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Gauge fixing, the transfer matrix, and confinement on a lattice*

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We use the transfer-matrix formalism of statistical mechanics to relate Wilson's Lagrangian approach and the Kogut-Susskind Hamiltonian approach to gauge theories on a lattice. As a preliminary we discuss gauge fixing in Wilson's theory. This process leaves invariant Green's functions of gauge singlet operators. Taking the timelike lattice spacing to zero, we extract the Kogut-Susskind Hamiltonian from the transfer matrix in the gauge $A_0 = 0$.

I. INTRODUCTION

A theory of quarks interacting with non-Abelian gauge mesons has recently attracted considerable attention as a potential theory of the strong interaction.[1] The esthetic appeal of such a theory is heightened by the successes of gauge theory elsewhere in particle physics, first and most dramatically in quantum electrodynamics and more recently in the renormalizable theories of the weak interaction.

The strong-interaction theory is conjectured to differ in one fundamental respect from previous applications of gauge theory. The physical spectrum of states should not contain isolated particle states corresponding to the fundamental fields, i.e., quarks and vector "gluons," but rather should include only bound states which are singlets under the gauge group. This conjectured "quark confinement" would provide a most elegant resolution of the successes of the quark model with the lack of observation of free quarks.

Unfortunately there is a dearth of theoretical evidence supporting confinement. Renormalization-group arguments indicate that, for low momenta, Green's functions reflect a large effective coupling constant.[1] This only means that perturbation theory is not a reliable tool for investigating widely separated quarks. Indeed, low orders of perturbation theory give no clear signal of a nascent confinement.[2]

To circumvent conventional perturbation theory, Wilson proposed placing the theory on a discrete space-time lattice and then perturbing in the kinetic term for the gauge field.[3] In this formulation gauge invariance remains an exact local symmetry of the action and naturally leads to confinement. The hope is that the artifice of going to a lattice is nothing but an ultraviolet cutoff. Unfortunately it is not known if the continuum limit exists and is Poincaré invariant.

Balian, Drouffe, and Itzykson have presented a further analysis of this theory.[4] One important feature of their work is a mean field-theory calculation indicating that in a sufficiently high number of space-time dimensions a phase transition will occur as the coupling constant is varied. For strong coupling one obtains the confined phase studied by Wilson whereas for small coupling one reverts to a quantum-electrodynamics–like theory of free quarks and gluons. Migdal has presented approximate arguments that for non-Abelian gauge theories the transition to the unconfined phase requires more than four space-time dimensions.[5] Presumably the transition can occur for Abelian theories in four dimensions and has occurred for quantum electrodynamics, where we certainly do not want confinement. From this point of view, the photon is the Goldstone boson of the ordered phase.[6]

Kogut and Susskind have pursued an alternative approach to lattice gauge theory.[7] Keeping time as a continuous variable, they make three-dimensional space discrete. Working in the gauge $A_0 = 0$, they define a Hamiltonian for quark- and gauge-field degrees of freedom on the space lattice. Confinement arises from the invariance of this Hamiltonian under time-independent gauge transformations. Recently, in conjunction with several other authors, they have investigated the spectrum of this Hamiltonian in a strong-coupling approximation.[8]

In this paper we investigate the relation between the Lagrangian approach of Wilson and the Hamiltonian approach of Kogut and Susskind. Our main tool is the transfer-matrix method of statistical mechanics. The use of the transfer matrix was suggested by Wilson[3] as a method of relating his Euclidean path integral with quantum mechanics in Minkowski space-time.

One of the virtues of Wilson's formulation is that it does not require a gauge selection for quantization; however, the gauge invariance does allow gauge fixing without altering the physics of gauge-invariant quantities. We will discuss this point and then go to the gauge $A_0 = 0$, as used by Kogut

and Susskind. In this gauge the transfer matrix assumes a particularly simple form. Once the transfer matrix and the Hilbert space in which it acts are obtained, we find the Hamiltonian for the continuous-time theory by taking the timelike lattice spacing to zero.

The plan of this paper is as follows. In the next section we review Wilson's theory and discuss gauge invariance and its relation to confinement. In Sec. III we discuss the possibility of gauge fixing in the theory. Section IV reviews the use of the transfer matrix to obtain conventional quantum mechanics from a discrete-time path integral. In Sec. V we find the transfer matrix and the continuous-time Hamiltonian for pure gauge fields. We study the Fermi fields in Sec. VI and give the complete coupled Hamiltonian in the concluding Sec. VII.

II. REVIEW OF WILSON'S THEORY

Electrodynamics can be formulated in terms of a path-dependent phase acquired by the wave function of a charged particle moving through an electromagnetic field.[9] For a non-Abelian gauge theory this phase is replaced by a rotation in the internal-symmetry space of the theory. This notion of a nonintegrable phase factor forms the basis of Wilson's prescription for formulating gauge fields on a lattice. Representing the gauge-field degrees of freedom, an internal-symmetry rotation is associated with each pair of nearest-neighbor sites on the lattice. The fermion degrees of freedom reside on the sites. Whenever a fermion moves from one site to a neighboring one, its wave function undergoes the corresponding internal-symmetry rotation.

