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Gauge-invariant theory of direct interaction between strings

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A manifestly gauge-invariant theory of direct interaction between strings is presented. The equations of motion found have no constraints in the type of interaction. The field theory associated with a large class of direct interactions is presented. We also discuss a "two-body" type of interaction between particles and strings.

I. INTRODUCTION

The purpose of this paper is to cast the theory of direct interaction between strings[1] in a manifestly gauge-invariant ("general-covariant") and Poincaré-covariant form. The main reason for carrying out this program is that the "general-covariant" approach[2] provides us with the most natural language to describe surfaces ("world sheets"). Thus, this approach will deepen our insight into the structure of the string theory. It is also important to know which objects are gauge invariant because these objects represent the "observables" of the theory. The requirement of gauge invariance will give us a clue to find the equations of motion for interacting strings without constraints on the types of interaction. The above-mentioned requirement leads us to the interesting result that the strings' end points are not slowed down by interactions, i.e., they move with the speed of light.

In Sec. II we present a summary of the "gauge-invariant" theory of direct interactions between particles.[3,4] We also discuss the class of field theories associated with a class of direct interactions between particles.[5]

In Sec. III we develop the gauge-invariant theory of direct interstring interaction following the lines given in Sec. II.

In Sec. IV we present the class of associated field theories that can be extracted from a class of direct interactions between strings.

In Sec. V we study a "two-body" type of interaction between particles and strings. We conclude by giving as an example the equations of motions that govern the gravitational interaction between a particle and an open string.

II. DIRECT INTERACTION BETWEEN PARTICLES

In this section we will study the "gauge-invariant" and Poincaré-covariant formalism that describes action-at-a-distance interactions between particles.[3,4] In the present section, "gauge invariance" means invariance with respect to a change in the parameter that describes the world line of each particle.

The metric of the space-time is the Minkowski metric

$$\eta_{\mu\nu}=0, \quad \text{if } \mu\neq\nu \tag{2.1}$$
$$\eta_{00}=-\eta_{11}=-\eta_{22}=-\eta_{33}=1.$$

The world line of the jth particle of proper mass m_j will be denoted by $z_j^\mu(t_j)$. t_j is a parameter that describes the world line; it may or may not be identical to the proper time s_j. Moreover, we do not assume that t_j has any particular property of invariance.

The action A of a system of N particles interacting through two-body forces is[3]

$$A=I_0+I_{\rm int}, \tag{2.2a}$$

where

$$I_0\equiv\sum_i \int m_i v_i dt_i, \tag{2.2b}$$

$$I_{\rm int}\equiv\sum_{i<j}\sum \int dt_i dt_j v_i v_j R_{ij}, \tag{2.2c}$$

$$R_{ij}=R_{ij}(v_i^\mu/v_i,\ v_j^\mu/v_j,\ z_i^\mu,\ z_j^\mu), \tag{2.2d}$$

$$v_i^\mu\equiv\frac{dz_i^\mu}{dt_i}, \quad v_i=(v_i^\mu v_{i\mu})^{1/2}. \tag{2.3}$$

The interaction has been restricted to be dependent only on the first derivatives of z_i^μ in order that we may end up with a second-order differential equation for z_i^μ. Note that the quantities v_i^μ/v_i, z_i^μ, and $v_i dt_i$ are invariants under the transformation

$$t_i\to t_i'=t_i'(t_i). \tag{2.4}$$

Thus A is gauge invariant and will be Poincaré invariant if its dependence on the indicated variables is through Poincaré invariants[5] formed with v_i^μ/v_i, v_j^μ/v_j, z_i^μ, and z_j^μ.

From the variational principle we obtain the equation of motion

$$\frac{m_i}{v_i}\frac{d}{dt_i}\frac{v_i^\mu}{v_i}=\tilde{\Lambda}_i^\mu \tilde{V}_i, \tag{2.5a}$$

where

$$\tilde{\Lambda}_i^\mu=\frac{\partial}{\partial z_{i\mu}}-\frac{1}{v_i}\frac{d}{dt_i}\left\{\frac{\partial}{\partial(v_{i\mu}/v_i)}+\frac{v_i^\mu}{v_i}\left[1-\frac{v_{i\rho}\partial}{v_i\partial(v_{i\rho}/v_i)}\right]\right\}, \tag{2.5b}$$

$$\tilde{V}_i=\sum_{j<i}\int dt_j v_j R_{ji}+\sum_{j>i}\int dt_j v_j R_{ij}. \tag{2.5c}$$

The multiplication of (2.5a) by v_i^μ gives $0=0$, i.e., we do not need further restrictions on R_{ij} to obtain a Minkowski force perpendicular to the velocity.

