PHYSICAL REVIEW D VOLUME 15, NUMBER 4 15 FEBRUARY 1977

Point transformations and renormalization in the unitary gauge: Renormalization effects

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(Received 2 July 1976)

We continue an analysis of two simple gauge theory models using point transformations rather than gauge transformations. The renormalization constants are examined directly in two gauges, the renormalization (Landau) and unitary gauges. Our result is that the individual coupling-constant renormalizations are identical when calculated in each of the above two gauges, although the wave-function and proper vertex renormalizations differ.

I. INTRODUCTION

In two previous papers,[1] which we denote by I and II, a new approach to the treatment of unified gauge theory models was introduced. In this approach one's attention is focused on two gauges only: one a specific renormalization gauge (in practice the Landau gauge) and the other the unitary gauge. In place of the group of gauge transformations which connect the different gauges used in the conventional treatments, the R and U gauges are connected by a point transformation of fields. This fact was used to give a formal proof of the renormalizability of the unitary-gauge Lagrangian to all orders. The approach was illustrated by means of two simple models, one Abelian and the other non-Abelian. In each of the models, the equality of the S-matrix elements constructed in the two gauges was demonstrated by means of an explicit calculation to fourth order.

For the calculations the canonical formalism was used rather than a dispersion approach,[2] as it allowed the insertion of renormalization counterterms in the Lagrangian in a simple gauge-invariant manner, and the calculation from first principles of the renormalization constants in each gauge, in addition to the Feynman diagrams with structure. The calculation itself was refined to the point where it was purely graphical, and one could tell by inspection that the S-matrix elements were identical, and how the divergences in the unitary gauge and the ghosts in the renormalization gauge (in the non-Abelian model) cancel in the evaluation of a physical amplitude.

The advantage of using the approach advocated in I and II lies in the direct comparison of the unitary gauge and one choice of renormalization gauge, rather than treating the unitary gauge as a limiting case. Thus we can derive relations between renormalization constants, for example, by working directly in the unitary gauge, as well as in a renormalization gauge. The disadvantage is that we lose the possibility of exploiting gauge dependence, as done if the ξ parameter is introduced when fixing the gauge.

In the treatment of I and II, not all of the renormalization constants were explicitly taken into account. Only the mass renormalization constants were considered, and it was shown that they were exactly the same in the R and U gauges, to second-order corrections. In other words, if the mass counterterms were chosen to cancel the self-energies as calculated in one gauge, then the self-energies in the other gauge were automatically canceled. On the strength of this result the self-energy contributions to the fourth-order S-matrix element were not considered, as they canceled out anyway. However, the other renormalizations needed at this level of perturbation theory, for example the wave-function and coupling-constant renormalizations, were not explicitly exhibited. It is known, for example, that certain of the wave-function renormalization constants are gauge dependent.[3,4]

From the calculations of I and II we see that the unrenormalized (bare) S-matrix elements are the same in the two gauges. What is not so evident, because the wave-function and coupling-constant renormalizations were not explicitly accounted for, is that these calculations have as a result the equality of the renormalized, or physical, S-matrix elements in the two gauges.

It is the purpose of this paper to point out that, while the wave-function and vertex renormalization constants are separately different in the two gauges, when combined together to form the renormalizations of the coupling constants (or charges) the anomalous parts cancel so that the resulting constants are identical in the two gauges, i.e., the charge renormalization constants are gauge independent. This result completes the treatment of I and II by exhibiting explicitly the renormalization properties of the wave functions and the coupling constants.

For the purposes of demonstrating the results we will examine the simpler of the two models considered, namely the Abelian model of I. The analysis has also been carried out for the non-Abelian model.[5] In this case no essentially new features occur; the calculation is more cumbersome, but the result follows nonetheless. In Sec. II we review the formalism developed for the Abelian model in I. In Sec. III we attack the renormalization constants of the model, in perturbation theory, and include the diagrammatic summaries of the relevant calculations.