We now formulate the theory precisely. Working on a hypercubical lattice of spacing a, we label sites on the corners of the hypercubes by an index i. We consider a four-component spinor field $\psi_i{}^\alpha$ for each site i and for every value of the internal-symmetry index α. The internal-symmetry group is a connected unitary group $\mathcal{G}$ of matrices $g^{\alpha\beta}$. For simplicity we assume that the field $\psi_i{}^\alpha$ transforms under the fundamental representation

$$\psi_i{}^\alpha \to g^{\alpha\beta}\psi_i{}^\beta , \tag{2.1}$$

where a sum over β is understood. For every pair of nearest-neighbor sites $\{i,j\}$ we introduce the gauge field $U_{ij}{}^{\alpha\beta}$ which is a matrix in the group $\mathcal{G}$. Under interchange of i and j we do not obtain a new degree of freedom but rather require that U become its inverse in the group

$$(U_{ji})^{\alpha\beta} = (U_{ij}{}^{-1})^{\alpha\beta} . \tag{2.2}$$

In terms of these degrees of freedom the Euclidean action defining the theory is

$$\begin{aligned} S = &-a^4 \sum_{i,j} \frac{A_{ij}}{2a} \bar{\psi}_i (1 - \gamma_\mu e_{ij}{}^\mu) U_{ij} \psi_j \\ &+ a^4 \sum_i \left(\frac{4}{a} - m \right) \bar{\psi}_i \psi_i \\ &- \frac{1}{8g^2} \sum_{ijkl} P_{ijkl} \operatorname{Tr}(U_{ij} U_{jk} U_{kl} U_{li}) , \end{aligned} \tag{2.3}$$

where we suppress internal-symmetry indices,

$$A_{ij} = \begin{cases} 1 \text{ if } i \text{ and } j \text{ are nearest neighbors} \\ 0 \text{ otherwise} , \end{cases} \tag{2.4}$$

$e_{ij}{}^\mu$ is a unit vector pointing from site i to site j, γ_μ are *Euclidean* Dirac matrices satisfying

$$\begin{aligned} &[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu} , \\ &\gamma_\mu^\dagger = \gamma_\mu , \end{aligned} \tag{2.5}$$

a Euclidean sum is understood in $\gamma_\mu e^\mu$, g is the bare gauge field coupling, and

$$P_{ijkl} = \begin{cases} 1 \text{ if } i,j,k,l \text{ run around a "plaquette," i.e., a square of side } a \text{ in the lattice} \\ 0 \text{ otherwise} . \end{cases} \tag{2.6}$$

A discussion of the reduction of Eq. (2.3) in the limit $a \to 0$ to the usual classical-gauge-theory action is contained in Refs. 3 and 4. The projection matrices $(1 - \gamma_\mu e_\mu)$ inserted in Eq. (2.3) ensure that free fermions will have low energy only for low momenta. Without these factors, one will have low-energy fermions with momenta of order π/a.

From the action in Eq. (2.3), Wilson defines Euclidean Green's functions by the path-integral formula

$$\begin{aligned} &G(\psi_i, \psi_j, \ldots, \bar{\psi}_l; U_{mn}, \ldots, U_{yz}) \\ &\quad = \frac{1}{Z} \int [d\psi dU] e^{-S} \psi_i \cdots \bar{\psi}_l U_{mn} \cdots U_{yz} , \end{aligned} \tag{2.7}$$

where

$$Z = \int [d\psi dU] e^{-S} . \tag{2.8}$$

The integral over fermion degrees of freedom is standard and will be defined precisely in Sec. VI. The measure $[dU]$ means that each independent U_{ij} is integrated over the internal-symmetry group with the Haar measure. This measure for compact groups has the properties

$$\int dg f(g) = \int dg f(gg_0) = \int dg f(g_0 g) = \int dg f(g^{-1}) \tag{2.9}$$

for any function $f(g)$ over the group and for any group element g_0. The measure is normalized

$$\int dg = 1 . \tag{2.10}$$

The formal argument for confinement is based on the local gauge symmetry of the action in Eq. (2.3). Given an arbitrary group element g_i for each site i, the action is invariant under

$$\begin{aligned} &\psi_i{}^{\alpha} \to g_i{}^{\alpha\beta}\psi_i{}^{\beta} , \\ &U_{ij}{}^{\alpha\beta} \to (g_i U_{ij} g_j{}^{-1})^{\alpha\beta} . \end{aligned} \tag{2.11}$$

The integration measure in Eq. (2.7) is also invariant under this transformation; consequently, if boundary conditions on the lattice can be ignored, we have

$$\begin{aligned} &G(\psi_i, \ldots; U_{mn}, \ldots) \\ &\qquad = G(g_i\psi_i, \ldots; g_m U_{mn} g_n{}^{-1}, \ldots) . \end{aligned} \tag{2.12}$$

This relation is true for arbitrary g_i; thus G can only depend on locally singlet combinations of the ψ_i and U_{ij}. In particular we conclude

$$G(\bar{\psi}_i, \psi_j) = 0 \quad \text{for } i \neq j , \tag{2.13}$$

which means that free quarks cannot propagate.