Usually one parametrizes the world line of each particle with its proper time s_i. In this case

$$u_i^\mu(s_i)\equiv v_i^\mu(t_i=s_i),\quad v_i(t_i=s_i)=1, \tag{2.6}$$

and Eqs. (2.5) take the simple form

$$m_i\frac{du_i^\mu}{ds_i}=\Lambda_i^\mu V_i, \tag{2.7a}$$

where

$$\Lambda_i^\mu=\frac{\partial}{\partial z_{i\mu}}-\frac{d}{ds_i}\left[\frac{\partial}{\partial u_{i\mu}}+u_i^\mu\left(1-u_{i\rho}\frac{\partial}{\partial u_{i\rho}}\right)\right], \tag{2.7b}$$

$$V_i=\sum_{j<i}\int ds_j R_{ji}+\sum_{j>i}\int ds_j R_{ij}. \tag{2.7c}$$

Equations (2.7) can also be obtained from a non-manifestly gauge-invariant action if instead of taking m_i to be the proper mass we regard m_i as a "mass" depending on the parameter s_i. In this case a multiplication of the equation of motion by u_i^μ will give us a differential equation for m_i. This differential equation can be easily solved, and the constant of integration corresponds to the proper mass.[3] In addition, the differential equations that $m_i(s_i)$ obeys can be found by demanding that the corresponding action be invariant under the transformation $s_i\to s_i+\delta s_i$.

If one takes a non-manifestly gauge-invariant action with a constant mass, the multiplication of the equation of motion by u_i^μ or the requirement of invariance of the action under $s_i\to s_i+\delta s_i$ will give us a constraint[1,6] on R_{ij}.

Now let us take the particular case of interaction given by

$$R_{ij}=q_iq_j(u_i^\mu u_{j\mu})^n G_j((z_i-z_j)^\mu(z_i-z_j)_\mu), \tag{2.8}$$

where n is an integer, and q_i and q_j are coupling constants. G_j is a Green's function that satisfies the second-order linear partial-differential equation

$$\mathfrak{D}G_j((x-z_j)_\mu(x-z_j)^\mu)=4\pi\delta^4(x^\rho-z_j^\rho). \tag{2.9}$$

When R_{ij} is given by (2.8), we can easily extract the field theory associated with this interaction as follows: From (2.7c) and (2.8) we have

$$V_i=q_iu_{i\alpha}u_{i\beta}\cdots u_{i\nu}\varphi_i^{\alpha\beta\cdots\nu}, \tag{2.10}$$

where

$$\varphi_i^{\alpha\beta\cdots\nu}(x_i)=\sum_{k\neq i}q_k\int u_k^\alpha u_k^\beta\cdots u_k^\nu \times G_k((x_i-x_k)^2)ds_k. \tag{2.11}$$

In this case we can regard the equations (2.7a), (2.7b), and (2.10) as the description of the motion of a particle in the presence of a field $\varphi_i^{\alpha\beta\cdots\nu}(x)$ created by the other particles, which obeys the field equation

$$\mathfrak{D}\varphi_i^{\alpha\beta\cdots\nu}(x)=4\pi J_i^{\alpha\beta\cdots\nu}(x), \tag{2.12}$$

$$J_i^{\alpha\beta\cdots\nu}(x)\equiv\sum_{j\neq i}q_j\int u_j^\alpha u_j^\beta\cdots u_j^\nu \times\delta^4(x^\rho-z_j^\rho)ds_j. \tag{2.13}$$

Special cases of theories of direct particle interaction with associated field theories are the Fokker principle of electrodynamics[7] ($n=1$, $G_j=\delta$) and the principles of scalar and vector mesodynamics[8] ($n=0$ or 1). The expression (2.8) is not the most general interaction that has an associated field theory. For a discussion of this point see Ref. 4.

