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Canonical methods in non-Abelian gauge theories

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The exploitation of unrenormalized canonical field equations and equal-time commutation relations is generally not fruitful because of the need for infinite renormalization. The asymptotically free non-Abelian gauge theories are much more benign in their ultraviolet behavior. Under certain smoothness assumptions, canonical methods applied to the $A_\mu^a(x)A_\nu^b(0)$ operator-product expansion are shown to lead to information on the exact singularity structure of the theory. As a result, we show (1) that the unrenormalized gauge coupling constant $g_0(K)$ as a function of the cutoff K has the behavior $g_0(K)\sim(\ln K)^{-1/2}$, (2) how to renormalize $\vec{a}^\mu\times\vec{a}^\nu$ (where $\vec{a}^\mu$ is the unrenormalized gauge field), and (3) that the field-strength tensor $\partial^\mu\vec{a}^\nu-\partial^\nu\vec{a}^\mu+g_0\vec{a}^\mu\times\vec{a}^\nu$ becomes proportional to $\vec{F}^{\mu\nu}=\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu$ upon renormalization. These conclusions agree with results obtained by the use of renormalization-group equations.

I. INTRODUCTION

The canonical formulation of quantum field theory is based on the canonical field equations and equal-time commutation relations satisfied by the unrenormalized field operators. However, the need to perform infinite renormalizations in perturbation theory completely destroys the utility of these relations. The (finite) renormalized field operators satisfy only very complicated field equations[1] and their products do not possess finite equal-time limits,[2] and the singular nature of the equation increases with the order of perturbation theory. In asymptotically free (AF) field theories, one can fortunately go beyond perturbation theory and exactly[3] calculate all of the renormalization constants and counterterms.[4,5] The results are always much less singular than in perturbation theory, and are often finite or even vanishing. The question to be studied in this paper is whether this state of affairs can reinstate the usefulness of the canonical formalism. We shall argue that the answer is affirmative.

The most physically interesting AF models[6] are the non-Abelian gauge theories (NAGT's) of Yang and Mills (YM) and their generalizations,[7] and we will be primarily concerned with these in the present paper. The canonical YM theories are particularly beautiful, having local gauge invariance and an elegant geometric interpretation.[8] It is a pity that much of this beauty is lost in the complicated perturbation theory expansions, which require the introduction of gauge-fixing terms and ghost fields.[9] The "quasicanonical" approach we will adopt is an attempt to resurrect some of the original simplicity of these theories.

In spite of its contradiction with perturbation theory, the canonical formalism was actively studied several years ago in connection with attempts to explain the observed Bjorken scaling behavior in electroproduction with a quantum field theory. A purely canonical approach, which incorporated reducible scale invariance, was proposed.[10,11] However, an explicit computational scheme was lacking. The AF theories are structurally quite similar to the quasicanonical ones, but have the advantage of computability via ordinary renormalized perturbation theory, as summed by the renormalization group. It should be stressed that there is at present absolutely no experimental support for the physical relevance of these theories. In particular, they shed no light on the observed precocity of the scaling in electroproduction. Nevertheless the *possibilities* they present for understanding Bjorken scaling, quark confinement, and related phenomena are sufficient motivation for their intensive study.

For application to electroproduction, one must determine how to renormalize the composite operators $\bar{\psi}\gamma_\alpha D_\beta\psi$, $G^a_{\alpha\nu}G^{a\mu}_\beta$, etc., which occur in the light-cone operator-product expansion of the product of electromagnetic currents.[12] Here

$$D_\beta=\partial_\beta-ig_0a^a_\beta T^a \tag{1.1}$$

and

$$G^a_{\mu\nu}=\partial_\mu a^a_\nu-\partial_\nu a^a_\mu+g_0f^{abc}a^b_\mu a^c_\nu\ , \tag{1.2}$$

with a^a_μ the (unrenormalized) NA gauge fields

($a = 1, \ldots, N$ for an N-dimensional gauge group) ψ the coupled spinor fields, and T^a the spinor representation matrices. A detailed discussion of this mixing problem is given in Ref. 13.

The fundamental gauge, ghost, and spinor fields A^a_μ, C^a_i, and ψ are the unique ones of their dimensions $(1, 1, \frac{3}{2})$ and quantum numbers and so are multiplicatively renormalizable. The composite operators

$$A^a_\mu A^b_\nu,\ g_{\mu\nu} C^a_1 C^b_2, \ldots , \tag{1.3}$$

etc. of dimension 2 are the next simplest set. The study of these operators, which all can mix upon renormalization, will be the main subject of this paper. The complete set of operators $\{O^{cd}_{\alpha\beta}\}$ which can mix with (1.3) is obtained from (1.3) by contraction with all possible Lorentz-invariant matrices $L^{\mu\nu}_{\alpha\beta}$ (constructed out of metric tensors $g_{\gamma\lambda}$) and all possible gauge-group-invariant matrices M^{cd}_{ab} (constructed out of δ^{ef}, structure constants f^{efg}, etc.).[14] In spite of the fact that these operators do not have the physical significance of those mentioned in the preceding paragraph, their study is still interesting. Their low dimension renders the mixing problem relatively simple and transparent, and the field product occurs in (1.2) and also occurs when the YM field equations are subjected to an R transformation $A^a_\mu(x) \to A^a_\mu(x) + r^a_\mu$.[15] It is in fact this latter circumstance which motivated our interest in this problem.

Because the models we consider are AF, the specified renormalization problem is exactly soluble via renormalization-group (RG) techniques. We have carried out the analysis for the general SU(n) gauge group elsewhere.[16] Our purpose here is to see how much can be deduced from canonical methods alone. Our knowledge of the exact singularity structure of the operators (1.3) will enable us to determine the correctness of the quasicanonical formalism in these models. It will be seen that all of our results support the validity of this formalism. However, all of the information provided by the RG analysis will *not* be obtained. Our analysis will nevertheless provide insights into the physical bases of AF field theories.

The most interesting specific results we obtain are the following: (1) the behavior

$$g_0(K) \sim (\ln K)^{-1/2} \tag{1.4}$$

of the unrenormalized coupling constant for large cutoff K, (2) the finiteness of the product

$$\frac{Z_1}{Z_3} f^{abc} A^b_\mu A^c_\nu , \tag{1.5}$$

which occurs in the renormalized version of (1.2), and (3) a lower bound for the singularity $E_{\mu\nu\rho\sigma}(x)$, namely

$$E_{\mu\nu\rho\sigma}\left(\frac{1}{K}\right) \gtrsim \frac{Z_3}{Z_1}(K) , \tag{1.6}$$

which occurs in the operator-product expansion (OPE)

$$A^a_\mu(x) A^b_\nu(0) \underset{x \to 0}{\sim} E^{abc}_{\mu\nu\rho\sigma}(x)\, \partial^\rho A^\sigma_c + \cdots . \tag{1.7}$$

The result (1.4) is a well-known consequence of the RG analysis[7] and we have previously[16] deduced the finiteness of (1.5) by RG methods. In the present paper, these results are deduced without use of the RG.

We review in Sec. II the operator-mixing formalism in a general context. In Sec. III we illustrate the quasicanonical approach in the context of $\lambda\phi^4$ theory. Section IV contains a summary of some field-theoretic aspects of NAGT's, and it also serves to fix the notation. In Sec. V we apply the quasicanonical method to NAGT's by evaluating the $[\ddot{a}, \dot{a}]$ commutator, and deducing a quasicanonical OPE. In Sec. VI we deduce some consequences of the quasicanonical OPE. In Sec. VII we conclude the paper with a summary of our results.