This argument can break down if boundary conditions imposed on the U_{ij} at the lattice edges affect the U_{ij} deep in the interior. In other words, for infinite lattices a phase transition to an ordered state may occur. This transition has been discussed in Refs. 4 and 5 and is expected to occur for the lattice version of quantum electrodynamics if this type of theory is to describe unconfined electrons and photons.

III. GAUGE FIXING AND THE LATTICE THEORY

A virtue of the Wilson approach is that we need not choose a gauge before quantizing. The integrals over the U_{ij} are finite because the group G is compact; consequently, no infinities arise from integrating over all gauges. On the other hand, gauge invariance of the action still permits working in a fixed gauge without affecting Green's functions of gauge-invariant operators. In this section we define gauge fixing in the lattice theory.

Let $P(\psi, \bar{\psi}, U)$ be some polynomial in the fields which is invariant under the general gauge transformation of Eq. (2.11). Associated with this gauge-invariant polynomial is a Green's function

$$G(P) = \frac{1}{Z} \int [d\psi dU] e^{-S} P(\psi, \bar{\psi}, U) . \tag{3.1}$$

We begin the gauge-fixing process by concentrating on a single link from site i to site j. A δ function $\delta(g', g)$ on the group $\mathcal{G}$ has the properties

$$\begin{aligned} &\int dg\, \delta(g', g) f(g) = \int dg\, \delta(g, g') f(g) = f(g') , \\ &\delta(g, g') = \delta(g_0 g g_1, g_0 g' g_1) \end{aligned} \tag{3.2}$$

for arbitrary g_0 and g_1. Consider the integral

$$I(P, g_0) = \frac{1}{Z} \int [d\psi dU]\, \delta(U_{ij}, g_0) e^{-S} P(\psi, \bar{\psi}, U) , \tag{3.3}$$

where g_0 is some group element. We have

$$\int dg_0 I(P, g_0) = G(P) . \tag{3.4}$$

If we now perform the gauge transformation of Eq. (2.11) on Eq. (3.3), we obtain

$$I(P, g_0) = I(P, g_i{}^{-1} g_0 g_j) . \tag{3.5}$$

Since g_i and g_j are arbitrary, $I(P, g_0)$ must be independent of g_0. Equation (3.4) then implies

$$\begin{aligned} G(P) &= I(P, g_0) \\ &= \frac{1}{Z} \int [d\psi dU]\, \delta(U_{ij}, g_0) e^{-S} P(\psi, \bar{\psi}, U) . \end{aligned} \tag{3.6}$$

Thus, to calculate a gauge-invariant Green's function we can set any particular U_{ij} to an arbitrary group element and only integrate over the remaining U's.

This process can be repeated to fix more U's. The final result is that we can arbitrarily fix any set of U's as long as this set contains no closed loops; i.e., the fixed U's form a tree (possibly disconnected). A gauge is completely determined by first choosing a maximal tree T, a tree to which no more links can be added without forming a loop. Then the U_{ij} on the tree are set to arbitrary elements g_{ij}. The Green's functions of gauge-invariant operators are then found by integrating over the remaining U's. The general formula is

$$\begin{aligned} G(P) = \frac{1}{Z} \int [d\psi dU] &\Big[\prod_{\{i,j\} \subset T} \delta(U_{ij}, g_{ij}) \Big] \\ &\times e^{-S} P(\psi, \bar{\psi}, U) . \end{aligned} \tag{3.7}$$

The notation $\{i,j\}$ means the link connecting sites i and j with arbitrary orientation. An example of a maximal tree is shown in Fig. 1.

The most natural element of G to use for fixing U's is the identity element. The simplest trees are those with many straight branches. The gauge we shall use for the rest of this paper corresponds to taking $A_0=0$ and was discussed briefly in Ref. 4. The tree for this case includes all timelike links, upon which the U_{ij} are set equal to the unit matrix. We are still free to make time-independent gauge transformations, which corresponds to the possibility of fixing a set of spacelike U's that link together the timelike branches of the tree. Such a tree is illustrated in Fig. 2.

The utility of the gauge $A_0=0$ is that the trace of four U's in a timelike plaquette becomes a trace of only two U's at subsequent times. We relabel the lattice sites with two indices i and t, where i represents the space coordinates and $a_0 t$ the time. Here we allow the timelike lattice spacing a_0 to differ from the spacelike spacing a. The action for the lattice gauge theory in the gauge $A_0=0$ then becomes

$$S=-a^3a_0\sum_{i,j,t}\frac{A_{ij}}{2a}\bar{\psi}_{i,t}(1-\vec{\gamma}\cdot\vec{e}_{ij})U_{ij,t}\psi_{j,t}-a^3a_0\sum_{i,t}\frac{1}{2a_0}[\bar{\psi}_{i,t+1}(1+\gamma_0)\psi_{i,t}+\bar{\psi}_{i,t}(1-\gamma_0)\psi_{i,t+1}]$$
$$+a^3a_0\sum_{i,t}\left(\frac{3}{a}+\frac{1}{a_0}-m\right)\bar{\psi}_{i,t}\psi_{i,t}-\frac{a}{2g^2a_0}\sum_{i,j,t}A_{ij}\,\mathrm{Tr}(U_{ij,t+1}{}^{-1}U_{ij,t})$$
$$-\frac{a_0}{8g^2a}\sum_{ijkl,t}P_{ijkl}\,\mathrm{Tr}(U_{ij,t}U_{jk,t}U_{kl,t}U_{li,t}). \tag{3.8}$$