III. DIRECT INTERACTION BETWEEN STRINGS

The motion of a string sweeps a two-dimensional curved timelike surface ("world sheet"), embedded in the physical Minkowski space with metric (2.1). We represent this surface as $x^\mu=x^\mu(\tau^A)$, where $\tau^A=(\tau^0,\tau^1)$. τ^0 is related to the timelike and τ^1 to the spacelike extensions of the world sheet.

In this section we will study the action-at-a-distance formalism for interaction between strings, stressing the gauge invariance and Poincaré covariance of the theory. By gauge invariance we mean invariance under the transformation

$$\tau^A\to\tau^{A'}=\tau^{A'}(\tau^B),\quad J\equiv\frac{\partial(\tau^{0'},\tau^{1'})}{\partial(\tau^0,\tau^1)}\neq 0. \tag{3.1a}$$

The range of the parameter τ^0 is $-\infty<\tau^0<+\infty$. The range of τ^1 is $\alpha\leq\tau^1<\beta$ for a closed string and $\overline{\alpha}\leq\tau^1\leq\overline{\beta}$ for an open string. For closed strings we will have $x^\mu(\tau^0,\alpha)=x^\mu(\tau^0,\beta)$. In the case of open strings, $\overline{\alpha}$ and $\overline{\beta}$ will denote the end points, and in this case we will require that

$$\frac{\partial \tau^{1'}}{\partial \tau^0}(\tau^A)\Big|_{\tau^1=\bar{\alpha},\bar{\beta}}=0, \qquad (3.1b)$$

in order that under the transformation (3.1a) the boundaries of the world sheet remain the boundaries.[2] Also (3.1) must satisfy some "physical conditions," e.g. (a) that $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ transform into finite quantities and $-\infty<\tau^0<+\infty \rightarrow -\infty<\tau^{0'}<+\infty$ and (b) that the timelike and the spacelike character of τ^0 and τ^1 be preserved, except at the end points.

In general we will not require that τ^0 and τ^1 be Lorentz-invariant quantities. The general covariance (invariance) under the transformation (3.1) will be referred to as g covariance (g invariance).

To obtain a g-invariant action analogous to (2.2) we must start studying the g-invariant quantities that can be formed with "geometrical objects" associated with one string. We will limit ourselves to objects that contain at most first derivatives in x^μ, because we want an action that gives us second-order differential equations.

The internal metric tensor of the string world sheet is

$$\gamma_{AB}=\eta_{\mu\nu}x^\mu_A x^\nu_B, \qquad (3.2)$$

$$x^\mu_A \equiv \frac{\partial x^\mu}{\partial \tau^A}. \qquad (3.3)$$

We also have the second-rank contravariant tensor density ϵ^{AB} with components

$$\epsilon^{01}=-\epsilon^{10}=1, \quad \epsilon^{00}=\epsilon^{11}=0. \qquad (3.4)$$

From (3.2)–(3.4) we can build the g-invariant quantities[2]

$$\sqrt{-\gamma}\,d^2\tau, \qquad (3.5)$$

$$T^{\mu\nu}\equiv\gamma^{AB}x^\mu_A x^\nu_B, \qquad (3.6)$$

$$S^{\mu\nu}\equiv\frac{\epsilon^{AB}}{\sqrt{-\gamma}}x^\mu_A x^\nu_B, \qquad (3.7)$$

where

$$\gamma\equiv\det\gamma_{AB}, \quad d^2\tau\equiv d\tau^0 d\tau^1. \qquad (3.8)$$

Note that (3.5) is a Poincaré scalar and (3.6) and (3.7) are Poincaré tensors even if the parameters τ^A are not Lorentz invariants. We will not consider in the action objects formed with the Riemann-Christoffel tensor formed with (3.2) because they produce in the equations of motion terms containing at least third derivatives of x^μ. It is important to realize that quantities formed with objects belonging to two different strings are in general not g invariants, e.g., $(-\gamma_a)^{-1/2}\epsilon^{AB}x^\mu_{aA}x^\nu_{bB}$, because each string in general will be parametrized in a different way. The tensor $T^{\mu\nu}$ is the energy-momentum tensor[2] and $S^{\mu\nu}$ spans the world sheet. From (3.6) and (3.7) we get the useful identities[2]

