PHYSICAL REVIEW D VOLUME 15, NUMBER 4 15 FEBRUARY 1977

Lattice gauge theories and the continuum limit in two dimensions

M. W. Roth
*Fermi National Accelerator Laboratory, Batavia, Illinois 60510**
(Received 18 June 1976)

Lattice gauge theories in two dimensions are studied with regard to investigating the continuum limit. The effective interaction is calculated for the lattice gauge theories for QED [U(1)] and SU(N) to all orders in the gauge coupling and is shown to reproduce the usual Schwinger and 't Hooft models, respectively, in the limit of zero lattice spacing. However, lattice gauge theories in strong coupling have, in general, qualitatively different S matrices than their expected continuum analogs. Except for the SU(N) lattice gauge theory in the formal limit $N \to \infty$, g^2N fixed, the lattice introduces additional four-or-more-body forces which are not present in the continuum.

I. INTRODUCTION

There is considerable interest in lattice formulations of gauge theories.[1-9] The major reason for this interest is that they provide a gauge-invariant description of quark confinement. A central question in their study is the nature of the continuum limit. While lattice gauge theories are designed to give the correct continuum limit for the classical lattice theory of quarks and gauge fields, the continuum limit of the *quantum* theory is a crucial problem.

In this paper, we study two-dimensional lattice gauge theories of quarks and lattice gauge fields in order to gain insight into the relationship to the continuum theory. In particular, we consider the QED [U(1)] and SU(N) lattice gauge theories as formulated by Wilson on a space-time lattice. The effective interaction is calculated and is shown to reproduce the usual continuum Schwinger[10-13] and 't Hooft[14-16] models respectively in the limit of zero lattice spacing when the correct physical quantities are held fixed. We show that if the lattice spacing a is taken to zero, holding the bare lattice coupling constant g_0 fixed, the lattice models become the usual continuum models with a coupling constant g_0. This behavior is expected because the two-dimensional continuum models are superrenormalizable and there is no coupling-constant renormalization. In addition, we define a physical two-body coupling constant g_R^2 and take the limit $a \to 0$ holding g_R^2 fixed. Then we show that $g_0^2(a) \to g_R^2$ as $a \to 0$ and the dimensionless two dimensional coupling constant $\alpha_0 = g_0^2(a)a^2/4\pi \to 0$ as $a \to 0$ as expected.

We show that in general the lattice introduces additional multibody forces which are not present in the continuum. The only expectation is the SU(N) lattice gauge model in the formal limit $N \to \infty$, g_0^2N fixed which is a theory of noninteracting bound states for any value of g_0^2N such as the 't Hooft model to leading order in N. But for finite N, the lattice introduces additional forces for sufficiently large distances. For a quark loop of area A, these multibody forces arise for areas of the order of $g_R^{-2}\ln N$ or larger. The presence of these forces implies two things. First, the S matrix computed in strong coupling is, in general, qualitatively different from the continuum S matrix. Second, the lattice does not merely act as an ultraviolet cutoff because the additional forces modify the infrared behavior.

In Sec. II we present a brief review of the space-time lattice gauge theory formalism. Its purpose is mostly to define the notations used and the Feynman graph rules. In Sec. III we formulate the 't Hooft and Schwinger models as potential theories on a lattice. We calculate the contribution of quark loops for the purpose of direct comparison with the lattice gauge formalism. Section IV presents a discussion of Migdal's theorem and how a comparison of the potential and gauge models leads to the definition of a renormalized charge. Interacting quark loops are studied in Sec. V. In Sec. VI we discuss the details and problems of the continuum limit. Finally, our conclusions are presented in Sec. VII. We include an appendix listing some properties of Fourier coefficents.

II. REVIEW OF LATTICE GAUGE THEORIES

As an approach to the solution of quantum chromodynamics, Wilson[1,2] formulated the gauge theory on a cubic space-time lattice as follows. First, change from Minkowski space to Euclidean space ($t \to it$), then discretize both space and time, $x_\mu = (n_0 a, n_1 a, n_2 a, n_3 a)$, $n_i = 0, \pm 1, \pm 2, \ldots$, where a is the lattice spacing. Of course, introduction of the lattice destroys Lorentz (i.e., Euclidean) and even rotational invariance, but we expect they will be restored as $a \to 0$. The restoration of Euclidean invariance in the continuum limit has been shown for free field theories[2] and the Ising model.[17]

However, if the classical continuum action is naively discretized using finite differences, then it would not be gauge invariant. Because of the vagaries of renormalization, the quantized theory might still lack gauge invariance in the limit $a \to 0$. A possibly better idea is to add terms to the action which vanish as $a \to 0$ but make the action gauge invariant for any a. The result is

$$A = -\sum_n \bar{\psi}_n \psi_n + K\sum_{n\mu}[\bar{\psi}_n(1-\gamma_\mu)U_{n,\mu}\psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}}(1+\gamma_\mu)U^\dagger_{n,\mu}\psi_n] + \frac{a^{d-4}}{2g_0^{\,2}}\sum_{n,\mu<\nu}\mathrm{Tr}(U_{n,\mu}U_{n+\hat{\mu},\nu}U^\dagger_{n+\hat{\nu},\mu}U^\dagger_{n,\nu}), \quad (2.1)$$

where $U_{n,\mu} = e^{ig_0 a A^a_\mu(n)T^a}$, T^a are the generators of the gauge group in the fermion representation, and d is the number of space-time dimensions. We have absorbed an irrelevant change of scale into the fermion fields. The terms $\bar{\psi}_n U_{n,\mu}\psi_{n+\mu}$ are included to make the free fermion energy single-valued within a single Brillouin zone. The constant K is related to the bare quark mass. Now consider the path-integral formulation of the generating functional for disconnected Green's functions. Since the action is periodic as the vector field ranges over the gauge group, we can restrict the integration to over only one cycle of the group. Thus we have