Here the definitions of A_{ij} and P_{ijkl} in Eqs. (2.4) and (2.6) are restricted to the spacelike lattice. Note that the term coupling U's at different times resembles the Hamiltonian for the statistical mechanics of a set of one-dimensional classical spin chains with nearest-neighbor interactions. The final term in Eq. (3.8) represents a 4-spin coupling between the chains. Note that in two-dimensional space-time there is no interchain coupling and the pure-gauge part of the theory is a trivial one-dimensional statistical-mechanics problem.

IV. TRANSFER MATRIX AND QUANTUM MECHANICS

In this section we review the use of the transfer matrix to relate a Feynman path integral to the conventional operator formulation of quantum mechanics. The discussion is similar to that given in Feynman's original paper.[10] A treatment of a conventional scalar field theory is in Ref. 11. We illustrate the method on the one-degree-of-freedom harmonic oscillator with Lagrangian

$$L=\tfrac{1}{2}\dot{x}^2-\tfrac{1}{2}\omega^2x^2. \tag{4.1}$$

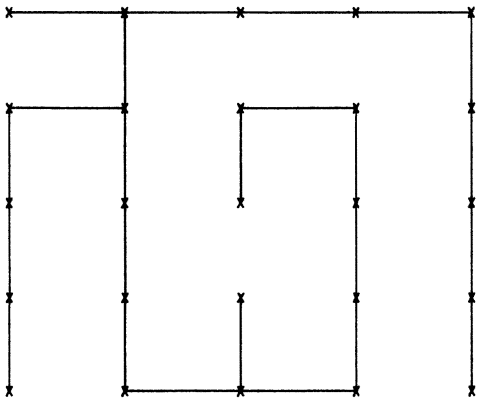

FIG. 1. An example of a maximal tree on a two-dimensional lattice. The U's corresponding to all links on the tree can be set to arbitrary group elements by the gauge-fixing process.

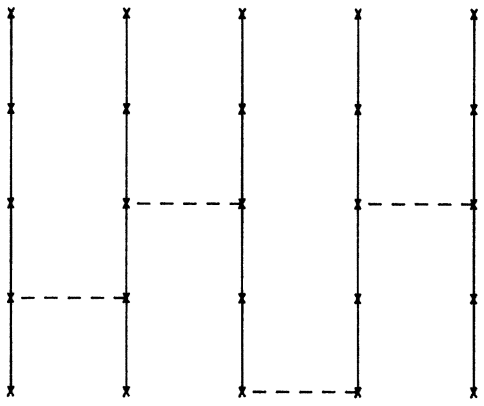

FIG. 2. A tree corresponding to the gauge $A_0=0$. Here the vertical direction represents time. The dashed links can be fixed by time-independent gauge transformations.

Going to an imaginary time lattice of spacing a leads to the action

$$S = a \sum_i \left[\tfrac{1}{2} \left(\frac{x_{i+1} - x_i}{a} \right)^2 + \frac{\omega^2}{2} x_i^2 \right] . \tag{4.2}$$

The path-integral prescription involves integrals such as

$$Z = \int [dx] e^{-S} . \tag{4.3}$$

The key to the transfer-matrix method is that the local nature of (4.2) allows Z to be written

$$Z = \int \prod_i (dx_i T_{x_{i+1} x_i}) , \tag{4.4}$$

where

$$T_{x',x} = \exp\left[- \frac{1}{2a} (x' - x)^2 - \frac{a\omega^2}{2} x^2 \right] . \tag{4.5}$$

Consider the Hilbert space of functions of fast decrease with inner product

$$\langle \psi' \mid \psi \rangle = \int dx \, \psi'^*(x) \psi(x) . \tag{4.6}$$

For notational convenience we expand states in this space in terms of the nonnormalizable basis $\{ \mid x \rangle \}$ so that

$$\begin{aligned} \mid \psi \rangle &= \int dx \, \psi(x) \mid x \rangle , \\ \langle x' \mid x \rangle &= \delta(x' - x) , \\ 1 &= \int dx \mid x \rangle \langle x \mid . \end{aligned} \tag{4.7}$$

Introduce the operators $\hat{p}$ and $\hat{x}$ with properties

$$\begin{aligned} \hat{x} \mid x \rangle &= x \mid x \rangle , \\ [\hat{p}, \hat{x}] &= -i , \\ e^{-i\hat{p}a} \mid x \rangle &\mid x + a \rangle . \end{aligned} \tag{4.8}$$

In this Hilbert space we define the operator T by

$$\langle x' \mid T \mid x \rangle = T_{x'x} , \tag{4.9}$$

where $T_{x'x}$ is given in Eq. (4.5). Taking a finite lattice of N sites and imposing periodic boundary conditions, we obtain

$$Z = \mathrm{Tr}(T^N) . \tag{4.10}$$

Green's functions of the theory are obtained by inserting polynomials of the x_i into Eq. (4.3). This corresponds to inserting the operator $\hat{x}$ in the appropriate places in the trace of Eq. (4.10).