II. OPERATOR MIXING

We consider a set of unrenormalized operators U_i, $i = 1, \ldots, N$, which are mixed upon renormalization, as described by the $N \times N$ renormalization constant matrix $\underline{Z}$. If Z_{ij} is the i,j matrix element of $\underline{Z}$, the renormalized operators are given by

$$R_i = \sum_{j=1}^{N} U_j (\underline{Z}^{-1})_{ji} \tag{2.1}$$

or

$$U_j = \sum_i R_i Z_{ij} . \tag{2.2}$$

Equation (2.1) should be written more precisely as

$$R_i(x) = \lim_{K^2 \to \infty} \sum_j U_j(x; K^2) (\underline{Z}^{-1}(K^2))_{ji} , \tag{2.3}$$

where we have now exhibited the spacetime dependence of R_i and U_j and the cutoff dependence of U_j and $\underline{Z}$. R_i is independent of the cutoff mass K^2 for $K^2 \to \infty$. The U_i are not multiplicatively renormalized into the R_i, but the matrix $\underline{V}$ which diagonalizes $\underline{Z}^{-1}$,

$$\underline{V}\,\underline{Z}^{-1}\underline{V} = \underline{z} = \mathrm{diag}(z_1, \ldots, z_n) , \tag{2.4}$$

can be used to construct the linear combinations

$$R'_i = \sum_j V_{ij} R_j , \quad U'_i = \sum_j V_{ij} U_j , \tag{2.5}$$

which *are* multiplicatively related:

$$R'_i = z_i U'_i . \tag{2.6}$$

In finite orders of perturbation theory, or exactly in AF theories, one has the form

$$\underline{Z}(K^2)=e^{(\ln t)\underline{d}}\underline{C}\ , \tag{2.7}$$

where

$$t=\ln(K^2/\mu^2)\ , \tag{2.8}$$

with μ^2 the renormalization mass parameter. Here $\underline{d}$ is the $N\times N$ effective anomalous dimension matrix and $\underline{C}$ is the $N\times N$ mixing matrix. $\underline{Z}$ can be made more explicit in terms of the projection operators $\underline{P}_l$ and eigenvalues d_l, $l=1,\ldots,N$, which decompose $\underline{d}$:

$$\underline{d}=\sum_l \underline{d}_l \underline{P}_l\ , \tag{2.9}$$

$$\underline{P}_i\underline{P}_j=\delta_{ij}\underline{I},\quad \sum_l \underline{P}_l=\underline{I}\ . \tag{2.10}$$

By using

$$e^{(\ln t)\underline{d}}=\sum_l e^{(\ln t)d_l}P_l = \sum_l t^{d_l}\underline{P}_l\ , \tag{2.11}$$

(2.7) becomes

$$\underline{Z}=\sum_l t^{d_l}\underline{P}_l\underline{C}, \tag{2.12}$$

so that (2.2) takes on the form

$$U_i=\sum_l t^{d_l}\sum_j R_j(\underline{P}_l\underline{C})_{ji}\ . \tag{2.13}$$

The leading divergence of U_i for $K^2\to\infty$ is thus given by the largest eigenvalue d_L of $\underline{d}$:

$$U_i(x;K^2)\xrightarrow[K^2\to\infty]{} t^{d_L}\sum_j R_j(\underline{P}_L\underline{C})_{ji}\ . \tag{2.14}$$

We will be interested in the cases where U_i is the product of unrenormalized fields a_i, b_i,

$$U_i(x)=a_i(x)\,b_i(x)\ . \tag{2.15}$$

Then in terms of the corresponding renormalized fields

$$A_i=Z_{A_i}{}^{-1}a_i,\quad B_i=Z_{B_i}{}^{-1}b_i\ , \tag{2.16}$$

(2.2) becomes

$$A_i(x)\,B_i(x)=\sum_j R_i(x)\tilde{Z}_{ji}(K^2)\ , \tag{2.17}$$

where

$$\tilde{Z}_{ij}=Z_{ij}Z_{A_i}{}^{-1}Z_{B_i}{}^{-1}. \tag{2.18}$$

Note that although (2.16) are cutoff independent, the local product (2.17) is not in general cutoff independent but rather has singularities for $K^2\to\infty$ given by $\underline{\tilde{Z}}(K^2)$. The point-separated product $A_i(x)\,B_i(0)$, on the other hand, *is* cutoff independent for $K^2\to\infty$ but has singularities for $x\to 0$. If the $K^2\to\infty$ and $x\to 0$ limits *commute*, then (2.17) can be recast as the OPE[17]

$$A_i(x)\,B_i(0)\underset{x\to 0}{\sim}\sum_j R_j(x)\tilde{Z}_{ji}(1/x^2)\ . \tag{2.19}$$

It is shown in Ref. 16 that (2.19) is in fact the correct OPE. If

$$Z_{A_i}{}^{-1}Z_{B_i}{}^{-1}\to t^{\delta_i}\ , \tag{2.20}$$

then, with (2.12), (2.19) gives

$$A_i(x)\,B_i(0)\underset{x\to 0}{\sim}\sum_l \rho^{d_l+\delta_i}\sum_j R_j(x)\,(\underline{P}_l\underline{C})_{ji}\ , \tag{2.21}$$

with

$$\rho=\ln(-x^2\mu^2)\ . \tag{2.22}$$

It is worthwhile, for later reference, to illustrate the above for the case $N=2$. We take $\underline{d}$ to be triangular so that (2.9) becomes

$$\underline{d}=\begin{pmatrix} u & w\\ 0 & v\end{pmatrix}=u\underline{P}_1+v\underline{P}_2, \tag{2.23}$$

where the projection matrices are

$$\underline{P}_1=\begin{pmatrix} 1 & \dfrac{w}{u-v}\\ 0 & 0\end{pmatrix},\quad \underline{P}_2=\begin{pmatrix} 0 & \dfrac{-w}{u-v}\\ 0 & 1\end{pmatrix}\ . \tag{2.24}$$

Now (2.13) becomes

$$U_i=t^u\sum_j R_j(\underline{P}_1\underline{C})_{ji}+t^v\sum_j R_j(\underline{P}_2\underline{C})_{ji} = t^uR_i\left(C_{1i}+\frac{w}{u-v}C_{2i}\right)+t^v\left(\frac{-w}{u-v}R_1+R_2\right)C_{2i}\ . \tag{2.25}$$

Of special interest is the case $u=v$ of equal eigenvalues. Then (2.25) becomes[18]

$$U_i=t^u(R_1C_{2i}w\ln t+R_1C_{1i}+R_2C_{2i})\ . \tag{2.26}$$

If t were K^2/μ^2 instead of $\ln(K^2/\mu^2)$, (2.26) would define a two-dimensional reducible but not completely reducible representation of the dilatation group. Such representations were previously discussed in detail.[19,10] When t is (2.8), (2.26) defines instead the "effective" version of the concept. As in Ref. 10, Eq. (2.26) can be inverted to give

$$R_1=\lim_{x\to 0}\frac{U_1}{C_{21}w\rho^u\ln\rho} = \lim_{x\to 0}\frac{U_2}{C_{22}w\rho^u\ln\rho}\ , \tag{2.27}$$
$$R_2=\lim_{x\to 0}(U_1-R_1C_{21}w\rho^u\ln\rho,-R_1C_{11}\rho^u)/C_{21}\rho^u\ ,$$

where we have reverted to x space, using (2.22).

III. QUASICANONICAL APPROACH

In this section we shall motivate and illustrate our quasicanonical approach with the study of the renormalization of composite operators in scalar field theories. Our discussion will be notationally simpler and more transparent than the corresponding gauge theory analysis which will be given in the following sections. However, the greater complexity of the NAGT's will give rise to much stronger results.