$$T^{\mu\nu}T_\nu{}^\lambda=T^{\mu\lambda}, \qquad (3.9a)$$

$$S^{\mu\nu}S_\nu{}^\lambda=T^{\mu\lambda}, \qquad (3.9b)$$

$$S^{\mu\nu}T_\nu{}^\lambda=S^{\mu\lambda}, \qquad (3.9c)$$

$$S^{\mu\nu}S_{\mu\nu}=T^{\mu\nu}T_{\mu\nu}=2. \qquad (3.9d)$$

Now the analog of (2.2) for strings is

$$\mathcal{A}=I^0+I^{\text{int}}, \qquad (3.10a)$$

where

$$I^0\equiv\sum_a\int M_a(-\gamma_a)^{1/2}d^2\tau_a, \qquad (3.10b)$$

$$I^{\text{int}}\equiv\sum_{a<b}\sum\int d^2\tau_a d^2\tau_b(-\gamma_a)^{1/2}(-\gamma_b)^{1/2}R_{ab}, \qquad (3.10c)$$

$$R_{ab}=R_{ab}(T^{\mu\nu}_a, T^{\mu\nu}_b, S^{\mu\nu}_a, S^{\mu\nu}_b, x^\nu_a, x^\nu_b). \qquad (3.10d)$$

The indices a and b run from 1 to S, S being the number of strings. M_a is a set of constants. Note that despite the identity (3.9b) we have included $T^{\mu\nu}$ in the dependence of (3.10d). The reason is that $T^{\mu\nu}$ has a clear physical meaning and when particular cases are studied it is easier to handle only one symbol than $S^{\mu\lambda}S_\lambda{}^\nu$.

The action (3.10) is manifestly g invariant and it will be Poincaré invariant if the dependence of R_{ab} on the indicated variables is through Poincaré-invariant quantities[9] built with $T^{\mu\nu}_a$, $T^{\mu\nu}_b$, $S^{\mu\nu}_a$, $S^{\mu\nu}_b$, x^μ_a, and x^μ_b.

The equations of motion of the strings are obtained by demanding that the action $\mathcal{A}$ be stationary under the variation

$$x^\mu_a(\tau^A_a)\rightarrow x^\mu_a(\tau^A_a)+\delta x^\mu_a(\tau^A_a), \qquad (3.11a)$$

$$\delta x^\mu_a(\tau_a=\pm\infty, \tau^1_a)=0, \qquad (3.11b)$$

$$\delta x^\mu_a(\tau^0_a, \tau^1_a=\alpha_a)=\delta x^\mu_a(\tau^0_a, \tau^1_a=\beta_a), \qquad (3.11c)$$

$$\delta x^\mu_a(\tau^0_a, \tau^1_a=\bar{\alpha}_a, \bar{\beta}_a)\neq 0. \qquad (3.11d)$$

The variations $\delta x^\mu(\tau^0_a, \tau^1_a=\bar{\alpha}_a)$, $\delta x^\mu(\tau^0_a, \tau^1_a=\bar{\beta}_a)$, and $\delta x^\mu(\tau^A_a)$ in general are independent. Furthermore we will assume that the closed strings are smooth, i.e.,

$$\frac{\partial x^\mu_a}{\partial \tau^1_a}\Big|_{\tau^1_a=\alpha_a}=\frac{\partial x^\mu_a}{\partial \tau^1_a}\Big|_{\tau^1_a=\beta_a}. \qquad (3.12)$$

From (3.10), (3.11c), and (3.12) we find

$$\delta I^0 = \sum_a \int d^2\tau_a M_a \left\{ \frac{\partial}{\partial \tau_a^A}[(-\gamma_a)^{1/2} x^A_{a\mu}\delta x_a^\mu] - (\Box_a{}^2 x_{a\mu})\delta x_a^\mu \right\}, \tag{3.13}$$

$$\delta I^{\rm int} = \sum_{a<b}\sum \int d^2\tau_a d^2\tau_b (-\gamma_b)^{1/2}\left(\frac{\partial}{\partial x_a^\mu} - \frac{\partial}{\partial \tau_a^A}\frac{\partial}{\partial x_{aA}}\right)[(-\gamma_a)^{1/2}R_{ab}]$$

$$+ \sum_{a<b}{}'\sum \int d^2\tau_b d\tau_a^0 \left\{\frac{\partial}{\partial x^\mu_{a1}}[(-\gamma_a)^{1/2}R_{ab}]\right\}_{\tau_a^1=\overline{\alpha}_a}^{\tau_a^1=\overline{\beta}_a}, \tag{3.14}$$

where $\sum'$ denotes summation only over open strings and

$$\Box_a{}^2 x_a^\mu \equiv (-\gamma_a)^{1/2}\nabla_A^a \nabla^{aA} x_a^\mu$$
$$= \frac{\partial}{\partial \tau_a^A}[(-\gamma_a)^{1/2}\gamma^{AB} x^\mu_{aB}]. \tag{3.15}$$