Connection with the usual Hamiltonian is made by taking the lattice spacing a to zero. Using Eqs. (4.8), we write T in terms of $\hat{p}$ and $\hat{x}$

$$\begin{aligned} T &= \int d\Delta \, e^{-(1/2a)\Delta^2} e^{-i\hat{p}\Delta} e^{-a\omega^2 \hat{x}^2/2} \\ &= (2\pi a)^{1/2} e^{-a\hat{p}^2/2} e^{-a\omega^2 \hat{x}^2/2} \\ &= (2\pi a)^{1/2} \exp[-aH + O(a^2)] , \end{aligned} \tag{4.11}$$

where

$$H = \tfrac{1}{2} (\hat{p}^2 + \omega^2 \hat{x}^2) \tag{4.12}$$

is the usual harmonic-oscillator Hamiltonian.

The procedure for going from a path-integral to a Hilbert-space formulation of quantum mechanics consists of two steps. First, construct the transfer matrix T and the space on which it acts. Second, take the logarithm of the transfer matrix and identify the coefficient of the linear term in the lattice spacing as the Hamiltonian. Physically, the transfer matrix propagates the system from one time to the next, and this is the role played by the exponentiated Hamiltonian in the Hilbert-space formulation. In statistical mechanics the thermodynamic properties of a system are determined by the largest eigenvalue of the transfer matrix. In ordinary quantum mechanics the corresponding eigenvector has the lowest eigenvalue of the Hamiltonian, i.e., it is the vacuum or ground state of the system. In the remainder of this paper we will construct the transfer matrix for the lattice gauge theory in the gauge $A_0 = 0$.

V. PURE GAUGE FIELDS

We postpone discussion of the quark fields to the next section and consider here only the pure-gauge part of the action in Eq. (3.8)

$$\begin{aligned} S = &- \frac{a}{2g^2 a_0} \sum_{ij,t} A_{ij} \, \mathrm{Tr}(U_{ij,t+1}^{-1} U_{ij,t}) \\ &- \frac{a_0}{8g^2 a} \sum_{ijkl,t} P_{ijkl} \, \mathrm{Tr}(U_{ij,t} U_{jk,t} U_{kl,t} U_{li,t}) . \end{aligned} \tag{5.1}$$

By analogy with the last section we wish to find a transfer matrix T such that

$$Z = \int [dU] e^{-S} = \mathrm{Tr}(T^N) , \tag{5.2}$$

where N is the number of discrete times and we have imposed periodic boundary conditions.

The space in which T operates is a direct product of spaces of square-integrable functions over the group G. A state $\mid \psi \rangle$ in this space is specified by a set of square-integrable functions $\psi_{ij}(g)$ over the group, one such function corresponding to each different pair of nearest-neighbor sites $\{i,j\}$ on a

spacelike lattice. A pair is not considered as being different if i and j are interchanged. The inner product in this space is

$$\langle\psi'\,|\,\psi\rangle = \prod_{\{i,j\}}\left[\int dg\,\psi'^{*}_{ij}(g)\,\psi_{ij}(g)\right], \tag{5.3}$$

where the product runs over all different nearest-neighbor pairs $\{i,j\}$. We expand the general state in the nonnormalizable basis $\{|\,U\rangle\}$, where a state in this basis is determined by a group element U_{ij} for each nearest-neighbor pair $\{i,j\}$. The U_{ij} satisfy a condition similar to Eq. (2.2),

$$U_{ij} = U_{ji}^{-1}. \tag{5.4}$$

The overlap in this basis is

$$\langle U'\,|\,U\rangle = \prod_{\{i,j\}}\delta(U'_{ij},U_{ij}). \tag{5.5}$$

Completeness is written

$$1 = \int [dU]\,|\,U\rangle\langle U\,|. \tag{5.6}$$

The general state is then expanded

$$|\,\psi\rangle = \int [dU]\,|\,U\rangle \prod_{\{i,j\}}\psi_{ij}(U_{ij}). \tag{5.7}$$

Working in this Hilbert space, we can immediately write down an operator T satisfying Eq. (5.2)

$$\langle U'\,|\,T\,|\,U\rangle = \exp\left[\frac{a}{2g^2a_0}\sum_{ij}A_{ij}\,\mathrm{Tr}U'^{-1}_{ij}U_{ij}\right]\exp\left[\frac{a_0}{8g^2a}\sum_{i,j,k,l}P_{ijkl}\,\mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li})\right]. \tag{5.8}$$

Just as we expressed T for the harmonic oscillator in terms of the operators $\hat{p}$ and $\hat{x}$, we would like to write this T in terms of some simple operators in the present Hilbert space. We begin by defining a set of matrix operators $\hat{U}^{\alpha\beta}_{ij}$ and unitary operators $R_{ij}(g)$,

$$\hat{U}^{\alpha\beta}_{ij}\,|\,U\rangle = U^{\alpha\beta}_{ij}\,|\,U\rangle, \tag{5.9}$$

$$R_{ij}(g)\,|\,U\rangle = |\,U'\rangle, \tag{5.10}$$

where

$$\begin{aligned} U'_{kl} &= U_{kl}, \quad \text{for } \{k,l\}\neq\{i,j\} \\ U'_{ij} &= gU_{ij}. \end{aligned} \tag{5.11}$$