$$Z(\bar{\eta},\eta) = \int \prod_{n\mu} dv_{n\mu} \prod_n d\bar{\psi}_n d\psi_n \exp\Big\{-\sum_n \bar{\psi}_n\psi_n + K\sum_{n\mu}[\bar{\psi}_n(1-\gamma_\mu)U(v_{n\mu})\psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}}(1+\gamma_\mu)U^\dagger(v_{n\mu})\psi_n] + \frac{a^{d-4}}{2g_0^{\,2}}\sum_{n,\mu<\nu}\chi_q(v_{n\mu}v_{n+\hat{\mu},\nu}v^\dagger_{n+\hat{\nu},\mu}v^\dagger_{n\nu}) + \sum_n(\bar{\psi}_n\eta_n + \bar{\eta}_n\psi_n)\Big\}, \quad (2.2)$$

where $v_{n\mu}$ is an element of SU(N), $dv_{n\mu}$ is an invariant measure normalized to unit volume, and $U(v)$ and $\chi_q(v)$ are the matrix and character, respectively, in the fundamental (quark) representation corresponding to the element v. The implications of this new theory are many. First of all, if we do not fix a gauge (and we do not in all that follows) then the expectation value of any gauge-noninvariant quantity is zero. For example,[7]

$$\langle U(v_{n\mu})\rangle = Z(0)^{-1}\int \prod dv \prod d\bar{\psi}\, d\psi\, U(v_{n\mu})e^A \equiv 0\,. \quad (2.3)$$

Likewise we have

$$S_{n0} = \langle \bar{\psi}_n \psi_0\rangle \propto \delta_{n0}\,. \quad (2.4)$$

S_{n0} resembles a free propagator only if we set all $U(v) \to 1$. In general, only local color-singlet states propagate through the lattice. Hence, they are the only physically observable states.

Note that the fact that only color singlets propagate does not mean quark confinement since it is true even without any gauge field self-interaction term. The essence of confinement is the suppression of quark loops with large enclosed areas. Without this suppression the quark's exotic flavor quantum numbers could be detected.

In order to calculate Green's functions we write down a set of spatial lattice Feynman rules for color-singlet sources. Expand the exponential of the fermion kinetic action in powers of K and integrate over the fermion fields. The rules then are:

(1) Diagrams consist of quark loops. A quark loop consists of a set of connected quark links. A quark link is a line segment connecting nearest-neighbor sites and is labeled with an arrow.

(2) For a quark link from n to $n+\hat{\mu}$ write $K\times(1+\gamma_\mu)U^\dagger_{n,\mu}$. For a quark link from $n+\hat{\mu}$ to n write $K(1-\gamma_\mu)U_{n,\mu}$.

(3) A factor Γ at site n for insertion of a source $\bar{\psi}_n\Gamma\psi_n$.

(4) Take both γ matrix and color traces.

(5) A factor $1/l$ for an internal loop containing l links.

(6) Integrate

$$\int \prod_{\text{links } l}\left[dv_l \exp\left[\frac{a^{d-4}}{2g_0^{\,2}}\sum_{n,\mu<\nu}\chi_q(v_{n\mu}v_{n+\mu\nu}v^\dagger_{n+\nu,\mu}v^\dagger_{n\nu})\right]\right].$$

(7) Sum over all possible quark loops containing the sources.

III. 't HOOFT AND SCHWINGER MODELS ON A LATTICE

In this section, we write down a discretized version of the continuum 't Hooft and Schwinger models. Because they are just superrenormalizable potential models, we make the very reasonable assumption that the continuum limits of the discretized models are just the usual continuum models. We regard this assumption as harmless because the form of discretization we use is merely a way of giving a concrete definition to the functional path-integral formalism.[18] The only way in which these discretized potential models differ from just a discretization of the path-integral formalism is the probably necessary way in which the fermion fields are included following Wilson. We use these discretized potential models to make direct comparison to the lattice gauge theories.

The Lagrangian density of the continuum theory is given by

$$\mathcal{L} = -\tfrac{1}{4}(G_{\mu\nu}^{ij})^2 + \bar{\psi}(i\not{\partial} - m)\psi + ig\bar{\psi}^i A^{ij}\psi^j , \tag{3.1}$$

where $A_\mu^{ij}(x) = T_a^{ij} A_\mu^a(x)$ and $G_{\mu\nu}^{ij} = \partial_\mu A_\nu^{ij} - \partial_\nu A_\mu^{ij} + g[A_\mu, A_\nu]^{ij}$. Choosing the gauge $A_1 = 0$ we find the equation of motion

$$\partial_1{}^2 A_0^{ij} = -g\bar{\psi}^i \gamma_0 \psi^j . \tag{3.2}$$

Thus, in two dimensions, the above theory is equivalent to

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) + \tfrac{1}{2}g^2 \int d^2y\, \bar{\psi}^i(x)\gamma_0\psi^j(x)\Delta(x-y)\bar{\psi}^j(y)\gamma_0\psi^i(y) , \tag{3.3}$$

where

$$\Delta(x-y) = \tfrac{1}{2}\delta(x_0 - y_0)\,|x_1 - y_1| .$$

Now we discretize this Lagrangian in the manner of Wilson and write for the action of this lattice potential model

$$A = -\sum_n \bar{\psi}_n \psi_n + K \sum_{n\mu}[\bar{\psi}_n(1-\gamma_\mu)\psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}}(1+\gamma_\mu)\psi_n] + \tfrac{1}{2}g^2a^2 \sum_{nm}\sum_{ij} \bar{\psi}_n^i \gamma_0 \psi_n^j \Delta_{nm} \bar{\psi}_m^j \gamma_0 \psi_m^i , \tag{3.4}$$

where $\Delta_{nm} = \frac{1}{2}\delta_{n_0 m_0}|n_1 - m_1|$. We have neglected the possible background electric field for the Schwinger model[13] because it is not present in the usual lattice gauge theories.