II. THE ABELIAN MODEL

In this section we will review some of the formalisms introduced in I for the Abelian model. The model consists, initially, of a single vector-meson field $V_\nu(x)$, with axial-vector couplings, a fermion field $\psi(x)$, and a scalar-pseudoscalar complex doublet $\phi(x)=\phi_1(x)+i\,\phi_2(x)$. The interactions are chosen invariant under a γ_5-type U(1) group of local gauge transformations. We denote the Yukawa coupling constant by g and the gauge coupling by f. The scalar-field self-interaction term is chosen to be of the Goldstone form.[6] After applying the usual analysis of shifting the scalar fields by a c-number amount, we find that the vector meson acquires a mass denoted by μ, the fermion acquires a mass given by m, while one of the scalar fields, ϕ_1, acquires a mass κ and the other remains massless. The parameters of the theory are related by

$$\frac{f}{\mu}=\frac{g}{2m}, \tag{1}$$

which follows from the analysis of the Goldstone potential. The Lagrangian describing the model then takes the form

$$\mathfrak{L}\left(V,\psi,\phi+\frac{m}{g}\right)=-\tfrac{1}{4}[F_{\mu\nu}(V)]^2+\tfrac{1}{2}\mu^2V^2+\bar{\psi}(i\not\partial-m)\psi+\tfrac{1}{2}|\partial_\nu\phi+2ifV_\mu\phi|^2-f\bar{\psi}\not{V}\gamma_5\psi-g\bar{\psi}(\phi_1-i\gamma_5\phi_1)\psi$$
$$+\mu V^\nu(\partial_\nu\phi_2+2fV_\nu\phi_1)-\frac{\kappa^2}{2}\left(\frac{g}{2m}\right)^2\left[|\phi|^4+\frac{4m}{g}|\phi|^2\phi_1+\left(\frac{2m}{g}\right)^2\phi_1{}^2\right]. \tag{2}$$

This Lagrangian is no longer invariant under the original symmetry group; however, the effects of the symmetry are still felt as we can write down transformation laws under which $\mathfrak{L}$ is invariant, called "broken" gauge transformations,[7]

$$\begin{aligned}&\psi(x)\to e^{-if\gamma_5\Lambda(x)}\psi(x)\,,\\&\phi(x)+\frac{m}{g}\to e^{-2if\Lambda(x)}\left[\phi(x)+\frac{m}{g}\right],\\&V_\nu(x)\to V_\nu(x)+\Lambda_{,\nu}(x)\,.\end{aligned} \tag{3}$$

To introduce the point transformations we concentrate our attention on two gauges. As our renormalization (R) gauge we choose the Landau gauge, in which the vector-meson field is purely transverse, and our unitary gauge is chosen conventionally as that in which the unphysical field $\phi_2(x)$ does not appear explicitly. If we use the Stückleberg form for the vector-meson field

$$\begin{aligned}&V_\nu(x)=U_\nu(x)+\frac{1}{\mu}\theta_{,\nu}(x)\,,\\&\partial^\nu U_\nu=0\,,\quad \Box\theta=\mu\partial^\nu V_\nu\,,\end{aligned} \tag{4}$$

then we see that we can formally define the R gauge by the condition

$$\theta=0$$

and the Ugauge by

$$\phi_2=0\,. \tag{5}$$

As in I, we choose as counterterms

$$\begin{aligned}\delta L\left(V,\psi,\phi+\frac{m}{g}\right)=&\;\delta m\bar{\psi}\left(1+\frac{g}{m}(\phi_1-i\gamma_5\phi_2)\right)\psi\\&+\frac{\delta k_1-\delta k_2}{2}\left(\frac{g}{2m}\right)\left|\phi+\frac{m}{g}\right|^4\\&+\tfrac{1}{2}(-\tfrac{1}{2}\delta k_1+\tfrac{3}{2}\delta k_2)\left|\phi+\frac{m}{g}\right|^2\\&-\tfrac{1}{2}\delta\mu U^2\,.\end{aligned} \tag{6}$$

These are manifestly invariant under the "broken" gauge transformations and provide mass counterterms δm, $\delta\mu$, δk, and δk_2 for the fields ψ, U, ϕ_1, and ϕ_2, respectively. The explicit charge renormalization terms included are not separately invariant.