The operators $R_{ij}(g)$ satisfy the group-representation property

$$R_{ij}(g)R_{ij}(g') = R_{ij}(gg'). \tag{5.12}$$

In terms of these operators we express T as

$$T = \left(\prod_{\{i,j\}}\left\{\int dg\,R_{ij}(g)\exp\left[\frac{a}{2g^2a_0}\mathrm{Tr}(g+g^{-1})\right]\right\}\right) \times\exp\left(\frac{a_0}{8g^2a}\sum_{ijkl}P_{ijkl}\hat{U}_{ij}\hat{U}_{jk}\hat{U}_{kl}\hat{U}_{li}\right). \tag{5.13}$$

This form is not yet simple enough to discuss the limit $a_0\to 0$. To proceed we must review some group theory. We parametrize the elements of our unitary group in the standard form

$$g = g(x) = e^{i\Sigma_m x_m\Lambda_m} \equiv e^{ix\cdot\Lambda}, \tag{5.14}$$

where $\{\Lambda_m\}$ is a set of Hermitian matrices that generate the group. We orthonormalize them such that

$$\mathrm{Tr}(\Lambda_m\Lambda_n) = \delta_{mn}. \tag{5.15}$$

The Λ_α satisfy an algebra

$$[\Lambda_l,\Lambda_m] = if_{lmn}\Lambda_n, \tag{5.16}$$

where $f_{\alpha\beta\gamma}$ is totally antisymmetric in its indices. The group integration measure takes the form

$$d(g(x)) = J(x)\prod_m dx_m, \tag{5.17}$$

where the Jacobian function $J(x)$ is determined by the group multiplication law. It satisfies

$$J(x) = J(-x) \tag{5.18}$$

and in a neighborhood of $x_m = 0$ it is regular and nonvanishing.

Because of the representation property in Eq. (5.12), $R_{ij}(g)$ can be written

$$R_{ij}(g(x)) = e^{i\Sigma_m x_m l^m_{ij}} = e^{ix\cdot l_{ij}}, \tag{5.19}$$

where the $l_{ij}{}^m$ are Hermitian operators with the following properties:

$$[l^l_{ij},l^m_{ij}] = if_{lmn}l^n_{ij}, \tag{5.20}$$

$$[l^m_{ij},\hat{U}_{ij}] = -\Lambda^m\hat{U}_{ij}, \tag{5.21}$$

$$[l^m_{ij},\hat{U}_{ji}] = \hat{U}_{ji}\Lambda^m, \tag{5.22}$$

$$[l_{ij}{}^2,l^m_{ij}] = 0 = [l_{ij}{}^2,R_{ij}(g)], \tag{5.23}$$

where

$$l_{ij}{}^2 = l_{ji}{}^2 = \sum_m l^m_{ij}\,l^m_{ij} \tag{5.24}$$

is the quadratic Casimir operator for the group. In our Hilbert space of square-integrable functions over the group, the $l_{ij}{}^m$ represent differential operators in the group parameters.

Using Eqs. (5.14), (5.17), and (5.19), we write

$$T=\prod_{\{i,j\}}\left\{\int\left(\prod_m dx^m\right)J(x)\,e^{il_{ij}\cdot x}\exp\left[\frac{a}{2g^2a_0}\,\mathrm{Tr}(2\cos\Lambda\cdot x)\right]\right\}\exp\left[\frac{a_0}{8g^2a}\sum_{ijkl}P_{ijkl}\,\mathrm{Tr}(\hat{U}_{ij}\hat{U}_{jk}\hat{U}_{kl}\hat{U}_{li})\right]. \tag{5.25}$$

When $a_0\to 0$, the integral over x is dominated by x near the maximum of $\mathrm{Tr}(2\cos\Lambda\cdot x)$. For a unitary group this maximum always occurs near $x=0$; consequently, we have

$$\mathrm{Tr}(2\cos\Lambda\cdot x)=2n-x^2+O(x^4), \tag{5.26}$$

where n is the dimension of the group matrices. Inserting Eq. (5.26) into Eq. (5.25), we do the Gaussian x integrations with the result

$$T=N\exp[-a_0H+O(a_0^2)], \tag{5.27}$$

where

$$H=\frac{g^2}{2a}\sum_{\{i,j\}}l_{ij}^2-\frac{1}{8g^2a}\sum_{ijkl}P_{ijkl}\,\mathrm{Tr}(\hat{U}_{ij}\hat{U}_{jk}\hat{U}_{kl}\hat{U}_{li}) \tag{5.28}$$

and N is an irrelevant constant factor. This is the gauge-field part of the Hamiltonian used by Kogut and Susskind.