Since we know that the physical sector of the 't Hooft and Schwinger models is the (color) charge-zero sector of the theory (the charge-nonzero sector may not even exist), we may concern ourselves in the lattice potential model with Green's functions of color-singlet sources only. To calculate these Green's functions, we write down a set of space-time Feynman rules. By taking the generating functional in the path-integral representation, expanding the exponential in powers of K and g, and integrating over the fermion fields, we obtain the following rules:

(1) Diagrams consist of quark loops and gluon exchanges. A quark loop consists of a set of nearest-neighbor quark links.

(2) For a quark link from site n to $n+\hat{\mu}$ write $\delta_{ij}K(1-\gamma_\mu)$. For a quark link from site $n+\hat{\mu}$ to n write $\delta_{ij}K(1+\gamma_\mu)$. i and j are color indices.

(3) A factor Γ at site n for the insertion of a source $\bar{\psi}_n\Gamma\psi_n$.

(4) A factor $1/l$ for an internal loop with l links.

(5) A factor $ga\gamma_0$ for every gluon-quark-quark vertex. A factor $\delta_{ij}\Delta_{nm}$ for a gluon exchange between vertices at n and m.

(6) A factor $1/p!$ for p gluon exchanges.

(7) Take both γ matrix and color traces.

(8) Sum over all possible quark loops containing the sources and over all possible gluon exchanges.

Now we proceed to calculate quantities with which we will make direct comparison to the lattice gauge theory. Consider a single quark loop as shown in Fig. 1. We write for the contribution of this quark loop without any gluon exchange

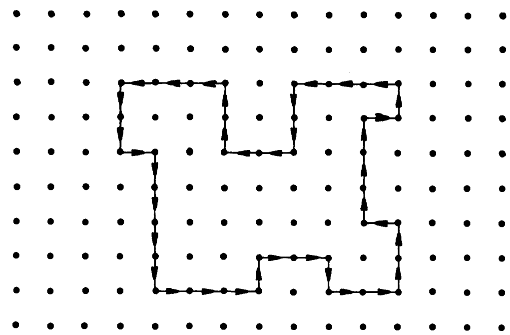

FIG. 1. A single quark loop on the lattice.

$$\mathfrak{M}_0(v) = NK^{P_v}\,\mathrm{Tr}[(1\pm\gamma_{\mu_1})(1\pm\gamma_{\mu_2})\cdots(1\pm\gamma_{\mu_{P_v}})] , \tag{3.5}$$

where v denotes the particular quark loop and P_v is the number of quark links in the quark loop.

The amplitude for the exchange of one gluon from point n to m within the quark loop v is given by

$$\mathfrak{M}_1^{nm}(v) = K^{P_v}\,\mathrm{Tr}[(1\pm\gamma_{\mu_1})\cdots(1\pm\gamma_{\mu_n})\gamma_0(1\pm\gamma_{\mu_{n+1}})\cdots(1\pm\gamma_{\mu_m})\gamma_0(1\pm\gamma_{\mu_{m+1}})\cdots(1\pm\gamma_{\mu_{P_v}})]N^2g^2a^2\Delta_{nm} . \tag{3.6}$$

Now observe that with only a few exceptions, we can eliminate the γ_0's in the γ-matrix trace and write $\mathfrak{M}_1^{nm}(v)$ in terms of the no-gluon amplitude

$$\mathfrak{M}_1^{nm}(v) = \mathfrak{M}_0(v) N g^2 a^2 \eta_n \eta_m \Delta_{nm}, \tag{3.7}$$

where $\eta_n = 0, \pm 1$ depending only on where the vertex at n is within the quark loop. For example, for a vertex between two links in the x_0 direction we have a factor

$$(1+\gamma_0)\gamma_0(1+\gamma_0) = +1(1+\gamma_0)(1+\gamma_0) \tag{3.8}$$

and $\eta = +1$ for this vertex. Note that we have a Euclidean metric for γ matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. A list of vertices for which η exists and the respective value of η is shown in Figs. 2(a)–2(c). For a one-gluon exchange graph, η does not exist for the configurations in Fig. 2(d). Also, when a vertex is at the same site as a source then η exists if the vertex is adjacent to a link in the $\pm x_0$ direction and equals ± 1, respectively, otherwise η does not exist. Consequently, if we consider a quark-loop configuration for which η exists along every site along the quark path, we can write the sum over all single-gluon exchanges as

$$\mathfrak{M}_1(v) = \mathfrak{M}_0(v) g^2 a^2 N \tfrac{1}{2} \sum_{\substack{nm \\ \text{in } v}} \eta_n \eta_m \Delta_{nm}. \tag{3.9}$$

Contributions for which η does not exist merely add small correction terms to this result. Consider a quark loop v with large fixed area A_v. For fixed g and vanishing a, the sum in Eq. (3.9) gives a contribution proportional to A_v, but the corrections give contributions proportional to aL', where L' is some fixed length. Consequently, the correction terms due to vertices for which η does not exist are negligible for large loop areas or equivalently small lattice spacing.

For L gluon exchanges, we can write

$$\mathfrak{M}_L(v) = \mathfrak{M}_0(v) \frac{1}{L!} \left(\tfrac{1}{2} g^2 N a^2 \sum_{\substack{nm \\ \text{in } v}} \eta_n \eta_m \Delta_{nm} \right)^L. \tag{3.10}$$

This expression is valid for QED and the 't Hooft model for $N \to \infty$, $g^2 N$ fixed. For finite N, this formula neglects terms down in $1/N$. It also neglects all small correction terms because once again they are negligible for large areas.

Consequently, the sum over all gluon exchanges for a single-quark loop v is

$$\mathfrak{M}(v) = \sum_{L=0}^{\infty} \mathfrak{M}_L(v)$$

$$= \mathfrak{M}_0(v) \exp\left(\tfrac{1}{2} g^2 N a^2 \sum_{\substack{nm \\ \text{in } v}} \eta_n \eta_m \Delta_{nm} \right). \tag{3.11}$$

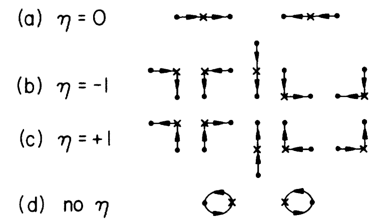

FIG. 2. The value of η for insertion of a $g\gamma_0$ vertex (represented by an $\times$).

