribution to the K^0-K^+ mass difference. On the other hand, owing to the term u_3, the combination of the equation of motion and the chiral current gives the current source J_a a mixed-G-parity spectrum, and this in turn leads to having the admixture of the ϕ_0 and ϕ_8 with ϕ_3 in the π^0, η, and η' states. We see in Eq. (9), for instance, that the pion field consists of the usual ϕ_3 field and a $G = +$ parity part ϕ_L, with the mixing coefficient ϵ of the order of e^2, which contributes only in a G-parity-violating process such as this η decay. (We do not know if the closeness of the ϵ value to the square of the electric charge is accidental, or due to an intimate link between the tadpole u_3 and the electromagnetism.) The main difference between the present treatment and the soft-pion approach is exhibited by the term B in Eq. (13a). The effect of this term on the decay rate [through the parameter a in Eq. (15a)] is appreciable.

Our result, Eq. (16), shows that the η decay rate depends sensitively on the r value, or on the field-mixing coefficient $\epsilon = 0.58e^2$. The main approximation in determining r is pole dominance and partial consideration of the background continuum in the matrix elements of the products of the currents in the amplitude (12).[23] To get some idea of the degree of accuracy to the present treatment, we note that Eq. (6b), with $r = 0.021$ given in Eq. (19), yields an η decay constant $c_\eta = 1.28c_\pi$ as compared to the value $c_\eta = 1.37c_\pi$ deduced from the exact chiral-limit expressions for c_K and c_η.[24] We also see that the $\Delta m^2_{K,t}$ value given in Eq. (19) is close to that estimated by making use of Dashen's sum rule.[19] It also agrees with the $\Delta m^2_{K,t}$ value, which is obtained from the pole dominance application in the η decay amplitude which is given by Langaker and Pagels.[21] Furthermore, the branching ratio of the η decaying into $3\pi^0$ and $\pi^0\pi^+\pi^-$, and the ratio of the η and η' decay rates, Eqs. (20a) and (20b), agree with experiment.

Finally we note that the decay rate in Eq. (13a) is identical in form with the amplitude (21) which is derived from the U(3)⊗U(3) symmetry and the tree-graph method in Ref. 4. Thus by comparing these relations we obtain some information on Weinberg's light boson, L. We see from Eq. (9) that the L boson does not represent a particle in our formalism, since it is the $G = 1$ part of the pion-field spectrum, Eq. (2c). Second, we note from Eqs. (16) and (21) that for obtaining the observed decay rate we must have $m_L = m/\sqrt{3}$.[22] Without the m_L term in the amplitude (21), or the B term in Eq. (13), the calculated rate would be three times less than the observed one.[4] Considering these points we observe the following: On the one hand, if we wish to apply the usual PCAC algebra to this process, we must realize the smoothness of the off-shell amplitude by suppressing the m_L contribution through a mechanism such as that proposed in Ref. 4; on the other hand, in doing this we partially eliminate the ϕ_0-ϕ_8-ϕ_3 admixture which is a part of the mechanism for this G-parity-violating interaction, and so consequently we find an unacceptable result.[25]

The on-mass-shell formalism as presented here avoids the above difficulty and the problem with the soft-pion extrapolation. Thus it offers a more successful formula for this decay; and within the errors involved in pole dominance and in the observed η decay rate, it gives a reasonable value for the u_3/u_8 ratio, or for the strength of this G-parity interaction, ϵ.

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APPENDIX

Here we give details of how the amplitude (13) is found from Eq. (12). What we need in Eq. (12) are the quantities

$$I_3 = m^{-2}\langle 2\pi, P|\,\partial_\mu J_3'^{\mu}\,|\pi^0, p'\rangle_{\vec{p}'=\vec{p}}\,,$$
$$I_8 = m^{-2}\langle 2\pi, P|\,\partial_\mu J_8'^{\mu}\,|\eta, k'\rangle_{\vec{k}'=-\vec{p}}\,. \tag{A1}$$

We note, according to Eqs. (7) and (2), that

$$c_\pi K_\pi \phi^-_{\pi^0} = \partial_\mu J_3'^{\mu}\,,$$
$$c_\eta K_\eta \phi^+_\eta = \partial_\mu J_8'^{\mu}\,. \tag{A2}$$