We close this section with a brief discussion of the remaining gauge freedom of the theory. As we have only specified $A_0=0$, we can still do time-independent gauge transformations. An operator that performs such a transformation is $\prod_i J_i(g_i)$, where g_i are arbitrary group elements and

$$J_i(g(x))=\exp\left(ix\cdot\sum_j A_{ij}\,l_{ij}\right) \tag{5.29}$$

generates a local gauge transformation at the site i. Using the invariance of the group integration measure and the cyclic properties of the trace, one can show

$$[T,J_i(g)]=0=[H,J_i(g)]; \tag{5.30}$$

consequently, $J_i(g)$ generates a symmetry of the theory.

VI. QUARK FIELDS

In this section we apply the transfer-matrix formalism to the quark fields. This discussion does not exactly parallel the treatments in Secs. IV and V for two reasons: (1) The quark Lagrangian is only linear in the time derivatives of the quark field, and (2) the quark fields obey anticommutation relations. In addition, technical differences in Wilson's approach to fermions preclude our obtaining exactly the Kogut-Susskind quark Hamiltonian. Wilson uses a four-component field at each site, whereas in Ref. 8 a one-component field suffices.

In the gauge $A_0=0$ the coupling term between the quarks and gauge fields only involves fields at equal times; consequently, it will be trivial to find the coupled Hamiltonian from the free-fermion Hamiltonian. Thus, for simplicity we extract from Eq. (3.8) the action for free lattice fermions

$$S=-a^3a_0\sum_{i,t}\frac{1}{2a_0}[\bar{\psi}_{i,t+1}(1+\gamma_0)\psi_{i,t}+\bar{\psi}_{it}(1-\gamma_0)\psi_{i,t+1}]-a^3a_0\sum_{i,j,t}\frac{A_{ij}}{2a}\bar{\psi}_{i,t}(1-\vec{\gamma}\cdot\vec{e}_{ij})\psi_{j,t}$$
$$+a^3a_0\sum_{i,t}\left(\frac{3}{a}+\frac{1}{a_0}-m\right)\bar{\psi}_{i,t}\psi_{i,t}. \tag{6.1}$$

The transfer matrix will act on a Hilbert space of fermions on a three-space lattice. The fundamental spinor operators $\chi_i{}^\alpha$ in this space satisfy

$$[\chi_i{}^\alpha,\chi_j{}^{\beta\dagger}]_+=a^3\delta_{ij}\delta_{\alpha\beta},\qquad [\chi_i{}^\alpha,\chi_j{}^\beta]_+=0. \tag{6.2}$$

To simplify the following formulas, we introduce the four-by-four projection matrices

$$P_\pm=\tfrac{1}{2}(1\pm\gamma_0). \tag{6.3}$$

We define a "bare vacuum" state $|0\rangle$ by

$$P_+\chi_i{}^\alpha|0\rangle=0,\qquad \bar{\chi}_i{}^\alpha P_-|0\rangle=0,\qquad \langle 0|0\rangle=1. \tag{6.4}$$

The general state in the Hilbert space is generated by application of polynomials in $\chi_i{}^\alpha$ and $\bar{\chi}_i{}^\alpha$ to this vacuum.

We now consider a set of anticommuting objects

$\psi_i{}^\alpha$ and $\bar{\psi}_i{}^\alpha$ and form a "Fermi-coherent" state[12]

$$|\psi\rangle = \exp\left[a^3 \sum_i (\bar{\chi}_i P_+ \psi_i + \bar{\psi}_i P_- \chi_i)\right] |0\rangle . \tag{6.5}$$

This state has the properties

$$\begin{aligned} &\langle 0|\psi\rangle = 1 , \\ &P_+ \chi_i{}^\alpha |\psi\rangle = P_+ \psi_i{}^\alpha |\psi\rangle , \\ &\bar{\chi}_i{}^\alpha P_- |\psi\rangle = -\bar{\psi}_i{}^\alpha P_- |\psi\rangle . \end{aligned} \tag{6.6}$$

The overlap between two of these states is

$$\langle \psi' | \psi\rangle = \exp\left[a^3 \sum_i (\bar{\psi}_i' P_+ \psi_i + \bar{\psi}_i P_- \psi_i')\right] , \tag{6.7}$$

where we have assumed that ψ_i and ψ_i' anticommute. Note that if we set $\psi_i = \psi_{i,t}$ and $\psi_i' = \psi_{i,t+1}$, then the argument of this exponential is just the time-changing term in Eq. (6.1). The states $|\psi\rangle$ form an overcomplete set just as the coherent states for boson fields do.[13] In analogy with the boson case, we define a completeness relation

$$1 = \int [d\psi d\psi^\dagger] |\psi\rangle \langle \psi| e^{-a^3 \Sigma_i \bar{\psi}_i \psi_i} . \tag{6.8}$$

This equation can be considered as defining the integral over the anticommuting objects ψ and $\psi^\dagger$ at one time. Integrals of various polynomials in ψ and $\psi^\dagger$ can be defined from this equation using Eq. (6.6). Note that the argument of the exponential in Eq. (6.8) is the remaining term in the action that does not contain a factor of a_0.