We can now introduce the point transformation of fields which relates the R and U gauges. To this end let us denote the R-gauge fields by U, ψ, and ϕ and those in the U gauge by $U_\nu+(1/\mu)\theta_{,\nu}$, η, and σ. Then the Lagrangians are connected by the point-equivalence result

$$L_R\left(U,\psi,\phi+\frac{m}{g}\right)=L_U\left(U+\frac{1}{\mu}\theta,\eta,\sigma+\frac{m}{g}\right), \tag{7}$$

where

$$\begin{aligned}&\psi(x)=e^{i(f/\mu)\gamma_5\theta(x)}\eta(x)\,,\\&\phi(x)+\frac{m}{g}=e^{2i(f/\mu)\theta(x)}\left[\sigma(x)+\frac{m}{g}\right],\end{aligned} \tag{8}$$

and the Stückleberg form of V_ν is used.

It is necessary to take account of the fact that the gauge function is a quantized field. We note from the point transformation that the fields cannot all consistently have zero vacuum expectation values, as required of second-quantized fields. To correct for this it was proposed in I to modify the point transformation so that the fields may consistently have zero vacuum expectation values. This is achieved by considering in place of Eqs. (8)

$$\psi(x) = e^{i(f/\mu)\gamma_5\theta(x)}\eta(x)\,,$$
$$\phi(x) + \frac{m}{g} = e^{2i(f/\mu)\theta(x)}\left[\sigma(x) + \frac{\tilde{m}}{g}\right], \tag{9}$$

where

$$\tilde{m}\left\langle \cos\frac{2f}{\mu}\right\rangle = m\,. \tag{10}$$

We see that this modification is equivalent to a mass renormalization in the unitary gauge only, and for this reason it is called an anomalous mass renormalization. This renormalization plays an important role in the calculations of this paper, as well as those considered previously.

Combining all these points together we can write the Lagrangian as

$$L(V,\psi,\pi) = -\tfrac{1}{4}[F_{\mu\nu}(V)]^2 + \bar{\psi}(i\not{\partial} - M - f\not{V}\gamma_5)\psi - G\bar{\psi}(\phi_1 - i\gamma_5\phi_2)\psi$$
$$-\tfrac{1}{2}|\partial_\nu\pi + 2ifV_\nu\pi|^2 + \tfrac{1}{2}\kappa_1\left(\frac{g}{2m}\right)^2|\pi|^4 + \tfrac{1}{2}\kappa_2|\pi|^2 - \frac{\delta\mu}{2}U^2\,, \tag{11}$$

with

$$G = g\left(1 - \frac{\delta m}{m}\right),$$
$$\kappa_1 = -\kappa^2 + \delta k_1 - \delta k_2\,,$$
$$\kappa_2 = \tfrac{1}{2}\kappa^2 - \tfrac{1}{2}\delta k_1 + \tfrac{3}{2}\delta k_2\,,$$

and in the U gauge,

$$\pi = \phi_1 + \frac{\tilde{m}}{g}\,, \tag{12}$$

while in the R gauge,

$$\partial^\nu V_\nu = 0\,, \quad \pi = \phi_1 + \frac{m}{g} + i\phi_2\,. \tag{13}$$

These equations define the model.

III. RENORMALIZATION CONSTANTS IN THE ABELIAN MODEL

Let us now turn to the problem which is at the heart of this paper, namely the comparison of the renormalization constants in the two gauges. We will show that, in general, the values of the wave-function and vertex renormalizations are different when calculated in the R and U gauges, but that they combine to form gauge-independent coupling-constant renormalizations.

In the derivation of this result we do not evaluate any Feynman integrals; in other words, we do not explicitly compute the various renormalization constants in the different gauges and deduce the result. Rather we approach the problem from the opposite point of view: We try to prove the result without having to compute the constants, and as such we use the graphical techniques introduced in I and II. In the spirit of this approach we examine the difference between the R and U gauge values of the renormalization constant (or S-matrix element, for that matter) under consideration. We consider the Feynman diagrams (and the corresponding Feynman integrals) which contribute, in either or both of the gauges, and focus on the contributions of each individual diagram to the overall difference We reexpress and/or expand each term into a convenient form in which the contributing factors can be reinterpreted as diagrams (we refer to this as a Δ expansion later). For this purpose it was necessary in I to introduce auxiliary vertices; we shall not have to extend that list. When we combine the Δ expansions of the various diagrams we see that their sum is zero—thus giving us the required result.