Observe that the gluon contribution is almost identical to the expectation value of a single-quark loop in an Abelian continuum gauge theory (in Euclidean space)

$$\left\langle \exp\left[ie \oint dx^\mu A^\mu(x) \right] \right\rangle \Big|_{\text{gauge fields}}$$

$$= \int [dA^\mu] \exp\left[ie \oint dx^\mu A^\mu(x) \right]$$

$$\times \exp\left(\int d^4x \tfrac{1}{4} G_{\mu\nu}{}^2 \right)$$

$$= \exp\left[-e^2 \oint dx^\mu \oint dy^\mu \Delta_{\mu\nu}(x-y) \right], \tag{3.12}$$

where $\Delta_{\mu\nu}(x-y)$ is the free gluon propagator. By comparison, it is easy to see now that the role of the η factor is as a unit vector tangent to the quark path.

Finally, for a large nonoverlapping single-quark loop with area $A_v \gg a^2$, we have for the lattice potential theory

$$\tfrac{1}{2} \sum_{\substack{nm \\ \text{in } v}} N_n N_m \Delta_{nm} \cong -\tfrac{1}{2} A_v / a^2. \tag{3.13}$$

And the total contribution of this single-quark loop is

$$\mathfrak{M}(v) = \mathfrak{M}_0(v) e^{-(1/2) g^2 N A_v}. \tag{3.14}$$

IV. MIGDAL'S THEOREM AND THE RENORMALIZED CHARGE

In this section we calculate the contribution of a quark loop in the lattice gauge theory to all orders in the gauge coupling. A comparison to the lattice potential models will then lead to a definition of a renormalized charge.

Consider a single box as shown in Fig. 3(a). We write its contribution to the generating functional as

$$Z^{(1)}(v) = \exp\left\{ \frac{1}{2 g_0{}^2 a^2} [\chi_q(v) + \chi_q^*(v)] \right\}$$

$$= \sum_p Z_p d_p \chi_p(v), \tag{4.1}$$

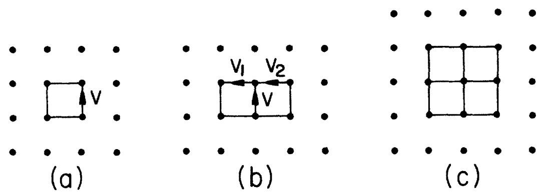

FIG. 3. Combining boxes to give Migdal's theorem.

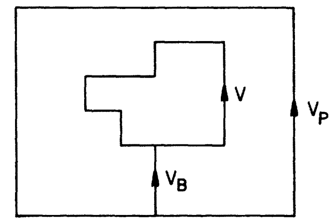

FIG. 4. Contours for calculating $\langle\chi_p(v^\dagger)\rangle$.

where Z_p is the Fourier component of $Z^{(1)}(v)$ in the pth irreducible representation of the lattice gauge group [U(1) or SU(N)] and $d_p=\chi_p(I)$ is the dimension of the pth representation. Z_p is given by

$$Z_p=d_p^{-1}\int dv\,\chi_p^*(v)Z^{(1)}(v)$$
$$=Z_p\left(\frac{1}{g_0^2a^2}\right). \quad (4.2)$$

Now consider what happens when we combine two connected boxes and integrate over the common link as in Fig. 3(b):

$$Z^{(2)}\equiv\int dv\,Z^{(1)}(v_1v)Z^{(1)}(v^\dagger v_2)$$
$$=\sum_{p_1p_2}Z_{p_1}d_{p_1}Z_{p_2}d_{p_2}\int dv\,\chi_{p_1}(v_1v)\chi_{p_2}(v^\dagger v_2)$$
$$=\sum_p Z_p^2d_p\chi_p(v_1v_2)\,. \quad (4.3)$$

We have used the orthogonality theorem for characters

$$\int dv\,\chi_{p_1}(v_1v)\chi_{p_2}(v^\dagger v_2)=\delta_{p_1p_2}d_{p_1}^{-1}\chi_p\,(v_1v_2)\,. \quad (4.4)$$

Consequently, if we combine four boxes as in Fig. 3(c) by integrating out the internal links we get

$$Z^{(4)}(v)=\sum_p Z_p^4d_p\chi_p(v)\,, \quad (4.5)$$

where v is the product contour around the perimeter. Thus, for an arbitrary combination of n connected boxes

$$Z^{(n)}(v)=\sum_p Z_p^{A_n/a^2}d_p\chi_p(v)\,, \quad (4.6)$$

where A_n is the area enclosed by the perimeter product contour v. This is Migdal's theorem which is very useful for studying the two-dimensional problem.

Our boundary conditions are chosen as a large box of area A. Then the contribution of the gauge fields to the generating functional in the absence of sources is

$$Z=\int\prod_{\text{links }l}dv_l\exp\left\{\frac{1}{2g_0^2a^2}\sum_{\text{boxes }b}[\chi_q(v_b)+\chi_q^*(v_b)]\right\}$$
$$=\int dv_p\sum_{p_1}Z_{p_1}^{A/a^2}d_{p_1}\chi_{p_1}(v_p)$$
$$=Z_0^{A/a^2}\,. \quad (4.7)$$

