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Comment on asymptotic planarity

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A qualitative discussion of the approach to planarity (i.e., the t behavior of the nonplanar contributions) in the framework of the topological expansion is presented. It is pointed out that—under certain assumptions—it is possible to connect this problem with the t behavior of the average multiplicity. The asymptotic-planarity condition is opposite to the prediction of the random-walk model.

A very promising new way to study the strong interactions—the so-called topological expansion—has recently been proposed by Veneziano[1] and further developed by Ciafaloni, Marchesini, and Veneziano.[2] (A slightly different and more pragmatic approach has been devised by Chan *et al.*[3,4])

If one assumes a certain field theory with a U(N) symmetry (particles—mesons—assigned to the N^2 adjoint representation) with interaction

$$\sum_i \frac{g_i}{i}\,\mathrm{Tr}(M^{(1)}\cdots M^{(i)}),\quad \mathrm{Tr}(M^{(i)}M^{(j)})=2\delta_{ij}\,, \tag{1}$$

where $M_\alpha^{(i)\beta}$ ($\alpha,\beta=1,2,\cdots,N$) is the tensor corresponding to the ith meson, then the "topological expansion" is the following expansion of the physical amplitudes ($g_i^2N^{i-2}=\gamma_i^2$):

$$A_n^{\text{phys}}=\sum_b \mathrm{Tr}^{(b)}(M\cdots M)N^{-b}A_n(N,g_i) \tag{2a}$$

$$=\sum_b \mathrm{Tr}^{(b)}(M\cdots M)N^{1-n/2}N^{1-b}\sum_h N^{-2h}G_h(\gamma_i,p_i)\,. \tag{2b}$$

Here h (denoting "handle") is the topological genus of the graph corresponding to A_n^{phys}, while b denotes "boundaries" (i.e., quark lines which connect groups of external particles). The "first term" in Eq. (2) has $h=0$ and $b=1$ and it is the planar contribution. The second-order contribution (2^{n-2} terms) has $h=0$ and $b=2$. It may be visualized as a cylinder which communicates along its axis (i.e., the t channel) only with states with zero additive quantum numbers. This contribution is nothing but the bare Pomeron previously discussed by Lee,[5] by Veneziano,[6] and by Chan and Paton.[7] For an analysis of higher-order contributions see the papers quoted in Ref. 2.

In a further study of the topological expansion, Chew and Rosenzweig[8,9] advanced the hypothesis that the contribution of the cylinder and all other nonplanar terms decreases for $t>0$ and becomes negligible for $t\to\infty$. This behavior—called asymptotic planarity—has been studied by Chew and Rosenzweig[10] and by Bishari.[11] Both papers rely on a planar bootstrap equation derived by Veneziano *et al.*[12-14] In essence, this bootstrap equation is nothing but the Dyson equation for the Reggeon propagator with the dressed vertex approximated by the bare one and with a certain ansatz for the full propagators. Taking into account that for the twisted links the signature factors are replaced by 1, Chew and Rosenzweig[10] suggested that the average value of $\cos\pi[\alpha_{j_1}(t_1)-\alpha_{j_2}(t_2)]$ (for identical trajectories this quantity becomes $\cosh 2\pi\alpha' t^{1/2}w(t)$, $t>0$, where $w(t)$ is an appropriately defined function) may be considered as a measure of the suppression of the twisted loops. For t near zero, they obtained a "cylinder quenching interval"

$$t_c\simeq\frac{1}{2\pi^2\alpha'^2\langle w^2\rangle_{t=0}}\,, \tag{3}$$

while for t large, the quenching rate slows down to

$$t_c\simeq\frac{1}{\pi w_{\max}t^{1/2}}\,. \tag{4}$$

(The quantities $w_{\max}$ and $\langle w^2\rangle_{t=0}$ have been estimated as being $\simeq m_\rho$ and $\simeq\frac{1}{5}m_\rho^2$, respectively.) By using the same bootstrap equation and a specific factorization for the triple-Reggeon vertex, Bishari[11] obtained a similar result for small t.