Thus the variational principle, Eqs. (3.11), (3.13), and (3.14), give us the equations of motion

$$M_a \Box_a{}^2 x_{a\mu} = \left(\frac{\partial}{\partial x_a^\mu} - \frac{\partial}{\partial \tau_a^A}\frac{\partial}{\partial x^\mu_{aA}}\right)[(-\gamma_a)^{1/2}\phi_a], \tag{3.16a}$$

$$\left\{ M_a(-\gamma_a)^{1/2}x^1_{a\mu} + \frac{\partial}{\partial x^\mu_{a1}}[(-\gamma_a)^{1/2}\phi_a]\right\}_{\tau_a^1=\overline{\alpha}_a,\overline{\beta}_a} = 0, \tag{3.16b}$$

where

$$\phi_a = \sum_{b<a}\int d^2\tau_b(-\gamma_b)^{1/2}R_{ba}$$
$$+ \sum_{b>a}\int d^2\tau_b(-\gamma_b)^{1/2}R_{ab}. \tag{3.16c}$$

Equation (3.16b) applies only to open strings and tells us that the end points of the string have their proper interactions. Equations (3.16) can also be obtained by assuming instead of (3.11b)–(3.11d) the relations

$$\delta x_a^\mu(\tau_a^0 = \pm\infty, \tau_a^1) = 0, \tag{3.17a}$$

$$\delta x_a^\mu(\tau_a^0, \tau^1 = \alpha_a, \overline{\alpha}_a, \beta_a, \overline{\beta}_a) = 0 \tag{3.17b}$$

and demanding that the part of the "canonical" momentum[10] that crosses the edges of the world sheet be zero at the edges.

Now we will use the fact that R_{ab} is given by (3.10d) to cast (3.16) in a form similar to (2.5). From (3.6), (3.7), and (3.9) we get

$$\frac{\partial S^{\alpha\beta}}{\partial x_E^\mu} = -S^{\alpha\beta}x^E_\mu + 2(-\gamma)^{-1/2}\epsilon^{EA}\delta_\mu^{[\alpha}x_A^{\beta]}, \tag{3.18}$$

$$\frac{1}{2}\frac{\partial T^{\alpha\beta}}{\partial x_E^\mu} = -T^{\alpha\beta}x^E_\mu + x^{E(\alpha}\delta_\mu^{\beta)} + (-\gamma)^{-1/2}\epsilon^{EA}x_A^{(\alpha}S_\mu^{\beta)}, \tag{3.19}$$

where

$$a^{[\mu}b^{\nu]} \equiv \tfrac{1}{2}(a^\mu b^\nu - a^\nu b^\mu),$$
$$a^{(\mu}b^{\nu)} \equiv \tfrac{1}{2}(a^\mu b^\nu + a^\nu b^\mu). \tag{3.20}$$

The identities (3.18) and (3.19) can easily be obtained if one uses the fact that in a two-dimensional Riemannian space we have[2]

$$\gamma = \tfrac{1}{2}\epsilon^{AC}\epsilon^{BD}\gamma_{AB}\gamma_{CD}, \tag{3.21}$$

$$\gamma^{AB} = \gamma^{-1}\epsilon^{AC}\epsilon^{BD}\gamma_{CD}. \tag{3.22}$$

Hence from (3.16), (3.10d), (3.18), and (3.19) we obtain

$$M_a \Box_a{}^2 x_{a\mu} = \mathfrak{L}_{a\mu}\phi_a, \tag{3.23a}$$

$$[M_a(-\gamma_a)^{1/2}x^1_{a\mu} + \mathfrak{B}^1_{a\mu}\phi_a]_{\tau_a^1=\overline{\alpha}_a,\overline{\beta}_a} = 0, \tag{3.23b}$$