We consider first a renormalized massless scalar field $\phi(x)$, which is assumed to satisfy the field equation

$$\Box\phi(x) = J(x) \tag{3.1}$$

for some finite local operator source $J(x)$. It is convenient to introduce a regularization, symbolized by a cutoff parameter K, so that

$$\phi(x) = \lim_{K\to\infty} \phi_K(x)\,, \tag{3.2}$$

and similarly for the other renormalized operators in the theory. We also assume that the typical equal-time commutation relations

$$[\phi_K(x)\,,\dot\phi_K(0)]\,\delta(x_0) = \sigma_1(K)\,i\delta^4(x)\,, \tag{3.3}$$

$$[\dot J_K(x),\,\phi_K(0)]\,\delta(x_0) = \sigma_2(K)\,ij_K(x)\,\delta^4(x) \tag{3.4}$$

are valid. Here the $\sigma_i(K)$ are constants, perhaps divergent for $K\to\infty$, and

$$j(x) \equiv \lim_{K\to\infty} j_K(x)$$

is a renormalized current operator. In finite orders of perturbation theory, and exactly in AF theories,[4,6] one has the form

$$\sigma_i(K) = \sigma_i\,[\ln(K/\mu^2)]^{\tau_i}\,,\quad i = 1,2 \tag{3.5}$$

for constants σ_i, τ_i.[20] In these theories, $\sigma_1 = Z_3^{-1}$.

Equations (3.1), (3.3), and (3.4) imply the fixed-K short-distance (SD) expansion[10]

$$\phi_K(x)\phi_K(0) \underset{x\to 0}{\sim} \sigma_1(K)\,\Delta_+(x)I + \sigma_2(K)\,\Gamma_+(x)\,j(0) + R(K;x,0)\,, \tag{3.6}$$

where

$$\Delta_+(x) = \frac{1}{4\pi^2}\,\frac{1}{x^2 - i\epsilon x_0}\,,\qquad \Gamma_+(x) = \frac{1}{16\pi^2}\ln(-x^2 + i\epsilon x_0) \tag{3.7}$$

have discontinuities $i\Delta \equiv \Delta_+ - \Delta_-$, $i\Gamma \equiv \Gamma_+ - \Gamma_-$, which satisfy

$$\dot\Delta(x)\,|_{x_0=0} = \delta^3(\vec{x}),\quad \dddot\Gamma(x)\,|_{x_0=0} = \delta^3(\vec{x})\,, \tag{3.8}$$

and the remainder $R(K;x,0)$ is less singular than $(\ln x)$ in x. Suppose now that the theory is sufficiently smooth that the $x\to 0$ and $K\to\infty$ limits commute in the sense that *the $K\to\infty$ limit of (3.6) gives the correct $x\to 0$ behavior after removal of the cutoff when the substitution $K^2 \to 1/x^2$ is made.* Then

$$\begin{aligned}\phi(x)\phi(0) \underset{x\to 0}{\sim} &\ \sigma_1(-\ln x^2\mu^2)^{\tau_1}\Delta_+(x)I + \sigma_2(-\ln x^2\mu^2)^{\tau_2}\,\Gamma_+(x)\,j(0) + R\left(\frac{1}{x^2};x,0\right)\\ &\equiv \sigma_1'(-\ln x^2\mu^2)^{\tau_1}\left(\frac{1}{x^2}\right)I + \sigma_2'(-\ln x^2\mu^2)^{\tau_2+1}j(0) + R\ ,\end{aligned} \tag{3.9}$$

where

$$\sigma_1' = \sigma_1/4\pi^2,\quad \sigma_2' = \sigma_2/16\pi^2\ . \tag{3.10}$$

If we suppose further that R in (3.9) is no more singular than the exhibited term $(\ln x^2)^{\tau_2+1}$, we learn how to renormalize the composite operator ϕ^2:

$$j(0) = \lim_{x\to 0}\frac{\phi(x)\,\phi(0) - \sigma_1(\ln x^2)^{\tau_1}\Delta_+(x)I}{\sigma_2(\ln x^2)^{\tau_2}\,\Gamma_+(x)}\ . \tag{3.11}$$

More formally, this can be written

$$j = \rho^{-1}\phi^2 = \lim_{K\to\infty}\rho^{-1}(K)\,\phi_K{}^2\ , \tag{3.12}$$

where

$$\rho(K) = \sigma_2(\ln K^2)^{\tau_2}\,\Gamma_+\left(\frac{1}{K}\right), \tag{3.13}$$

and we have not indicated the c-number subtraction which appears in (3.11). The assumptions we made in order to deduce (3.11) will be seen to be valid in AF theories.

If, instead of the form (3.4), we have

$$[\dot J_K(x),\,\phi_K(0)]\,\delta(x_0) = \sigma_3(K)\,i\phi_K{}^2(0)\,\delta^4(x)\,, \tag{3.14}$$

then (3.6) is replaced by

$$\phi_K(x)\,\phi_K(0) \underset{x\to 0}{\sim} \sigma_1(K)\,\Delta_+(x)I + \sigma_3(K)\Gamma_+(x)\phi_K{}^2(0) + R\,. \tag{3.15}$$

Now, if the above smoothness assumptions are correct, we can deduce as a consistency condition that

$$(\ln x^2\mu^2)^{\tau_3}(\ln x^2\mu^2) \underset{x\to 0}{\longrightarrow} \text{const}\ , \tag{3.16}$$

or

$$\tau_3 = -1\ , \tag{3.17}$$

but we have not learned how to renormalize ϕ^2. The NAGT's will be seen to be sufficiently more intricate than the above one so that we will be able to deduce the analogs of *both* (3.11) and (3.16).

In practice, the source will have the form

$$J_K = \sigma_4(K)\,\phi_K{}^3 + \cdots \,, \tag{3.18}$$

and (3.4) will be evaluated using (3.3),

$$[\dot{J}(x), \phi(0)]\,\delta(x_0) = 3\sigma_1(K)\,\sigma_4(K)\,\phi_K{}^2 i\delta^4(x)\,, \tag{3.19}$$

so that

$$\sigma_3(K) = 3\sigma_1(K)\,\sigma_4(K), \quad \tau_3 = \tau_1 + \tau_4\,. \tag{3.20}$$

This procedure requires the further assumption that equal-time and $K \to \infty$ limits can be interchanged. In previous investigations, this assumption was found to be sometimes valid and sometimes not.[2] We will see that the assumption is valid in AF theories.

The smoothness assumptions encountered above are often not valid in renormalized perturbation theory, probably because the $\sigma_i(K)$ are too singular, e.g., $\sigma_3(K) \sim g^{2n}(\ln K)^n$ in $2n$th order. For example, in $\lambda\phi^4$ theory one would find

$$\phi_0(x)\,\phi_0(0) \underset{x\to 0}{\sim} \Delta_+(x) I - 3\lambda_0 \Gamma_+(x)\,\phi_0{}^2(x) + R_0\,, \tag{3.21}$$

where ϕ_0 is the unrenormalized field and λ_0 is the unrenormalized coupling constant. If R_0 were $\leqslant$const $\phi_0{}^2(0)$, this would imply that

$$\lambda_0(K) \sim (\ln K)^{-1}\,, \tag{3.22}$$

which is wrong in perturbation theory, where

$$\lambda_0(K) = \sum_{n=0}^{\infty} \lambda^{2n}[c_n(\ln K)^n + \cdots]\,. \tag{3.23}$$

The behavior of (3.23) is presumably just too singular for finite n. But can the sum in (3.23) be less singular? For positive renormalized coupling constant $\lambda > 0$, this is not known. However, for $\lambda < 0$ the theory is AF[4] and then one obtains precisely (3.22). This suggests that our assumptions are valid for the exact AF$\lambda < 0$ theory. The YM theories will be seen to behave similarly.