Using Eqs. (A2) in (A1) and making use of the fact that the state $|\pi^0\rangle$ or $|\eta\rangle$ is mixed according to Eq. (9), we find that

$$I_3 = i(M-m)(2E+m-M)m^{-2}c_\pi g_{\pi^0}\,,$$
$$I_8 \simeq i(M^2-E^2)m^{-2}c_\eta g_\eta\,, \tag{A3}$$

where we have set $E \equiv p_0 m^{-1}$, and used $k_0' \simeq M$ in the physical range of interest for E. We have also defined the coupling constants

$$ig_{\pi^0} = \langle 2\pi|\,\phi_3\,|\pi^0\rangle\,, \tag{A4}$$

$$ig_\eta = \langle 2\pi|\,\phi_8\,|\eta\rangle\,. \tag{A5}$$

It can be verified that the momentum dependence of the coupling constants g_{π^0} and g_η in the range of interest for E, which is $m \leq E \leq M-2m$, is negligible.[26] Also we note that the g's can be read off a part of the Lagrangian $\mathcal{L}_M^I$, in Eq. (3d), which is

$$\mathcal{L} = \rho\phi^4 + \tfrac{2}{3} d_{abm} d_{mcn}\phi_a\phi_b\phi_c\phi_\eta. \tag{A6}$$

From Eqs. (A4) and (A5) we obtain

$$g_{\pi^0} = \lambda,$$
$$g_\eta = \tfrac{1}{3}\lambda \tag{A7}$$

for $\eta \to \pi^0\pi^+\pi^-$, and

$$g_{\pi^0} = 3(\rho + \tfrac{1}{2}\lambda),$$
$$g_\eta = \tfrac{1}{3}\lambda \tag{A8}$$

for $\eta \to 3\pi^0$.[20] We see from Eqs. (A7), (A8), (21) how the mass parameter m_L in the U(3)⊗U(3) approach is related to the coupling constants g_{π^0} and g_η in our work. We also note from Eq. (A8) and the work in Ref. 14 that the parameters, which appears only in the $\eta \to 3\pi^0$ decay, is intimately related to the coupling constants of the ϵ and s scalar mesons.

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[1]See J. S. Bell and D. G. Sutherland, Nucl. Phys. B4, 315 (1968), and note that the view on the electromagnetic perturbation applied to the present decay has not changed since 1968; see also our Ref. 25. In fact, it is seen that even the modified current algebra that explains the decay cannot contribute appreciably to this decay; see E. S. Abers, D. A. Dicus, and V. I. Teplitz, Phys. Rev. D 2, 485 (1971) and our following references.

[2]For instance, W. Hudnall and J. Schechter, Phys. Rev. D 7, 2111 (1974).

[3]G. Cicogna, F. Stocchi, and R. V. Caffarelli, Lett. Nuovo Cimento 1, 25 (1974).

[4]S. Weinberg, Phys. Rev. D 11, 3583 (1975).

[5]J. Kogut and I. Susskind, Phys. Rev. D 10, 3468 (1974); see also Ref. 4.

[6]The same conclusion but based on a different argument can be reached from the work of K. Wilson, Phys. Rev. 179, 1499 (1969).

[7]M. Gell-Mann and M. Lêvy, Nuovo Cimento 52A, 23 (1967).

[8]The Lagrangian in Ref. 7 has also been used for formulating the on-mass-shell current algebra which we will use in this paper; see A. A. Golestaneh and C. E. Carlson, Nuovo Cimento 13, 514 (1973).

[9]Note that while the current source J_a^μ is a nonet, only its octet part is identical to the chiral current whose algebra is given in Ref. 8. Also in writing J_a^μ in Ref. 8, the ϕ_0-ϕ_8 mixing was neglected. Now this and other mixing due to the ϕ_0-ϕ_3 and ϕ_8-ϕ_3 are included in the spectrum as given in Eq. (2b).

[10]This division of the current-source spectrum according to Eq. (2) is analogous to the division of the current into the first- and second-class operators which was originally proposed by S. Weinberg, Phys. Rev. 112, 4 (1958); 112, 1375 (1958).

[11]See our Ref. 26 below and the list of the previous work in it.

[12]Note that the mixing of the ϕ_0-ϕ_8-ϕ_3 fields will not be affected by adding the electromagnetic terms to the Lagrangian in order to produce the "nontadpole" part of the K^0-K^+ mass difference and the π^0-π^+ mass splitting. To see the structure of this part of the Lagrangian see, for instance, R. Socolow, Phys. Rev. 137, B1221 (1965).