In this fermion Hilbert space we define the operator

$$T = \exp(-a_0 H) , \tag{6.9}$$

where

$$\begin{aligned} H = a^3 \sum_{i,j} &\left[\frac{1}{2a} A_{ij} \bar{\chi}_i (1 - \vec{\gamma} \cdot \vec{e}_{ij}) \chi_j\right] \\ &- a^3 \sum_i \left(\frac{3}{a} - m\right) : \bar{\chi}_i \chi_i : . \end{aligned} \tag{6.10}$$

Here the normal-ordering symbol : : means with respect to the vacuum of Eq. (6.3). Matrix elements of T between the Fermi-coherent states are

$$\begin{aligned} \langle \psi_{t+1}| T |\psi_t\rangle = \exp\Big[&+ a_0 a^3 \sum_{ij} \frac{1}{2a} A_{ij} (\bar{\psi}_{i,t+1} P_+ + \bar{\psi}_{i,t} P_-)(1 - \vec{\gamma} \cdot \vec{e}_{ij})(P_+ \psi_{j,t} + P_- \psi_{j,t+1}) \\ &- a_0 a^3 \sum_i \left(\frac{3}{a} - m\right) (\bar{\psi}_{i,t+1} P_+ + \bar{\psi}_{i,t} P_-)(P_+ \psi_{i,t} + P_- \psi_{i,t+1})\Big] \langle \psi_{t+1}|\psi_t\rangle . \end{aligned} \tag{6.11}$$

To proceed we consider small a_0 and assume

$$\psi_{i,t+1} - \psi_{i,t} = O(a_0) . \tag{6.12}$$

This implies

$$\begin{aligned} P_+ \psi_{j,t} + P_- \psi_{j,t+1} &= \psi_{j,t} + P_- (\psi_{j,t+1} - \psi_{j,t}) \\ &= \psi_{j,t} + O(a_0) . \end{aligned} \tag{6.13}$$

In Eq. (6.11) this gives

$$\begin{aligned} &\langle \psi_{t+1}| T |\psi_t\rangle \\ &= \exp\Big[+ a_0 a^3 \sum_{i,j} \frac{1}{2a} A_{ij} \bar{\psi}_i (1 - \vec{\gamma} \cdot \vec{e}_{ij}) \psi_j \\ &\quad - a_0 a^3 \sum_i \left(\frac{3}{a} - m\right) \bar{\psi}_i \psi_i + O(a_0^2)\Big] \langle \psi_{t+1}|\psi_t\rangle . \end{aligned} \tag{6.14}$$

Up to the $O(a_0^2)$ term, the argument of this exponential gives the remaining terms in the action of Eq. (6.1). Combining Eqs. (6.7), (6.8), and (6.14), we obtain

$$Z = \int [d\psi d\psi^\dagger] e^{-S} = \mathrm{Tr}(\{\exp[-a_0 H + O(a_0^2)]\}^N) \tag{6.15}$$

with H given by Eq. (6.10).

VII. CONCLUSIONS

Combining the results of Secs. V and VI, we can easily construct the full interacting Hamiltonian

$$\begin{aligned} H = &\frac{g^2}{2a} \sum_{\{i,j\}} l_{ij}{}^2 - \frac{1}{8g^2 a} \sum_{ijkl} P_{ijkl} \,\mathrm{Tr}(\hat{U}_{ij} \hat{U}_{jk} \hat{U}_{kl} \hat{U}_{li}) \\ &+ a^3 \sum_{i,j} \left[\frac{1}{2a} A_{ij} \bar{\chi}_i (1 - \vec{\gamma} \cdot \vec{e}_{ij}) \hat{U}_{ij} \chi_j\right] \\ &- a^3 \sum_i \left(\frac{3}{a} - m\right) : \bar{\chi}_i \chi_i : . \end{aligned} \tag{7.1}$$

This differs from the Hamiltonian of Kogut and Susskind only in that we treat the fermions with Wilson's projection-operator technique. We regard this as a technical point, although a careful discussion of the chiral invariance of the theory for $m = 0$ may reveal something more.[14] Indeed, the complications in placing fermions on a lattice are intimately related to the Adler-Bell-Jackiw anomaly in current algebra[15] and deserve further study.

In summary, modulo these technical points in the treatment of fermions, we have derived the Kogut-Susskind Hamiltonian for lattice gauge theories from Wilson's Lagrangian formulation.

This required addition of a gauge-fixing term to Wilson's theory. In the gauge $A_0 = 0$, the gauge-field part of the theory is equivalent to a statistical-mechanical system of coupled one-dimensional spin chains. The nature of the phase transition to the unconfined phase needs further study. In particular, ordinary quantum electrodynamics should be obtained in a continuum limit from the ordered phase.

If the Wilson and the Kogut-Susskind approaches are equivalent, which is preferable? The answer is clearly a matter of taste. In the Wilson form, space-time symmetry is more apparent, as is the relation of the possible phase transition to a statistical-mechanics problem. The particle spectrum is given by the singularity structure of Green's functions. In the Kogut-Susskind approach one deals with a generalization of conventional quantum mechanics with a well-defined Hamiltonian operator. The spectrum of the theory is the spectrum of this Hamiltonian, and the phase transition should be related to a level crossing in the infinite-volume limit.

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