Our final illustrative example will be an exactly scale-invariant model.[10,11] The assumed basic equations

$$\begin{aligned} &\Box\phi = \lambda J\,,\\ &[\phi(x), \dot{\phi}(0)]\,\delta(x_0) = i\delta^4(x),\\ &[\dot{J}(x), \phi(0)]\,\delta(x_0) = j(0)\,\delta^4(x) \end{aligned} \tag{3.24}$$

are *not* consistent with conventional irreducible scale invariance but require reducible scale invariance[10]:

$$\phi(x) \to \rho\phi(\rho^{-1}x)\,, \tag{3.25}$$

$$j(x) \to \rho^2 j(\rho^{-1}x)\,, \tag{3.26}$$

$$k(x) \to \rho^2[k(\rho^{-1}x) + \ln\rho\, j(\rho^{-1}x)]\,. \tag{3.27}$$

Here $\{j, k\}$ constitute a two-dimensional, not completely reducible, representation of the dilatation group. They exactly correspond to the renormalized operators $\{R_1, R_2\}$, encountered at the end of Sec. II, which mix according to (2.26) and which have equal (vanishing) anomalous dimensions ($u = v = 0$). The relations (3.24) imply the OPE

$$\phi(x)\,\phi(0) \underset{x\to 0}{\sim} \Delta_+(x) I + \left(1 - \frac{i\lambda}{4\pi^2}\ln x^2\right) j(0) + k(0)\,, \tag{3.28}$$

and the consequent expressions (ignoring the c-number subtraction)

$$j(x) = Z\,\phi^2(x), \quad k(x) = \phi^2(x) - Z^{-1} j(x)\,, \tag{3.29}$$

for the renormalized operators with

$$Z^{-1}(K) = 1 - \frac{i\lambda}{4\pi^2}\ln K\,. \tag{3.30}$$

The representations (3.29) are precisely the same as (2.27) with the identifications

$$\begin{aligned} &R_1 = j, \quad R_2 = k, \quad U_1 = \phi^2,\\ &u = 0, \quad w = i\lambda/4\pi^2, \quad C_{11} = C_{21} = 1\,. \end{aligned} \tag{3.31}$$

To illustrate how our arguments work in this model, we replace the third commutator in (3.24) by

$$[\dot{J}(x), \phi(0)]\,\delta(x_0) = Z\,\phi^2\,\delta^4(x)\,, \tag{3.32}$$

with Z unknown and $Z\phi^2$ not assumed to be finite. We can then deduce the OPE

$$\phi(x)\,\phi(0) \underset{x\to 0}{\sim} \frac{-i\lambda}{4\pi^2}(\ln x^2) Z\,\phi^2 + R\,, \tag{3.33}$$

and conclude from consistency that $Z \sim [(-i\lambda/4\pi^2) \times (\ln x^2)]^{-1}$. According to (3.30), this conclusion is correct. We note that we have not been able to deduce the finiteness of $Z\phi^2$ and we recall that the equal-time and $K \to \infty$ limits often do *not* commute in this model.[10] The YM models, because of their greater complexity and less singular behaviors, are nicer in both of these aspects.

IV. NON-ABELIAN GAUGE THEORIES

In this section we review some field-theoretic apparatus used in studying non-Abelian gauge theories (NAGT's). The classical theory is specified by

$$\mathfrak{L}_{\text{classical}}(x) = -\tfrac{1}{4} G^a_{\mu\nu}(x)\, G_a^{\mu\nu}(x)\,, \tag{4.1}$$

where

$$G^a_{\mu\nu}(x)=\partial_\mu a^a_\nu(x)-\partial_\nu a^a_\mu(x)+g_0 f^{abc}a^b_\mu(x)\,a^c_\nu(x)\,, \tag{4.2}$$

and f^{abc} are the group-theoretic structure constants. For second quantization, a gauge-fixing term and the attendant ghost fields are admitted, and the Lagrangian is then

$$\mathcal{L}(x)=\mathcal{L}_{\text{classical}}(x)-\frac{1}{2\alpha_0}[\partial\cdot a(x)]^2 +\partial_\mu c^a_1(x)\,[\delta^{ab}\partial^\mu+g_0 f^{acb}a^c_\mu(x)]\,c^b_2(x)\,, \tag{4.3}$$

where α_0 is the (unrenormalized) gauge parameter, and c_1 and c_2 are anticommuting scalar ghost fields. It is then possible to write down canonical equal-time commutation relations (ETCR's) by identifying the canonically conjugate momenta to $a^a_\mu(x)$ to be

$$\pi^a_\mu=-G^a_{a\mu}-\frac{1}{\alpha_0}g_{0\mu}\partial_\lambda a^{a\lambda}\,. \tag{4.4}$$

The ETCR's are

$$\delta(x^0-y^0)\,[G^{0i}_a(x),a^j_b(y)]=i\delta^{ab}g^{ij}\delta^4(x-y)\,, \tag{4.5}$$

$$\delta(x^0-y^0)\left[\frac{1}{\alpha_0}\partial^0 a_a(x),a^0_b(y)\right]=i\delta^{ab}\delta^4(x-y)\,. \tag{4.6}$$

The Lagrangian (4.3) also gives the field equations

$$0=[\delta^{ab}\partial^\nu+g_0 f^{abc}a^\nu_c(x)]\,G^b_{\mu\nu}(x) -\frac{1}{\alpha_0}\partial_\mu\partial\cdot a_a(x)+g_0 f^{abc}[\partial_\mu c^b_1(x)]\,c^c_2(x)\,, \tag{4.7}$$

$$[\delta_{ab}\partial^\mu+g_0 f^{acb}a^\mu_c(x)]\,\partial_\mu c^b_1(x)=0\,, \tag{4.8}$$

$$\partial_\mu[\delta_{ab}\partial^\mu+g_0 f^{acb}a^\mu_c(x)]\,c^b_2(x)=0\,. \tag{4.9}$$

We always use lower-case letters to denote the unrenormalized fields, and use an index 0 to denote unrenormalized parameters. Renormalized fields are denoted by capital letters. Thus,

$$\vec{a}^\mu=Z_3^{1/2}\vec{A}_\mu\,, \tag{4.10}$$

$$\vec{c}_i=Z_3^{1/2}\vec{C}_i\,,\quad i=1,2 \tag{4.11}$$

$$g_0=\frac{Z_1}{Z_3^{3/2}}g\,, \tag{4.12}$$

$$\alpha_0=Z_3\alpha\,. \tag{4.13}$$

The field equations (4.7)–(4.9) in terms of renormalized quantities become

$$0=\left[\delta^{ab}\partial^\nu+\frac{Z_1}{Z_3}gf^{acb}A^\nu_c(x)\right] \times\left[\partial_\mu A^b_\nu(x)-\partial_\nu A^b_\mu(x)+\frac{Z_1}{Z_3}gf^{bcd}A^c_\mu(x)A^d_\nu(x)\right] -\frac{1}{Z_3\alpha}\partial^\mu\partial\cdot A^a(x)+\frac{\tilde{Z}_1}{Z_3}gf^{abc}\partial^\mu C^b_1(x)\,C^c_2(x)\,, \tag{4.14}$$

$$0=\left[\delta^{ab}\partial^\mu+\frac{Z_1}{Z_2}gf^{acb}A^\mu_c(x)\right]\partial_\mu C^b_1(x)\,, \tag{4.15}$$

$$0=\partial_\mu\left[\delta^{ab}\partial^\mu+\frac{Z_1}{Z_3}gf^{acb}A^\mu_c(x)\right]C^b_2(x)\,. \tag{4.16}$$