As we do not need to evaluate explicitly any divergent integral, we do not need to use a regularization prescription explicitly. However, to follow the procedure outlined above we need to make use of the invariance of the Feynman integrals under translations in momentum space—this corresponds to a relabeling of the momenta in any given diagram. Any form of regularization which preserves this symmetry property will suffice, and we merely note that examples are known to exist.[8]

The Feynman rules which are needed for these calculations are easily derived from the interaction Hamiltonian (1). In deriving the interaction Hamiltonian from the Lagrangian (11), in the unitary gauge, we must take note to include the Lee-Yang (or Faddeev-Popov) logarithm term.[9] In Fig. 1 we list those rules we need, as well as the auxiliary vertices we use. We have normalized $g/2m$ to unity, for ease of calculation.

The couplings we consider to demonstrate the result are the physical trilinear couplings, i.e., the trilinear couplings which occur in the U-gauge Lagrangian, namely the $\bar{\psi}\psi\sigma$, $\bar{\psi}\psi V$, $V^2\sigma$, and σ^3 cou-

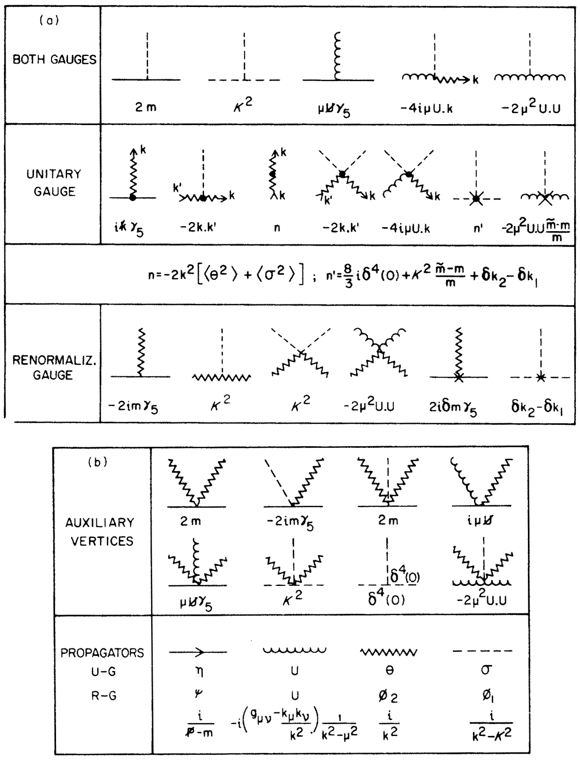

FIG. 1. The Feynman rules for vertices and propagators in both unitary and renormalization gauges, calculated from the interaction Lagrangian, and the auxiliary vertices which are used in the calculations.

plings. To examine the renormalization constants, we look at all the one-loop corrections to these vertices, with all external momenta on the mass shell, after the self-energy contributions have been subtracted. Since we are going to consider the *difference* between U- and R-gauge contributions, we need only consider those diagrams which are manifestly different in the two gauges. By separating the one-particle-reducible and one-particle-irreducible parts, we can glean information about the vertex and wave-function renormalization constants at the same time as studying the charge renormalizations.

The calculations for the $\bar{\psi}\psi\sigma$ and $\bar{\psi}\psi V$ couplings are actually already included in paper I. However, it is not obvious from the treatment given there that the result follows; by regrouping the relevant diagrams, and putting the external momenta on the mass shell we see that the result follows immediately. This is done in Fig. 2, and one can see by inspection how the cancellations occur. We see that the wave-function renormalization constants $Z^{\sigma\sigma}$ and $Z^{\psi\psi}$ are gauge dependent, whereas Z^{VV} is gauge independent, and that both vertex constants $Z^{\psi\psi\sigma}$ and $Z^{\psi\psi V}$ are also gauge dependent. By combining, we see at once that the anomalous terms, which denoted gauge dependence, cancel to give us our result for these two cases.