where

$$\mathfrak{L}_{a\mu} \equiv (-\gamma_a)^{1/2}\frac{\partial}{\partial x_a^\mu} - \frac{\partial}{\partial \tau_a^A}\left[(-\gamma_a)^{1/2}x^A_{a\mu}\left(1 - S_a^{\alpha\beta}\frac{\partial}{\partial S_a^{\alpha\beta}} - 2T_a^{\alpha\beta}\frac{\partial}{\partial T_a^{\alpha\beta}}\right) + 2(-\gamma_a)^{1/2}x_a^{A\beta}\frac{\partial}{\partial T_a^{\beta\mu}}\right.$$
$$\left. + 2\epsilon^{AB}x^\alpha_{aB}\left(\frac{\partial}{\partial S_a^{\mu\alpha}} + S^\beta_{a\mu}\frac{\partial}{\partial T_a^{\alpha\beta}}\right)\right], \tag{3.23c}$$

$$\mathfrak{B}^E_{a\mu} \equiv (-\gamma_a)^{1/2}x^E_{a\mu}\left(1 - S_a^{\alpha\beta}\frac{\partial}{\partial S_a^{\alpha\beta}} - 2T_a^{\alpha\beta}\frac{\partial}{\partial T_a^{\alpha\beta}}\right) + 2(-\gamma_a)^{1/2}x_a^{E\beta}\frac{\partial}{\partial T_a^{\alpha\beta}} + 2\epsilon^{EA}x^\alpha_{aA}\left(\frac{\partial}{\partial S_a^{\mu\alpha}} + S^\beta_{a\mu}\frac{\partial}{\partial T_a^{\alpha\beta}}\right). \tag{3.23d}$$

The equations of motion (3.23) are manifestly invariant under the gauge transformation (3.1) and they will be Poincaré covariant if ϕ_a is a Poincaré scalar.

The multiplication of Eq. (3.23a) by x_{aC} gives the identity $0=0$ (see Appendix A). The multiplication of (3.23b) by x_{aC} gives

$$\gamma_a(\tau_a^0, \tau_a^1 = \overline{\alpha}_a, \overline{\beta}_a) = 0 \tag{3.24}$$

(see Appendix B). Thus we have that Eqs. (3.23) are valid for all R_{ab} of the form (3.10d).

From Eq. (3.24) one concludes that the end points of open strings are not slowed down by interactions, i.e., they move with the speed of light (see Appendix B). Also (3.24) tells us that Eqs. (3.16b) and (3.23b) do not have a clear meaning as they are written. We must understand them as limits (see Appendix B).

An equivalent set of equations to Eqs. (3.23) can be obtained by setting $M_a = M_a(\tau_a^A)$ in (3.10b) and replacing

$$R_{ab} \rightarrow (\gamma_a\gamma_b)^{-1/2} R'_{ab}((-\gamma_a)^{1/2} S_a^{\mu\nu}, (-\gamma_b)^{1/2} S_b^{\mu\nu}, x_a^\mu, x_b^\mu) \tag{3.25}$$

in (3.10c). The equation of motion obtained in this case, when multiplied by x_{aC}, gives a differential equation for $M_a(\tau_a^A)$ that can be easily solved; the constant of integration corresponds to the constant that appears in Eqs. (3.23). Also the differential equation that $M_a(\tau_a^A)$ obeys can be found by demanding that the corresponding action be invariant under the transformation $\tau_a^A \rightarrow \tau_a^A + \delta\tau_a^A$. If one uses a constant M_a and a noninvariant action, the multiplication of the corresponding equation of motion by x_{aC} gives a constraint on the interaction.[1] In summary, each different possibility to derive the equations of motion of particles that we discussed in Sec. II has an analog in the string case.

IV. THE FIELD THEORY ASSOCIATED WITH A GIVEN INTERACTION

As we have seen in Sec. II, to extract the field theory associated with a given interaction between particles we must restrict the type of interaction. The situation in the string case will be richer, because we can form a larger class of interactions that have associated field theories. To ensure Poincaré invariance of the theory the dependence of R_{ab} on its arguments must be through Poincaré scalars. Examples of Poincaré scalars constructed with objects belonging to two different strings are