RG techniques sum up the perturbation series in the ultraviolet limit for asymptotically free NAGT's. The behavior of the combination Z_1/Z_3 of renormalization constants in the $K\to\infty$ limit is then known from such analyses.[5] The result is that

$$\lim_{K\to\infty}\frac{Z_1}{Z_3}(K)=0\,. \tag{4.17}$$

On the other hand, the operator products $A^a_\mu(x)A^b_\nu(x)$ are defined with some regularization K or with separated points. The singularities in the $K\to\infty$ or the $x\to 0$ limit have also been studied by RG techniques.[16] The operator product $A^a_\mu(x)A^b_\nu(0)$ possesses an expansion in terms of all operators with the same quantum numbers, and for dimension-2 operators there are three types: (1) $:AA:$, (2) ∂A, (3) $:C_1C_2:$. Thus the expansion takes the form

$$A^a_\mu(x)A^b_\nu(0)\underset{x\to 0}{\sim} D^{abcd}_{\mu\nu\rho\sigma}(x):A^\rho_c A^\sigma_d: +E^{abc}_{\mu\nu\rho\sigma}(x)\,\partial^\rho A^\sigma_c+F^{abcd}_{\mu\nu}(x):C^c_1C^d_2:\,. \tag{4.18}$$

In Ref. 16, we evaluated the singular functions on the right-hand side lowest nontrivial order in perturbation theory, and used the result as input to the RG summation. Up to technical details, the summation does not present any more difficulty. The true singularities of (4.18) are thus exactly knowable.

V. QUASICANONICAL METHODS FOR NON-ABELIAN GAUGE THEORIES

In this section we shall use the quasicanonical methods developed in Sec. III for the case of YM theories. Of particular interest is the behavior of $A^m_\mu(x)A^n_\nu(y)$ at short distance. The quasicanonical method incorporates information from the ETCR's and the field equations, and can be used to infer how to renormalize the composite operator involved.

To proceed we shall first write down the basic canonical field ETCR's. The canonical momentum conjugate to the YM field $\vec{a}_\mu$ is as given by Eq. (4.4),

$$\pi^a_\mu\equiv-G^a_{0\mu}-\frac{1}{\alpha_0}g_{0\mu}\partial_\lambda a^{a\lambda}\,, \tag{5.1}$$

and satisfies

$$\delta(x^0-y^0)\,[a^a_\mu(x),\pi^b_\nu(y)]=+ig_{\mu\nu}\delta^{ab}\delta^4(x-y)\,, \tag{5.2}$$

and so after some rearrangement,

$$\delta(x^0-y^0)[a_0^a(x),\dot a_0^b(y)]=-i\alpha_0\delta^{ab}\delta^4(x-y)\,, \quad (5.3)$$

$$\delta(x^0-y^0)[a_i^a(x),\dot a_j^b(y)]=-ig_{ij}\delta^{ab}\delta^4(x-y)\,, \quad (5.4)$$

$$\delta(x^0-y^0)[\dot a_i^a(x),\dot a_j^b(y)]=-2ig_0f^{abc}a_0^c g_{ij}\delta^4(x-y)\,, \quad (5.5)$$

$$\delta(x^0-y^0)[\dot a_0^a(x),\dot a_i^b(y)]=-i(\alpha_0-1)\delta^{ab}\partial_i\delta^4(x-y) -i\alpha_0 g_0 f^{acb}a_i^c\delta^4(x-y)\,, \quad (5.6)$$

$$\delta(x^0-y^0)[\dot a_0^a(x),\dot a_0^b(y)]=0\,. \quad (5.7)$$

To learn how to renormalize $A_\mu^a A_\nu^b$ we need the logarithmic singularity in the OPE $a_\mu^a(x)a_\nu^b(y)$, and this will be inferred from the commutator $[\ddot a,\dot a]$. We distinguish three kinds of (q-number) contributions to the commutator: (1) terms bilinear in a, (2) terms involving one derivative of a, (3) terms with derivatives of δ functions. Thus, we write

$$\delta(x^0-y^0)[\ddot a_\mu^a(x),\dot a_\nu^b(y)]=iO_{\mu\nu}^{ab}(x)\delta^4(x-y) +iP_{\mu\nu}^{ab}(x)\delta^4(x-y) +iQ_{\mu\nu k}^{ab}(x)\partial_x^k\delta^4(x-y)\,, \quad (5.8)$$

and by the use of (5.3)–(5.7) and the field equation (4.7), we have after a very tedious calculation

$$O_{00}^{ab}=0\,, \quad (5.9a)$$

$$O_{0i}^{ab}=-\alpha_0 g_0^2 f^{ace}f^{bde}a_0^c a_i^d\,, \quad (5.9b)$$

$$O_{i0}^{ab}=-\alpha_0 g_0^2 f^{ace}f^{bde}a_i^c a_0^d\,, \quad (5.9c)$$

$$O_{ij}^{ab}=-g_0^2 f^{ace}f^{bde}(a_\lambda^c a^{d\lambda}-4a_0^c a_0^d)g_{ij} +g_0^2(\alpha_0 f^{ace}f^{bde}-f^{abe}f^{cde}+f^{ade}f^{bce})a_i^c a_j^d\,; \quad (5.9d)$$

$$P_{00}^{ab}=-\alpha_0 g_0 f^{abc}\partial^k a_k^c\,, \quad (5.10a)$$

$$P_{0i}^{ab}=-g_0 f^{abc}(\alpha_0\partial_0 a_i^c-2\partial_i a_0^c)\,, \quad (5.10b)$$

$$P_{i0}^{ab}=g_0 f^{abc}(2\alpha_0\partial_0 a_i^c-\alpha\partial_i a_0^c)\,, \quad (5.10c)$$

$$P_{ij}^{ab}=g_0 f^{abc}[(\alpha_0-2)\partial_i a_j^c+2\partial_j a_i c-2_\lambda a^{c\lambda}g_{ij}]\,; \quad (5.10d)$$

$$Q_{00k}^{ab}=-2\alpha_0 g_0 f^{abc}a_k^c\,, \quad (5.11a)$$

$$Q_{0ik}^{ab}=Q_{i0k}^{ab}=-(\alpha_0-2)g_0 f^{abc}a_0^c g_{ik}\,, \quad (5.11b)$$

$$Q_{ijk}^{ab}=g_0 f^{abc}[\alpha_0(a_i^c g_{jk}+a_j^c g_{ik})-2g_{ij}a_k^c]\,. \quad (5.11c)$$

It is amusing to note some consistency conditions that have to be satisfied by the ETCR's (5.8)–(5.11). We start for example from Eq. (5.6) for the ETC $[\dot a,\dot a]$. We define

$$x=u+z\,,\qquad y=u-z\,, \quad (5.12)$$

and differentiate (5.6) with respect to u_0 to get

$$\delta(x^0-y^0)\frac{\partial}{\partial u_0}[\dot a_0^a(x),\dot a_i^b(y)]=-i\alpha_0 g_0 f^{abc}\dot a_i^c(x)\delta^4(x-y)\,. \quad (5.13)$$

Now the left side (5.13) is just

$$\delta(x^0-y^0)\frac{\partial}{\partial u_0}[\dot a_0^a(x),\dot a_i^b(y)]=\delta(x^0-y^0)[\ddot a_0^a(x),\dot a_i^b(y)] +\delta(x^0-y^0)[\dot a_0^a(x),\ddot a_i^b(y)]\,, \quad (5.14)$$

and making use of (5.8), we have

$$\begin{aligned}\delta(x^0-y^0)[\dot a_0^a(x),\ddot a_i^b(y)]&=-\delta(x^0-y^0)[\ddot a_i^b(y),\dot a_0^a(x)]\\ &=-iO_{i0}^{ba}(y)\delta^4(x-y)\\ &\quad-iP_{i0}^{ba}(y)\delta^4(x-y)\\ &\quad-iQ_{i0k}^{ba}(y)\partial_y^k\delta^4(x-y)\\ &=iO_{i0}^{ba}(x)\delta^4(x-y)\\ &\quad-iP_{i0}^{ba}(x)\delta^4(x-y)\\ &\quad+iP_{i0k}^{ba}(x)\partial_x^k\delta^4(x-y)\\ &\quad+i[\partial_x^k Q_{i0k}^{ba}(x)]\delta^4(x-y)\,.\end{aligned} \quad (5.15)$$