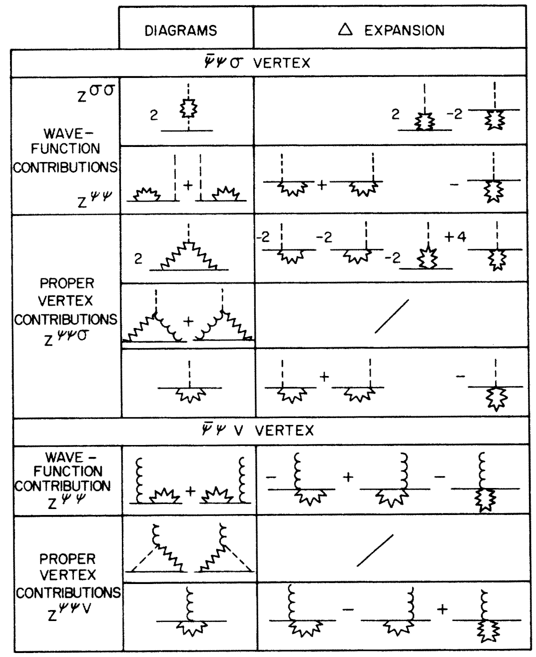

FIG. 2. Expansion of the diagrams contributing to the renormalization of the $\bar{\psi}\psi\sigma$ and $\bar{\psi}\psi V$ couplings, showing explicitly that the renormalization, in each case, is independent of the choice of gauge.

The $V^2\sigma$ and σ^3 vertices were not treated in I, as there it was a fermion-fermion scattering process which was considered and the corrections to these vertices would not contribute until higher orders were considered. The diagram summary is given in Figs. 3 and 4. Again we see explicitly that the

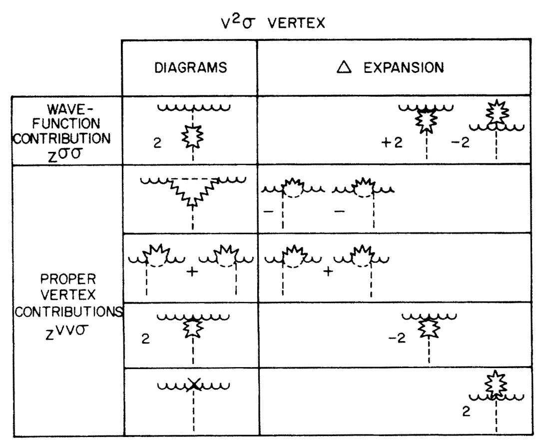

FIG. 3. Expansion of the diagrams contributing to the renormalization of the V_σ^2 couplings, showing explicitly that the renormalization, in each case, is independent of the choice of gauge.

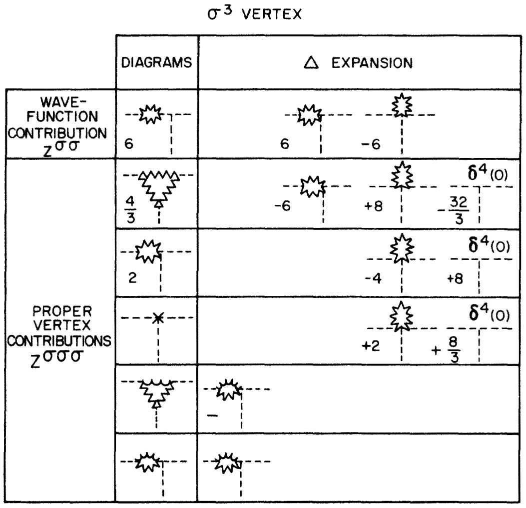

FIG. 4. Expansion of the diagrams contributing to the renormalization of the σ^3 couplings, showing explicitly that the renormalization, in each case, is independent of the choice of gauge.

wave-function and vertex renormalization contributions are gauge dependent, but in the renormalization of the charge cancellations occur to yield a gauge-independent result. We notice in Fig. 3 the contribution of the anomalous mass renormalization to the cancellation, in the final diagram, while in Fig. 4, for the σ^3 vertex, besides the anomalous mass renormalization term, we see also the contribution from the Lee-Yang $\delta^4(0)$ logarithm term in canceling divergences in the unitary gauge.

ACKNOWLEDGMENTS

I would like to express my gratitude to the Dublin Institute for Advanced Studies, where much of this work was carried out, for their hospitality and support. I would also like to thank Professor Lochlainn O'Raifeartaigh and Dr. G. B. Mainland for help and encouragement, and Dr. D. A. Ross for a helpful discussion.

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*Work supported in part by the U. S. Energy Research and Development Administration, under Contract No. E(40-1)3992.

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