[13]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); see also S. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[14]A. A. Golestaneh, Tech. Report No. 72-023, Maryland Univ., 1971 (unpublished). In this work the parameters of the Lagrangian are determined in order to have (a) the meson mass spectra, (b) the expected values of the coupling constants of the ϵ and s mesons decaying into 2π, and (c) the same $\pi\pi$ scattering lengths from the tree-graph and the on-mass-shell current-algebra methods. Consequently the Lagrangian is modified according to Eq. (3) of the text, where the parameter $\rho \neq \rho'$, and $\rho = 8$ is found.

[15]Note that as a result of $\rho \neq \rho'$ in Ref. 14, the PCAC operator $\partial_\mu A_a^\mu$ is modified by the terms containing ϕ_b with $b \neq a$, and thus A_a^μ acquires a spectrum similar

to the one for the J_a^μ current in Eq. (2b).

[16]The problem of the meson mass differences and its related problem of the baryon mass differences have not been satisfactorily understood. The tadpole part of the K^0-K^+ mass splitting, which is of special interest here, has been treated by several authors. See, for instance, S. Coleman and S. Glashow, Phys. Rev. 134, B671 (1964), Socolow in Ref. 12, and Langacker and Pagels in Ref. 21. From these treatments we find that $m_{K^0}{}^2 - m_{K^+}{}^2 = (5 \text{ to } 10.4) \times 10^{-3}\ \text{GeV}^2$, which is too large a range to be used for the present calculation.

[17]L. Li and H. Pagels, Phys. Rev. D 5, 1509 (1972); see also Ref. 21.

[18]Next to the one-particle state, n, is the two-particle state consisting of 2γ rays. The contributions of this and other states containing γ rays are obviously negligible. Next are the 3π continuum in Eq. (12b) and the $(2\pi, \eta)$ continuum in Eq. (12c), which we cannot evaluate because of the unknown coupling coefficients involved in the process. Although we neglect such states for the present low-energy process, we observed the following. The Z graph for $n \equiv 3\pi$, in Eq. (12b), vanishes,

$$\langle 2\pi | \partial_\mu J'^{\mu}_{\pi^0} | 3\pi \rangle = \langle 2\pi | \overline{2\pi} \rangle \langle 0 | \partial_\mu J'^{\mu}_{\pi^0} | \pi^0 \rangle + \cdots ,$$

and $\langle 0 | \partial_\mu J'^{\mu}_{\pi^0} | \pi^0 \rangle = 0$, which is due to definition (7). Similar is the case for $n \equiv (2\pi, \eta)$ in Eq. (12c). The contribution of a state n with high mass, or large numbers of pions, becomes negligible owing to energy denominators in Eq. (12).

[19]R. Dashen, Phys. Rev. 183, 1254 (1969). The sum rule (17) is exact at the chiral-symmetry limit. For the physical case, see Langacker and Pagels in Ref. 21. Note that relation (17) has also been used to calculate by P. Dittner, P. H. Dondi, and S. Eliezer, Phys. Rev. D 8, 2253 (1973).

[20]A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974).

[21]P. Langacker and H. Pagels, Phys. Rev. D 10, 2904 (1974). Note that the pole dominance in this work produces $\Delta m_{K,t}{}^2 = (5.3 \text{ to } 6.9) \times 10^{-3}\ \text{GeV}^2$, which is a range close to the one given by our Eqs. (18) add (19). However, the $\Delta m^2_{K,t}$ range presented by these authors is larger by a factor of $\frac{3}{2}$, as the result of a logarithmic correction term added to the pole terms, in the η decay amplitude.

[22]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[23]The figure $m/\sqrt{3}$ is the highest value of $|m_L|$ according to our treatment, whereas this limit is $m\sqrt{3}$ according to Ref. 4.

[24]See P. Langacker and H. Pagels in Ref. 21.

[25]The recent work by S. Raby [Phys. Rev. D 13, 2594 (1976)] also does not give an acceptable result, as it is based on a Lagrangian in which the ϕ mixing is not included, and on the PCAC algebra and soft-pion technique.

[26]The momentum dependence of the coupling constants is studied using the dispersion relation in which the imaginary part of the matrix element is found by the reduction formalism involving the present current algebra; see, for instance, A. A. Golestaneh, Phys. Rev. D 11, 3240 (1975).