Thus, from (5.13), (5.14), and (5.15) we must have equality for the coefficients of $\delta^4(x-y)$:

$$-i\alpha_0 g_0 f^{acb}\dot a_i^c(x)=i\{[O_{ij}^{ab}(x)-O_{ji}^{ba}(x)] +[P_{ij}^{ab}(x)-P_{ji}^{ba}(x)] +\partial_x^k Q_{i0k}^{ba}(x)\}\,. \quad (5.16)$$

We can easily check the correctness of (5.16) by substituting in (5.9b), (5.9c), (5.10b), (5.10c), and (5.11b) to get the right-hand side of (5.16) to be

$$\begin{aligned}\text{right-hand side}&=i\{-g_0 f^{abc}[\alpha_0\partial_0 a_i^c+(\partial_i a_0^c)(2-\alpha_0)]+(\alpha_0-2)g_0 f^{abc}\partial_i a_0^c\}\\ &=i\alpha_0 g_0 f^{acb}\partial_0 a_i^c=\text{left-hand side}\,.\end{aligned} \quad (5.17)$$

Similar consistency conditions can be checked for other components of the ETC.

Having thus checked the consistency of the commutation relations (5.8)–(5.11), we proceed to deduce their implications on the form of the $a_\mu^a(x)a_\nu^b(y)$ OPE. This OPE is restricted by the usual symmetry properties. There can be *seven* independent forms for operators bilinear in a, and *three* forms for single derivatives of a:

$$a_\mu^a(x)\, a_\nu^b(y) \underset{x\to y}{\longrightarrow} [e_1^{abcd}\, a_\mu^c(u)\, a_\nu^d(u)\, \Gamma_+(z) + e_2^{abcd}\, g_{\mu\nu}\, a_\lambda^c(u)\, a^{d\lambda}(u)\, \Gamma_+(z) + e_3^{abcd}\, g_{\mu\nu}\, a_\alpha^c(u)\, a_\beta^d(u)\, \Delta_+^{\alpha\beta}(z)$$
$$+ e_4^{abcd}\, a_\lambda^c(u)\, a^{d\lambda}(u)\, \Delta_{+\mu\nu}(z) + e_5^{abcd}\, a_\alpha^c(u)\, a_\beta^d(u)\, G_+^{\alpha\beta}{}_{\mu\nu}(z)$$
$$+ e_6^{abcd}\, a_\mu^c(u)\, a_\alpha^d(u)\, \Delta_{+\nu}{}^\alpha(z) + e_7^{abcd}\, a_\nu^c(u)\, a_\alpha^d(u)\, \Delta_{+\mu}{}^\alpha(z)]$$
$$+ [c_1^{abc}(\partial_\mu a_\nu^c(u) - \partial_\nu a_\mu^c(u))\, \Gamma_+(z) + c_2^{abc}(g_{\mu\alpha}\partial_\beta a_\nu(u) - g_{\nu\alpha}\partial_\beta a_\mu(u))\, \Delta_+^{\alpha\beta}(z)$$
$$+ c_3^{abc}(g_{\mu\alpha}\partial_\nu a_\beta(u) - g_{\nu\alpha}\partial_\mu a_\beta(u))\, \Delta_+^{\alpha\beta}(z)]\,, \tag{5.18}$$

where, as in (5.12),

$$x = u + z\,,\qquad y = u - z\,, \tag{5.19}$$

and

$$\Gamma_+(z) = \frac{1}{16\pi^2}\ln(-z^2 + i\epsilon z_0)\,, \tag{5.20}$$

$$\Delta_+^{\alpha\beta}(z) = \frac{1}{8\pi^2}\frac{z^\alpha z^\beta}{z^2 - i\epsilon z_0}\,, \tag{5.21}$$

$$G_+^{\alpha\beta\gamma\delta}(z) = -\frac{1}{4\pi^2}\frac{z^\alpha z^\beta z^\gamma z^\delta}{(z^2 - i\epsilon z_0)^2}\,. \tag{5.22}$$

The singular functions (5.20)–(5.22) are chosen to give the following limits at equal times ($z_0 = 0$):

$$\int d^3z\, \dddot{\Gamma}(z_0 = 0) = 1\,, \tag{5.23}$$

$$\int d^3z\, \dddot{\Delta}_{\alpha\beta}(z_0 = 0) = 4 g_{\alpha 0} g_{\beta 0} - g_{\alpha\beta}\,, \tag{5.24}$$

$$\int d^3z\, \dddot{G}_{\alpha\beta\gamma\delta}(z_0 = 0) = 24 g_{\alpha 0} g_{\beta 0} g_{\gamma 0} g_{\delta 0} - 4(g_{\alpha 0} g_{\beta 0} g_{\gamma\delta} + g_{\alpha 0} g_{\gamma 0} g_{\beta\delta} + g_{\alpha 0} g_{\delta 0} g_{\beta\gamma}$$
$$+ g_{\beta 0} g_{\gamma 0} g_{\alpha\delta} + g_{\beta 0} g_{\delta 0} g_{\alpha\gamma} + g_{\gamma 0} g_{\delta 0} g_{\alpha\beta}) + (g_{\alpha\beta} g_{\gamma\delta} + g_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\delta} g_{\beta\gamma})\,, \tag{5.25}$$

where the discontinuities are

$$i\Gamma = \Gamma_+ - \Gamma_-\,,\qquad i\Delta = \Delta_+ - \Delta_-\,,\qquad iG = G_+ - G_-\,. \tag{5.26}$$

A slight complication arises in the determination of the covariant tensor singularities from the ETC. Suppose that

$$[a_\mu^a(x), a_\nu^b(y)] \underset{x\to y}{\longrightarrow} \Gamma(z)\, \mathcal{S}_{\mu\nu}^{ab}(u)\,. \tag{5.27}$$

Then three differentiations with respect to time z_0 would give the operator $\mathcal{S}_{\mu\nu}^{ab}$ in terms of the ETC,

$$\{\partial_{z_0}{}^3 [a_\mu^a(x), a_\nu^b(y)]\}_{\mathrm{ET}} = \delta^3(\vec{x} - \vec{y})\, \mathcal{S}_{\mu\nu}^{ab}(u)\,, \tag{5.28}$$

while canonical calculation above furnishes the quantity

$$C_{\mu\nu}^{ab}(x, y) \equiv [\ddot{a}_\mu^a(x), \dot{a}_\nu^b(y)]_{\mathrm{ET}}\,. \tag{5.29}$$

We adopt the notation

$$[a_\mu^a(x), a_\nu^b(y)]_{\mathrm{ET}} \equiv \mathcal{A}\mathcal{B}\,,\qquad [\dot{a}_\mu^a(x), a_\nu^b(y)]_{\mathrm{ET}} \equiv \dot{\mathcal{A}}\mathcal{B}\,, \tag{5.30}$$

etc. It follows from the chain rule that

$$\partial_{z_0}{}^3 \mathcal{A}\mathcal{B} = \dddot{\mathcal{A}}\mathcal{B} - 3\ddot{\mathcal{A}}\dot{\mathcal{B}} + 3\dot{\mathcal{A}}\ddot{\mathcal{B}} - \mathcal{A}\dddot{\mathcal{B}}\,. \tag{5.31}$$