In what follows we shall present a new discussion of this topic by limiting ourselves to the bare Pomeron contribution P ($h=0$, $b=2$). As in Refs. 5–7, we shall consider P as the shadow of the production process, i.e.,

$$\mathrm{Im}A^P(s,t)\simeq\sum_{n_1,n_2=1}|A^R_{n_1,n_2}(s,t)|^2\,, \tag{5}$$

where $A^R_{n_1,n_2}$ is the planar amplitude for the process $2\to n_1+n_2$. Now, the nonlinear term on the right-hand side of Eq. (5) may be converted into a linear one by writing it as a missing-mass discontinuity contribution, namely

$$\mathrm{Im}A^P(s,t)\simeq\sum_{n_1}\mathrm{Disc}_{M^2}A^R_{n_1+2\to n_1+2}(s,t)\,. \tag{6}$$

As the quantity $\mathrm{Disc}_{M^2}A^R_{n_1+2\to n_1+2}$ is essentially Reggeon-dominated, then (modulo some nonleading singularities) (Ref. 2)

$$\mathrm{Im}A^P(s,t)\simeq s^{\alpha_P(t)}\exp\left[\sum_\kappa\frac{c_\kappa(t)}{\kappa!}y\right],\quad y\simeq\ln s\,, \tag{7}$$

i.e.,

$$\alpha_P(t)\simeq\alpha_R(t)+\sum_\kappa\frac{c_\kappa(t)}{\kappa!}\,. \tag{8}$$

Here $\alpha_i(t)$ $i=P,R$ are assumed to be linear trajectories. Equation (8) is the starting point of our considerations. First, let us observe that the coefficients $c_\kappa(t)$ may be interpreted as the cumulants of a certain probability distribution function, i.e.,

$$\sum_\kappa\frac{c_\kappa(t)}{\kappa!}=\ln\varphi(t,1)\,, \tag{9}$$

where $\varphi(t,z)$ is the corresponding characteristic function. Next, we shall introduce a cluster picture by writing $\varphi(t,z)$ as ($|z|\leq 1$) (Ref. 15)

$$\varphi(t,z)=\exp\{\lambda[\tilde\varphi(t,z)-1]\}\,. \tag{10}$$

Hence

$$\sum_\kappa\frac{c_\kappa(t)}{\kappa!}=\lambda[\tilde\varphi(t,1)-1]\,, \tag{11}$$

where λ is the average number of clusters and $\tilde\varphi(t,z)$ describes the decay of clusters. Therefore, Eq. (8) becomes

$$\alpha_P(t)-\alpha_R(t)\simeq\lambda[\tilde\varphi(t,1)-1]\,. \tag{12}$$

Now, we shall derive an inequality which will be useful for the subsequent developments. Observing that ($|z|\leq 1$)

$$|1-z|^2=(1-z)(1-\bar z)\leq 2(1-\mathrm{Re}z)\,, \tag{13}$$

with $\tilde\varphi(t,z)=\int e^{izn}\tilde P(t,n)dn$ one has

$$|\tilde\varphi(t,z)-1|=\left|\int(e^{izn}-1)\tilde P(t,n)dn\right| \leq\int|e^{izn}-1|\,\tilde P(t,n)dn \tag{14}$$

$$\leq\sqrt2\int[1-\mathrm{Re}(e^{izn})]^{1/2}\tilde P(t,n)dn\,. \tag{15}$$

As for real θ, $1-\cos\theta\leq\frac12\theta^2$, the inequality (15) becomes

$$|\tilde\varphi(t,z)-1|\leq\int(zn)\tilde P(t,n)dn\leq\langle(zn)\rangle\,, \tag{16}$$