The expectation value of a particular contour is (see Fig. 4 for notation)

$$\langle\chi_p(v^\dagger)\rangle=Z^{-1}\int\prod_l dv_l\chi_p(v^\dagger)\exp\left\{\frac{1}{2g_0^2a^2}\sum_b[\chi_q(v_b)+\chi_q^*(v_b)]\right\}$$
$$=Z_0^{-A/a^2}\int dv_pdv_Bdv\sum_{p_1}Z_{p_1}^{(A-A_v)/a^2}d_{p_1}\chi_{p_1}(v^\dagger v_Bv_pv_B^\dagger)\sum_{p_2}Z_{p_2}^{A_v/a^2}d_{p_2}\chi_{p_2}(v)\chi_p(v^\dagger)$$
$$=d_p(Z_p/Z_0)^{A_v/a^2}\,, \quad (4.8)$$

where A_v is the area inside v. This is Wilson's law of areas. Since $Z_p<Z_0$ large areas are exponentially suppressed. Now consider the contribution of a single-quark loop v to a Green's function. Using the

Feynman rules of Sec. II we write

$$\begin{aligned}\mathfrak{M}(v) &= K^{P_v}\mathrm{Tr}[(1\pm\gamma_{\mu_1})\cdots(1\pm\gamma_{\mu_{P_v}})]\\ &\quad\times\int\prod_l dU_l\,\mathrm{Tr}\{U_{n\mu_1}U_{n+\mu_1,\mu_2}\cdots U_{n+\mu_1+\cdots+\mu_{P_v'},-\mu_{P_v}}\}\exp\left\{\frac{1}{2g_0^2a^2}\sum_b[\chi_q(v_b)+\chi_q^*(v_b)]\right\}\\ &= N^{-1}\mathfrak{M}_0(v)\langle\chi_q(v)\rangle\\ &= \mathfrak{M}_0(v)e^{-(1/2)g_R^2NA_v}\,,\end{aligned}\tag{4.9}$$

where $\mathfrak{M}_0(v)$ is the same [Eq. (3.5)] for the lattice potential theory, and we have defined the renormalized charge

$$g_R^2N=-\frac{2}{a^2}\ln\frac{Z_q(1/g_0^2a^2)}{Z_0(1/g_0^2a^2)}\,.\tag{4.10}$$

The qualitative behavior of g_R^2 is shown in Fig. 5. By comparison with Eq. (3.14), we see that the contributions of single-quark loops in the lattice gauge theory are precisely the same as in a lattice potential model with coupling g_R^2. Therefore, for the quark-antiquark sector of the theory, the lattice gauge theory acts like the usual respective 't Hooft and Schwinger models with coupling g_R^2.

V. INTERACTING LOOPS

In this section we consider the contribution of overlapping loops in both the lattice gauge and potential models. First consider the lattice gauge theory. The contribution of two overlapping quark loops as in Fig. 6(a) is written

$$\mathfrak{M}_a(v_1,v_2)=N^{-2}\mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)\langle\chi_q(v_1)\chi_q(v_2)\rangle\,.\tag{5.1}$$

The expectation value can be written

$$\begin{aligned}\langle\chi_q(v_1)\chi_q(v_2)\rangle &= Z_0^{-(A_1+A_2+A_3)/a^2}\int dv_1^0dv_1^idv_2^0dv_2^i\sum_{p_1}Z_{p_1}^{A_1/a^2}d_{p_1}\chi_{p_1}(v_1^{0\dagger}v_1^{i\dagger})\sum_{p_2}Z_{p_2}^{A_2/a^2}d_{p_2}\chi_{p_2}(v_1^{i\dagger}v_2^i)\\ &\qquad\times\sum_{p_3}Z_{p_3}^{A_3/a^2}d_{p_3}\chi_{p_3}(v_2^{0\dagger}v_2^{i\dagger})\chi_q(v_1^0v_1^i)\chi_q(v_2^0v_2^i)\\ &=\left(\frac{Z_q}{Z_0}\right)^{(A_1+A_3)/a^2}\sum_p C_p^{qq}d_p(Z_p/Z_0)^{A_2/a^2}\,,\end{aligned}\tag{5.2}$$

where C_p^{qq} is the Clebsch-Gordan coefficient for finding the pth representation within the $q\otimes q$ representation. Consequently, we have for overlapping quark loops

$$\begin{aligned}\mathfrak{M}_a(v_1,v_2)&=\mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g_R^2N(A_1+A_3)}\\ &\quad\times N^{-2}\sum_p C_p^{qq}e^{-(1/2)g_p^2NA_2}d_p\,,\end{aligned}\tag{5.3}$$

where

$$g_p^2N=-\frac{2}{a^2}\ln\frac{Z_p}{Z_0}\quad(g_1^2\equiv g_R^2)\,.$$

Now consider the contribution of interacting loops for the latticized U(1) potential model. Define the charge configuration and distances as shown in Fig. 6(b). The potential energy for this configuration is

$$\begin{aligned}V_b&=-g^2[d_1-(d_1+d_2)+(d_1+d_2+d_3)\\ &\qquad+d_2-(d_2+d_3)+d_3]\\ &=-g^2(d_1+d_3)\,.\end{aligned}\tag{5.4}$$

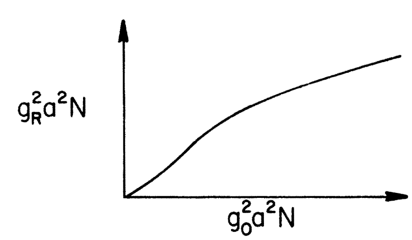

FIG. 5. Qualitative behavior of the renormalized charge.

And so this gives a contribution

$$\mathfrak{M}_b(v_1, v_2) = \mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g^2(A_1+A_3)}, \quad (5.5)$$

which is the same as in the U(1) lattice gauge theory. The same argument applies to Fig. 6(c), which gives the same as above which again agrees with the U(1) lattice gauge theory.