Next we note that

$$\dot{\mathcal{A}}\mathcal{B} \sim \delta^3(\vec{x} - \vec{y})\,, \tag{5.32}$$

$$\mathcal{A}\dot{\mathcal{B}} \sim \delta^3(\vec{x} - \vec{y})\,, \tag{5.33}$$

and so it follows by repeated differentiations of (5.32)–(5.33) that

$$0 = \dddot{\mathcal{A}}\mathcal{B} + 2\ddot{\mathcal{A}}\dot{\mathcal{B}} + \dot{\mathcal{A}}\ddot{\mathcal{B}} \tag{5.34}$$

and

$$0 = \ddot{\mathcal{A}}\dot{\mathcal{B}} + 2\dot{\mathcal{A}}\ddot{\mathcal{B}} + \mathcal{A}\dddot{\mathcal{B}}\,. \tag{5.35}$$

Equations (5.34) and (5.35) allow us to eliminate $\dddot{\mathcal{A}}\mathcal{B}$ and $\mathcal{A}\dddot{\mathcal{B}}$ in (5.31) to obtain

$$\partial_{z_0}{}^3 \mathcal{A}\mathcal{B} = 4\dot{\mathcal{A}}\ddot{\mathcal{B}} - 4\ddot{\mathcal{A}}\dot{\mathcal{B}}$$
$$= 4([\dot{a}_\mu^a(x), \ddot{a}_\nu^b(y)]_{\mathrm{ET}} - [\ddot{a}_\mu^a(x), \dot{a}_\nu^b(y)]_{\mathrm{ET}})\,. \tag{5.36}$$

We can simplify (5.36) further by noting that

$$[\dot{a}^a_\mu(x), \ddot{a}^b_\nu(y)]_{\mathrm{ET}} = -[\ddot{a}^b_\nu(y), \dot{a}^a_\mu(x)]_{\mathrm{ET}} = -C^{ba}_{\nu\mu}(y,x)\,. \tag{5.37}$$

Thus we obtain finally

$$\delta^3(\vec{x}-\vec{y})\, \mathcal{S}^{ab}_{\nu\mu}(u) = -4[C^{ab}_{\mu\nu}(x,y) + C^{ba}_{\nu\mu}(y,x)]\,. \tag{5.38}$$

Now we can take the ET limit of (5.18) and compare coefficients with (5.8)–(5.11) to determine the tensors $e_1, \ldots, e_7, c_1, c_2, c_3$:

$$e_1 = -[-\tfrac{1}{2}\alpha_0 h_1 + \tfrac{1}{6}(\alpha_0-2)h^s_1] + (\tfrac{1}{2}h_2 + \tfrac{1}{12}h^s_2)\,, \tag{5.39a}$$

$$e_2 = -\tfrac{1}{12}(1+\alpha_0)h^s_1 + \tfrac{1}{24}h^s_2\,, \tag{5.39b}$$

$$e_3 = -(\tfrac{5}{6} + \tfrac{1}{12}\alpha_0)h^s_1 + \tfrac{1}{24}h^s_2\,, \tag{5.39c}$$

$$e_4 = -\tfrac{1}{12}(1+\alpha_0)h^s_1 + \tfrac{1}{24}h^s_2\,, \tag{5.39d}$$

$$e_5 = \tfrac{1}{24}h^s_2 - \tfrac{1}{12}(\alpha_0-2)h^s_1\,, \tag{5.39e}$$

$$e_6 = -[-\tfrac{1}{6}(\alpha_0-2)h^s_1 + \tfrac{1}{4}\alpha_0 h_1] - (\tfrac{1}{4}h_2 - \tfrac{1}{12}h^s_2)\,, \tag{5.39f}$$

$$e_7 = -[\tfrac{1}{3}(\alpha_0+1)h^s_1 - \tfrac{1}{4}h_1] - (\tfrac{5}{12}h_2 - \tfrac{1}{4}h_2)\,, \tag{5.39g}$$

where

$$(h_1)^{abcd} = g_0^{\,2} f^{ace} f^{bde}\,, \tag{5.40a}$$

$$(h_2)^{abcd} = g_0^{\,2}(-\alpha_0 f^{ace} f^{bde} + f^{abe} f^{cde} - f^{ade} f^{bce})\,, \tag{5.40b}$$

and

$$h^s = \tfrac{1}{2}(h + \tilde{h})\,, \tag{5.41}$$

with

$$\tilde{h}^{abcd} \equiv h^{abdc}\,, \tag{5.42}$$

and

$$c_1 = -\tfrac{1}{4}(\alpha_0+5)g_0 f^{abc}\,, \tag{5.43a}$$

$$c_2 = -(\alpha_0-1)g_0 f^{abc}\,, \tag{5.43b}$$

$$c_3 = +\tfrac{1}{4}(\alpha_0-1)g_0 f^{abc}\,. \tag{5.43c}$$

Equations (5.18) and (5.39)–(5.43) give us then the form of the OPE $A^a_\mu(x)A^b_\nu(y)$ that follows from the unrenormalized basic field equations and ETCR's. It is significant that the independent forms $e_1, \ldots, e_7, c_1, c_2, c_3$ are *uniquely* determined by solving a set of coupled linear equations which follow from identifying coefficients from the ETC's (5.8)–(5.11). This means that every one of these independent coefficients $e_1, \ldots, e_7, c_1, c_2, c_3$ is affected by the particular form taken by the ETC's as follows from the canonical quantum field theory of the Lagrangian (4.3). Were this not the case, say if some of them were indeterminate unknowns in the set of coupled linear equations, those coefficients would be free from any restrictions imposed by the set of ETC's.

VI. QUASICANONICAL DETERMINATION OF OPERATOR-PRODUCT SINGULARITIES

We now make use of the fixed-K SD expansion (5.12) to study the SD singularity structure of the $A^a_\mu(x)A^b_\nu(y)$ operator product. Omitting Lorentz-internal indices, we schematically write (5.12) as

$$a(x)\,a(0) \underset{x\to 0}{\longrightarrow} \mathfrak{X} g_0^{\,2}(a\,a)\mathfrak{L} + \mathfrak{Y} g_0(\partial a)\mathfrak{L} + \bar{\mathfrak{X}} g_0^{\,2}(aa) + \bar{\mathfrak{Y}} g_0(\partial a) + R\,, \tag{6.1}$$

where $\mathfrak{L} = \ln x^2$, $\mathfrak{X}, \mathfrak{Y}$ denote terms in (5.12) with the coefficient Γ_+ proportional to $\ln x^2$, $\bar{\mathfrak{X}}, \bar{\mathfrak{Y}}$ denote those with coefficients with only direction-dependent singularities, and R denotes the remainder that is no more singular in the $K^2 \sim 1/x^2 \to \infty$ limit than the exhibited terms. By making a wave-function renormalization this is just equivalent to

$$A(x)A(0) \underset{x\to 0}{\longrightarrow} \mathfrak{X} g_0^{\,2}(AA)\mathfrak{L} + \mathfrak{Y} g_0(\partial A) Z_3^{-1/2}\mathfrak{L} + \bar{\mathfrak{X}} g_0^{\,2}(AA) + \bar{\mathfrak{Y}} g_0(\partial A) Z_3^{-1/2} + R Z_3^{-1}\,. \tag{6.2}$$

On the other hand, general principles[1] call for the existence of an OPE in terms of finite operators for $A(x)A(0)$:

$$A(x)A(0) \underset{x\to 0}{\longrightarrow} \xi(:AA:)\mathfrak{L}^D + \eta(\partial A)\mathfrak{L}^\delta\,. \tag{6.3}$$