i.e.,

$$|\alpha_P(t)-\alpha_R(t)|\leq\lambda\langle n(t)\rangle\,. \tag{17}$$

We summarize the above results. Using some general properties of the characteristic functions and assuming a cluster picture, we obtained an upper bound for the difference of the trajectories in terms of the average multiplicity per cluster. Hence, one can study the approach to planarity by investigating the $t\to\infty$ behavior of $\langle n(t)\rangle$. In order to do so we shall utilize a result due to Squires and Webber,[16] who showed that, for a fairly general Regge-cluster model (which includes the model considered here), the average multiplicity per cluster is defined by

$$\langle n(s,t)\rangle\simeq\sum_{R_1R_2}C_{R_1R_2}(t)\ln s\,, \tag{18}$$

where

$$C_{R_1R_2}(t)\simeq C_{R_1R_2}(0)e^{a_{R_1R_2}t} \tag{19}$$

and R_1 and R_2 denote Reggeons. The behavior of $\langle n(t)\rangle$ with respect to t depends on the sign of $a_{R_1R_2}$, which, in turn, depends on the t behavior of the triple-Regge vertex $f_{R_1R_2P}(t_1,t_2;t)$. If one writes $f_{R_1R_2P}(t_1,t_2;t)=f_1(t_1t_2)f_2(t)$ one can have two possible cases:

(a) $f_2(t)$ falls off rapidly for $t<0$, i.e., $a_{R_1R_2}>0$. This is the case assumed by Squires and Webber,[16] and it corresponds to an increase (decrease) of $\langle n(t)\rangle$ with $t>0$ ($t<0$). This result is predicted by the random-walk model.[17]

(b) $f_2(t)$ increases rapidly for $t<0$, i.e., $a_{R_1R_2}<0$. This corresponds to a decrease (increase) of $\langle n(t)\rangle$ with $t>0$ ($t<0$).

The above factorized form for the triple-Regge vertex has also been utilized by Bishari,[11] who observed that a decrease of $f_2(t)$ for $t>0$ leads to asymptotic planarity. [This observation was made in another context with no reference to the behavior of $\langle n(t)\rangle$.]

Therefore, not only is the factorization property of the triple-Regge vertex (a current theoretical prejudice) crucial in determining the t behavior of $\langle n(t)\rangle$, but so is the t behavior of $f_2(t)$ (i.e., the sign of $a_{R_1R_2}$). This assertion modifies the emphasis of the conclusion in Ref. 16. From these considerations one can conclude that the necessary condition to ensure asymptotic planarity is the validity of case (b). Hence, the asymptotic planarity condition is in contradiction with the random-walk model. A possible ansatz consistent with the case (b), i.e., with the asymptotic planarity is

$$|\alpha_P(t)-\alpha_R(t)|\lesssim\mathrm{const}\times\frac{e^{-t/\lambda_y}}{\lambda_y}\,,\quad\lambda_y>0\,. \tag{20}$$

From this relation one can understand the first quenching interval defined by Chew and Rosenzweig[10] as being proportional to the "correlation length" λ_y. With $\alpha_R(t)=\alpha_0+\alpha' t$, one gets $\lambda_y \simeq 1/\alpha'$. One may also write $\lambda_y \simeq b/\alpha'^2$ (with $b\simeq \bar{y}\alpha'$ connected to the cluster size), which is approximately in agreement with the results given in Refs. 10 and 11.

We conclude this paper with one more observation. It is well known that the missing-mass discontinuities of the $(4+2n)$-point functions satisfy the so-called unitarity sum rules, i.e., constraints imposed by energy-momentum conservation.[18,19] These constraints plus dual-amplitude approximation imply a transverse-momentum cutoff which can explain the asymptotic planarity.[20] As a matter of fact, the Rosenzweig-Veneziano bootstrap condition,[12] which has been essential in Bishari's derivation,[11] is nothing but a particular case of the above-mentioned unitarity sum rules.

After this paper was completed, we became aware of a paper by Veneziano[21] where a similar method to study the approach to planarity is suggested. The onset of planarity is also tentatively connected with the transverse-momentum cutoff.

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