However, consider a situation such as that in Fig. 6(d) with adjacent charges of the same sign. We have

$$\begin{aligned} V_d &= -g^2[-d_1 + (d_1 + d_2) + (d_1 + d_2 + d_3) \\ &\quad + d_2 + (d_2 + d_3) - d_3] \\ &= -g^2(d_1 + 4d_2 + d_3), \end{aligned} \quad (5.6)$$

and the amplitude for these overlapping quark loops is

$$\mathfrak{M}_{\text{potential}} = \mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g^2(A_1+4A_2+A_3)}. \quad (5.7)$$

This does not agree with the U(1) lattice gauge theory which is

$$\mathfrak{M}_{\text{gauge}} = \mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g_R^2(A_1+A_3)-(1/2)g_2^2A_2}. \quad (5.8)$$

Defining

$$g_{e2}^2 = 4g_R^2 - g_2^2 = \frac{2}{a^2}\ln\frac{Z_2}{Z_0}\left(\frac{Z_0}{Z_1}\right)^4,$$

we have

$$\mathfrak{M}_{\text{gauge}}/\mathfrak{M}_{\text{potential}} = e^{-(1/2)g_{e2}^2A_2}.$$

This new coupling is a consequence only of the lattice gauge formalism. It introduces a new four-quark force which is not present in the continuum whenever there is a quark-loop configuration such as that in Fig. 6(d).

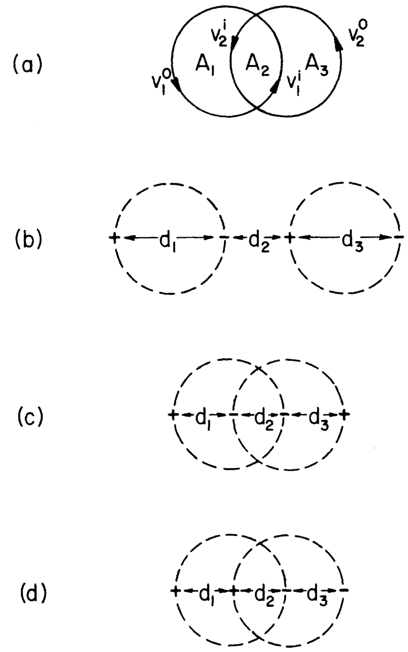

FIG. 6. U(1) interacting loops for (a) the lattice gauge theory and (b)–(d) the lattice potential model.

We can mock up a contribution to the continuum action which would reproduce this additional force as follows. Define $Q(z,t) = \int_{-\infty}^{\infty} dx\, \theta(z-x)\rho(x,t)$, where $\rho(x,t)$ is the charge density operator. Figure 6(d) contributes whenever $\langle Q\rangle = \pm 2$ in a spatial region with length equal to d_2. All the other configurations such as Figs. 6(b) and 6(c) give $\langle Q\rangle = 0$ or ± 1. So an interaction Lagrangian which would reproduce the amplitude $\mathfrak{M}_{\text{gauge}}$ is

$$\begin{aligned} L_{\text{extra}} &= \frac{g_{e2}^2}{12}\int_{-\infty}^{\infty} dz' Q^2(z',t)[Q^2(z',t)-1] \\ &= \frac{g_{e2}^2}{12}\left[\tfrac{1}{2}\int dx\,dy\,\rho(x,t)\rho(y,t)\,|x-y| + \int dw\,dx\,dy\,dz\,\rho(w,t)\rho(x,t)\rho(y,t)\rho(z,t)V(w,x,y,z)\right], \end{aligned} \quad (5.9)$$

where

$$V(w,x,y,z) \equiv \int_{-\infty}^{\infty} dz'\theta(z'-w)\theta(z'-x) \times \theta(y-z')\theta(z-z').$$

For $a\to 0$ with g_0^2 fixed, $g_{e2}^2 \to 0$; but for strong coupling $g_{e2}^2 \neq 0$ and gives an S matrix qualitatively different from the continuum S matrix. Likewise the lattice gauge interaction of three-or-more-quark loops introduces additional six-or-more-quark forces which are not present in the continuum. The additional couplings they introduce vanish as $a\to 0$ with g_0^2 fixed, but not for strong coupling.

Now consider the SU(N) lattice potential model. Since to leading order in N loops are noninteracting we have for the configuration in Fig. 6(a)

$$\mathfrak{M}(v_1, v_2) = \mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g^2(A_1+2A_2+A_3)}. \quad (5.10)$$

Because of the properties of the Fourier coefficients to leading N as discussed in the Appendix, the two-loop lattice gauge theory gives an identical answer but with coupling g_R^2. The lattice theory of SU(N) for leading N is a theory of noninteracting bound states such as the continuum theory because the lattice is also restricted to planar topologies for leading N like the continuum.[4] However, for finite N

$$\begin{aligned}\mathfrak{M}(v_1, v_2) &= \mathfrak{M}_0(v_1)\mathfrak{M}_1(v_2)e^{-(1/2)g_R^2 N(A_1+A_3)} \\ &\quad\times \sum_p C_p^{qq}\frac{d_p}{N^2}e^{-(1/2)g_p^2 A_2} \\ &= \mathfrak{M}_0(v_1)\mathfrak{M}_0(v_2)e^{-(1/2)g_R^2(A_1+A_3)} \\ &\quad\times(N^{-2}+e^{-(1/2)g_{N^2-1}^2 A_2}),\end{aligned} \tag{5.11}$$

so that for large areas of intersection

$$A_2 \cong g_R^{-2}\ln N, \tag{5.12}$$

the amplitude deviates from the potential model, which again shows the manifestation of additional forces as introduced by the lattice gauge formalism.

VI. THE CONTINUUM LIMIT

In this section we discuss the lattice gauge theory in the limit of zero lattice spacing ($a\to 0$). As mentioned in the Introduction the limit depends on the quantity being held fixed as $a\to 0$. For definiteness we will consider only the U(1) gauge model, although our conclusions apply equally well to any SU(N)-gauge model except in the limit $N\to\infty$, g^2N fixed.