We substitute the expansion (6.3) for AA on both sides of (6.2) to obtain

$$:A^e A^f:(\mathfrak{L}^D \xi^{abef}) + \partial A^e(\mathfrak{L}^\delta \eta^{abc}) \underset{x\to 0}{\longrightarrow} :A^e A^f:(g_0^{\,2}\mathfrak{L}^{D+1}\mathfrak{X}^{abcd}\xi^{cdef} + g_0^{\,2}\mathfrak{L}^D \bar{\mathfrak{X}}^{abcd}\xi^{cdef}) + \partial A^e(g_0^{\,2}\mathfrak{L}^{\delta+1}\mathfrak{X}^{abcd}\eta^{cde} + g_0 Z_3^{-1/2}\mathfrak{L}\mathfrak{Y}^{abc} + g_0^{\,2}\mathfrak{L}^\delta \bar{\mathfrak{X}}^{abcd}\eta^{cde} + g_0 Z_3^{-1/2}\bar{\mathfrak{Y}}^{abc}) + R^{ab} Z_3^{-1}\,. \tag{6.4}$$

We compare the singular coefficients of the finite operators on both sides of (6.4). The $:AA:$ coefficients give the relation

$$g_0^{\,2}\mathfrak{L}\mathfrak{X}^{abcd}\xi^{cdef} \cong \xi^{abef}\,, \tag{6.5}$$

with $\cong$ meaning here "as singular as." From (6.5) it follows that necessarily

$$g_0^{\,2}\mathfrak{L} \cong 1\,. \tag{6.6}$$

The equality sign cannot be inferred in such considerations because of the possible presence of R, as mentioned in Sec. III. The comparison of the ∂A coefficients yields

$$g_0^2 \mathfrak{L}^{\delta+1} \mathfrak{X}^{abcd} \eta^{cde} + g_0 Z_3^{-1/2} \mathfrak{L} \mathcal{Y}^{abe} \cong \mathfrak{L}^{\delta} \eta^{abe} . \tag{6.7}$$

Using the information (6.5) in (6.6) we obtain

$$\mathfrak{L}^{\delta} \mathfrak{X}^{abcd} \eta^{cde} + g_0 Z_3^{-1/2} \mathfrak{L} \mathcal{Y}^{abe} \cong \mathfrak{L}^{\delta} \eta^{abe} . \tag{6.8}$$

For (6.7) to hold the singularity $g_0 Z_3^{-1/2}\mathfrak{L}$ obviously cannot be greater than $\mathfrak{L}^{\delta}$, and so we have the bound

$$\mathfrak{L}^{\delta} \gtrsim g_0 Z_3^{-1/2} \mathfrak{L} . \tag{6.9}$$

Using (6.5) and recalling that $g_0 = Z_1 Z_3^{-3/2} g$, we get

$$\mathfrak{L}^{\delta} \gtrsim \mathfrak{L}^{1/2} Z_3^{-1/2} \tag{6.10}$$

and

$$\mathfrak{L}^{\delta} \geq Z_3 / Z_1 . \tag{6.11}$$

So far our quasicanonical methods have yielded the estimates (6.5), (6.9), and (6.10) of renormalization constants much the same as in the simpler $\lambda\phi^4$ theory of Sec. III. However, the occurence of two finite operators $:AA:$ and ∂A in the expansion enables us to conclude more about the AA OPE.

We consider again Eq. (6.2). $\mathcal{Y}^{abc}$ from our calculations is just proportional to the antisymmetric f^{abc}. We therefore have two equations from (6.2), according to whether the a, b indices are symmetrized or antisymmetrized:

$$\text{Symmetrize:}\quad [A(x)A(0)]^S \underset{x\to 0}{\longrightarrow} \mathfrak{X}^S g_0^2 (AA)\mathfrak{L} + \overline{\mathfrak{X}}^S g_0^2 (AA) + R^S Z_3^{-1} , \tag{6.12}$$

$$\text{Antisymmetrize:}\quad [A(x)A(0)]^A \underset{x\to 0}{\longrightarrow} \mathfrak{X}^A g_0^2 (AA)\mathfrak{L} + \mathcal{Y} g_0 (\partial A)^{-1/2} \mathfrak{L} + \overline{\mathfrak{X}}^A g_0^2 (AA) + \overline{\mathcal{Y}} g_0 (\partial A) Z_3^{-1/2} + R^A Z_3^{-1} . \tag{6.13}$$

Nothing interesting arises from the symmetrized relation (6.11). However, from (6.12) a little rearrangement gives (putting back in the internal indices)

$$\mathcal{Y}^{abc} \partial A^c \cong \left(\frac{Z_3^{1/2}}{g_0 \mathfrak{L}}\right) \{ [A^a(x) A^b(0)]^A - \mathfrak{X}^A_{abcd} g_0^2 A^c A^d \mathfrak{L} - \overline{\mathfrak{X}}^A_{abcd} g_0^2 A^c A^d + \overline{\mathcal{Y}}^{abc} g_0 \partial A^c Z_3^{-1/2} \} , \tag{6.14}$$

where $\cong$ again means "as singular as." Omitting terms smaller by a logarithm, we get

$$\mathcal{Y}^{abc} \partial A^c \cong \frac{Z_1}{Z_3} A^c A^d \left(\delta^{ac} \delta^{bd} + \frac{1}{b} \mathfrak{X}^{abcd} \right)^A , \tag{6.15}$$

where we need[7]

$$g_0(K) \underset{K\to\infty}{\longrightarrow} \frac{1}{(2b \ln K)^{1/2}} \tag{6.16}$$

and "A" again means antisymmetrization with respect to a and b. From (6.14) we can immediately conclude that

$$\frac{Z_1}{Z_3} f^{abc} A^b A^c \cong \partial A^a , \tag{6.17}$$

and is therefore a finite operator. It is particularly interesting to note that the field intensity $G^a_{\mu\nu}$ in NAGT's is linear in the gauge fields:

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu a^a_\nu - \partial_\nu a^a_\mu + g_0 f^{abc} a^b_\mu a^c_\nu \\ &= Z_3^{1/2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} \frac{Z_1}{Z_3} A^b_\mu A^c_\nu \right) \\ &\cong Z_3^{1/2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) . \end{aligned} \tag{6.18}$$

The important result (6.17) is in full agreement with the results of the RG analysis.[16] The behavior (6.16) for $g_0(K)$ is of course also a consequence of the RG analysis, but it is good to see it emerge in such a simple way here.

VII. CONCLUSION

It is interesting to observe the relationship of the ETC calculations to the calculation of singular functions in (4.18) in low orders performed in Ref. 16. In (5.18), the operator products on the right are at coincident spacetime points, and would have singularities beyond the free theory. But on the level of order-by-order (without summing) perturbation theory, the $\{e_i\}$ are already of second order, and so up to g^2, the $a(u)a(u)$ operator products are effectively free. In (4.18) the $:AA:$ are necessarily singularity-free, so that up to second order the singular contribution to D again multiplies an effectively free $:AA:$. Thus the computations for $\{D\}$ in perturbation theory performed elsewhere exactly coincide with the quasicanonical evaluation of the $\{e_i\}$ carried out here, provided that the usual identification of

$$K^2 \leftrightarrow 1/x^2 \tag{7.1}$$

is made. The agreement with the perturbative calculation serves as an invaluable check on the veracity of our quasicanonical evaluation.

The way to go beyond perturbation theory is quite different in the two approaches. The RG approach takes the second-order calculation and uses it as input to the RG to obtain the full behavior of the singular coefficient. The quasicanonical approach takes the SD expansion (5.18) as an exact consequence of the formal field equations and deduces information inaccessible by finite-order perturbation theory. This abbreviated approach cannot hope to replace the RG analysis, but its relative success indicates that AF theories are even more benign in their ultraviolet (UV) behavior than is generally expected. The other side of the coin is that this rather simple quasicanonical method continues to work amazingly well, and might merit further study.

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