The $2n$-quark coupling constant is given by

$$g_n^2 = -\frac{2}{a^2}\ln\frac{Z_n(1/2g_0^2a^2)}{Z_0(1/2g_0^2a^2)}. \tag{6.1}$$

The extra couplings generated solely by the lattice which are not present in the continuum are given by

$$g_{en}^2 = n^2 g_1^2 - g_n^2. \tag{6.2}$$

The dimensionless coupling is defined by $\alpha_0 \equiv g_0^2a^2/4\pi$.

Now we consider several cases where a different quantity is held fixed each time while taking the limit $a\to 0$. As a first example, hold the bare coupling g_0 fixed. Because the two-dimensional continuum has no coupling-constant renormalization, we expect that keeping g_0 fixed would give us the usual continuum, and indeed it does. In particular,

$$\lim_{a\to 0}\begin{Bmatrix} g_p^2 \\ g_{ep}^2 \\ \alpha_0\end{Bmatrix} = \begin{Bmatrix} p^2 g_0^2 \\ 0 \\ 0\end{Bmatrix}, \quad g_0^2 \text{ fixed} \tag{6.3}$$

all the additional lattice forces disappear, and the usual continuum theory is reproduced with coupling g_0.

What would happen if we held fixed a physical $2n$-quark coupling constant g_n^2 for some n? In that case, the bare coupling g_0 would become a function of the lattice spacing. The limit would give

$$\lim_{a\to 0}\begin{Bmatrix} g_0^2 \\ g_p^2 \\ g_{ep}^2 \\ \alpha_0\end{Bmatrix} = \begin{Bmatrix} g_n^2/n^2 \\ p^2 g_n^2/n^2 \\ 0 \\ 0\end{Bmatrix}, \quad g_n^2 \text{ fixed}. \tag{6.4}$$

Again, all the additional lattice forces disappear, and the usual continuum theory is reproduced with coupling g_n/n.

But what would happen if there were, say, some mass or coupling constant which vanished in the previous limit and was inadvertently held fixed? An example of this might occur in the four-dimensional theory if it really gave a pion mass which vanished in the "true" continuum. As a specific example, let one of the additional lattice couplings g_{en}^2 be held fixed. In this case

$$\lim_{a\to 0}\begin{Bmatrix} g_0^2 \\ g_p^2 \\ \alpha_0\end{Bmatrix} = \begin{Bmatrix} \infty \\ \infty \\ 0\end{Bmatrix}, \quad g_{en}^2 \text{ fixed}. \tag{6.5}$$

Notice that α_0 vanishes, but not fast enough to allow finite limits for g_0^2 and g_p^2. The resulting theory appears to be a perfectly well-defined Euclidean-invariant theory.

VII. CONCLUSIONS AND DISCUSSION

In this paper, we have shown that when either the bare coupling g_0^2 or a renormalized coupling g_p^2 is held fixed, the lattice gauge theories U(1) and SU(N) for $N\to\infty$, g^2N fixed in two dimensions become the usual continuum Schwinger and 't Hooft models, respectively, in the limit of vanishing lattice spacing. Our method consisted of calculating the contributions of quark loops to all orders in the gluon coupling and comparing them for the lattice gauge theory and for a latticized potential version of the continuum theory. We have, of course, assumed that the latticized potential theory becomes the continuum potential theory for vanishing lattice spacing holding its bare coupling fixed. However, we regard this last assumption as harmless because this form of discretization is a way of defining the continuum functional path integrals, except possibly for the (albeit necessary) method we used for discretizing the fermion fields. Our interest was whether the additional modifications to the discrete (lattice) theory made by the Wilson formalism (manifest gauge invariance of the action, integration over only one cycle of the group, etc.) would introduce any modification in comparison to continuum theory. We have shown that if the correct physical quantities are held fixed, there are no modifications in the two-dimensional theories in the limit $a \to 0$. We also showed that, in general, the lattice introduces additional forces not present in the usual continuum. These forces make the lattice S matrix qualitatively different from the continuum S matrix, particularly in strong coupling. The extra forces go away only for vanishing lattice spacing and holding a proper physical constant fixed. Consequently, in order to have a lattice theory which has the same physics as the continuum, additional terms must be added to the lattice gauge action to compensate these additional forces. For finite lattice spacing, the simple box plus quark link action does not have the same physics as the continuum.

As usual, the continuum limit of the lattice gauge theory is controlled by the existence of critical points in the renormalization group. The two-dimensional gauge theories are asymptotically free and the renormalization group is very simple. In this case, holding physical couplings fixed and sending the lattice spacing to zero forced the bare coupling to its ultraviolet (short-distance) fixed point. The existence of an ultraviolet fixed point plays a crucial role in the four-dimensional case. We expect that if the theory has an ultraviolet fixed point at the origin, then we expect that taking the continuum limit holding the right physical couplings fixed will force us to the critical behavior of the lattice theory.

ACKNOWLEDGMENTS

We thank all our colleagues at Fermilab for useful discussions. We especially thank W. A. Bardeen for his suggestion of this problem and his continual support.

APPENDIX

In this Appendix, we list some of the useful properties of the Fourier coefficients and the renormalized charge for the U(1) and SU(N) gauge groups. First consider the U(1) group. The character for the pth representation is given by

$$\chi_p(\theta) = e^{ip\theta} . \tag{A1}$$

Letting $x = 1/g_0^2 a^2$, the coefficients are

$$Z_p(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{ip\theta} e^{x\cos\theta} = I_p(x) , \tag{A2}$$

where $I_p(x)$ is a modified Bessel function of pth order with asymptotic properties

$$I_p(x) \sim \begin{cases} \dfrac{1}{p!}(x/2)^p + \cdots , & x \ll 1 \\ \dfrac{e^x}{(2\pi x)^{1/2}}\left(1 - \dfrac{4p^2-1}{8x} + \cdots\right) , & x \gg 1 . \end{cases} \tag{A3}$$

Consequently, the renormalized charge for U(1) has the behavior

$$g_R^2\,\big|_{\mathrm{U}(1)} \sim \begin{cases} g_0^2 , & g_0 a \ll 1 \\ \dfrac{2}{a^2}\ln g_0^2 a^2 , & g_0 a \gg 1 . \end{cases} \tag{A4}$$

Consider the SU(N) lattice gauge theory. We denote representations by their dimension d_p. The representation can also be characterized in terms of a set of N numbers $f_1, \ldots, f_N$ which determine p. The character of the pth representation is

$$X_p(\phi_1, \ldots, \phi_N) = \xi(l_1, \ldots, l_N)/\Delta , \tag{A5}$$

where $l_j = f_j + N - j$, $\xi(l_1, \ldots, l_N) = |\epsilon^{l_1}, \ldots, \epsilon^{l_N}|$, $\epsilon^\dagger = (\epsilon_1, \ldots, \epsilon_N)$, $\epsilon_i = e^{i\phi_i}$, and $\Delta = \xi(N-1, \ldots, 1)$ under the constraint $\sum_i \phi_i = 0$. The Fourier coefficients are given by

$$Z_p = d_p^{-1} V_N^{-1} \int_0^{2\pi} d\phi_1 \ldots d\phi_N\, \delta(\textstyle\sum_i \phi_i)\, |\Delta|^2 \times X_p^*(\phi) e^{x[X_N(\phi) + X_N^*(\phi)]} , \tag{A6}$$

where V_N is the group volume. We have the following relationships between the coefficients: $Z_N = Z_{\bar N}$, where $\bar N$ is the antiquark representation, $\partial Z_0/\partial x = NZ_N + NZ_{\bar N}$, or in general

$$\frac{\partial^n}{\partial x^n} Z_0 = \sum_p d_p Z_p \sum_{m=0}^{n} \binom{n}{m} C_p^{(n-m)N, m\bar N} , \tag{A7}$$

where $C_p^{\kappa N, l\bar N}$ is the Clebsch-Gordan coefficient for finding the pth representation in the $(N\otimes)^\kappa \otimes (\bar N\otimes)^l$ product representation. The renormalized charge has the behavior

$$g_R^2 N\Big|_{\mathrm{SU}(N)} \sim \begin{cases} g_0^2 N , & g_0 a \ll 1 \\ \dfrac{2}{a^2}\ln g_0^2 N , & g_0 a \gg 1 . \end{cases} \tag{A8}$$

Now let $N \to \infty$ keeping $g_0^2 N$ fixed and observe that the ratios of Fourier coefficients are all of the order of 1, $Z_p/Z_0 = f(g_0^2 N)$. We find that $\chi_{N(N-1)/2} = \chi_{N(N+1)/2}$, so that $Z_{N(N-1)/2} = Z_{N(N+1)/2}$, which is also equal to Z_{N^2-1}, and $\partial^2 Z_0/\partial x^2 = 4N^2 Z_{N^2-1}$. Now consider

$$\frac{\partial}{\partial x}\left(\frac{Z_N}{Z_0}\right) = 2N\left[\frac{Z_{N^2-1}}{Z_0} - \left(\frac{Z_N}{Z_0}\right)^2\right]. \quad \text{(A9)}$$

Since the left-hand side is $O(1/N)$ then the expression in the brackets is $O(1/N^2)$. Thus to leading order in $1/N$, $Z_{N^2-1}/Z_0 = (Z_N/Z_0)^2$. In general, if $O(d_{p_1}) = O(d_{p_2}) = O(N^\kappa)$ then $Z_{p_1} = Z_{p_2} = Z_0(Z_N/Z_0)^\kappa$.

---

*Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration.

[1]K. Wilson, in Recent Progress in *Lagrangian Field Theory and Applications,* proceedings of the Marseille Colloquium on Lagrangian Field Theory, 1974, edited by C. P. Korthals-Altes *et al.* (Centre de Physique Theorique, Marseille, 1975), p. 125; Phys. Rev. D 10, 2445 (1974); talk presented at the Coral Gables Conference, 1976 (unpublished).

[2]K. Wilson, Erice lecture notes, Report No. CLNS-321, 1975 (unpublished).

[3]R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D 10, 3376 (1974); 11, 2098 (1975); 11, 2104 (1975).

[4]C. P. Korthais-Altes, in *Recent Progress in Lagrangian Field Theory and Application* (Ref. 1), p. 102.

[5]L. Susskind, in *Trends in Elementary Particle Theory,* proceedings of the International Summer Institute for Theoretical Physics, Bonn, West Germany, 1974, edited by H. Rollnik and K. Dietz (Springer, New York, 1975), p. 234; J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); T. Banks, L. Susskind, and J. Kogut, *ibid.* 13, 1043 (1976); A. Carroll, J. Kogut, D. K. Sinclair, and L. Susskind, *ibid.* 13, 2270 (1976); L. Susskind, Phys. Rev. D (to be published).

[6]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 810 (1975) [Sov. Phys.—JETP 42, 413 (1976)]; 69, 1457 (1975) [42, 743 (1976)]; see also L. P. Kadanoff, Zurich IBM Research Laboratory report (unpublished).

[7]S. Elitzur, Phys. Rev. D 12, 3978 (1975).

[8]W. A. Bardeen and R. B. Pearson, Phys. Rev. D 14, 547 (1976).

[9]S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 1627 (1976).

[10]J. Schwinger, Phys. Rev. 125, 397 (1962); 128, 2425 (1962).

[11]J. Lowenstein and A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971).

[12]A. Casher, J. Kogut, and L. Susskind, Phys. Rev. Lett. 31, 792 (1973); Phys. Rev. D 10, 732 (1974).

[13]S. Coleman, Ann. Phys. (N.Y.) 101, 239 (1976); R. Jackiw, L. Susskind, and S. Coleman, *ibid.* 93, 267 (1975).

[14]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[15]C. Callan, N. Coote, and D. Gross, Phys. Rev. D 13, 1649 (1976).

[16]M. Einhorn, Phys. Rev. D 14, 3451 (1976).

[17]T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. B 13, 316 (1976).

[18]See, for example, E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).