$$(x_a - x_b)^2 \equiv (x_a^\mu - x_b^\mu)(x_{a\mu} - x_{b\mu}), \tag{4.1}$$

$$S_a^{\mu\nu} S_{b\mu\nu}, \tag{4.2}$$

$$\epsilon^{\mu\nu\lambda\sigma} S_{a\mu\nu} S_{b\lambda\sigma}, \tag{4.3}$$

$$T_a{}^{\mu\nu} T_{b\mu\nu}, \tag{4.4}$$

$$S_a{}^{\mu\nu} T_a{}^{\lambda\sigma} T_{b\nu\lambda} S_{b\sigma\mu}, \tag{4.5}$$

$$S_a{}^{\mu\nu} T_{b\nu\lambda} S_a{}^{\lambda\rho} T_{b\rho\mu}, \tag{4.6}$$

$$T_a{}^{\mu\nu} (x_a - x_b)_\mu (x_a - x_b)_\nu. \tag{4.7}$$

The scalars (4.1)–(4.5) are symmetric under the interchange of a and b. Obviously the relations (4.5) and (4.6) can be generalized to an arbitrary number of "elements."

Now we will consider the particular interaction formed with powers of Poincaré scalar quantities similar to (4.2)–(4.6) and its generalizations, i.e.,

$$R_{ab} = q_a q_b \left(\overset{n}{\underset{1}{\otimes}} S_a \otimes \overset{l}{\underset{1}{\otimes}} T_a\right) * \left(\overset{n'}{\underset{1}{\otimes}} S_b \otimes \overset{l'}{\underset{1}{\otimes}} T_b\right) G((x_a - x_b)^2), \tag{4.8}$$

where q_a and q_b are coupling constants,

$$\overset{n}{\underset{1}{\otimes}} A = A \otimes A \otimes \cdots \otimes A \quad (n \text{ factors}), \tag{4.9}$$

and the asterisk denotes the "total scalar product" of the tensors formed by the tensorial products indicated. G is the Green's function that satisfies Eq. (2.9). Note that n, l, n', and l' by construction are such that it is always possible to perform the "total contraction" $*$. And it can be done in several ways, as (4.2)–(4.6) indicate, which give rise to different interactions.

Now the equations of motion (3.23) read

$$M_a \square_a^2 x_{a\mu} = q_a \underset{a}{\mathfrak{L}}{}_\mu\left[\left(\overset{n}{\underset{1}{\otimes}} S_a \otimes \overset{l}{\underset{1}{\otimes}} T_a\right) * \psi_a(x_a)\right], \tag{4.10a}$$

$$\left\{M_a(-\gamma_a)^{1/2} x_{a\mu}^1 + q_a \mathfrak{B}_{a\mu}^1\left[\left(\overset{n}{\underset{1}{\otimes}} S_a \otimes \overset{l}{\underset{1}{\otimes}} T_a\right) * \psi_a(x_a)\right]\right\}_{\tau_a^1 \rightarrow \overline{\alpha}_a, \overline{\beta}_a} = 0, \tag{4.10b}$$

where

$$\psi_a(x_a) \equiv \sum_{b \neq a} q_b \int d^2\tau_b (-\gamma_a)^{1/2} \overset{n'}{\underset{1}{\otimes}} S_b \otimes \overset{l'}{\underset{1}{\otimes}} T_b \times G((x_a - x_b)^2). \tag{4.11}$$

Because G satisfies (2.9) we get the field equation

$$\mathfrak{D}\psi_a(x) = 4\pi J_a(x), \tag{4.12a}$$

where

$$J_a(x) \equiv \sum_{b \neq a} q_b \int d^2\tau_b (-\gamma_b)^{1/2} \overset{n'}{\underset{1}{\otimes}} S_b \otimes \overset{l'}{\underset{1}{\otimes}} T_b \delta^4(x^\mu - x_b^\mu). \tag{4.12b}$$

We can interpret the set of equations (4.10) as the equations of motion of a string that interacts with the tensorial field ψ_a produced by the other strings and obeying the field equation (4.12). The study of the field theory associated with the interaction (4.2) can be found in Ref. 1.

Also, associated field theories can be extracted from interactions that are formed with terms like powers of (4.7), but in this case we will not have a single field equation like (4.12); instead we will have to look for the field equation case by case.

V. DIRECT INTERACTION BETWEEN PARTICLES AND STRINGS

Let us consider the action

$$\mathfrak{A}^{\text{tot}} = I_0 + I_{\text{int}} + I^0 + I^{\text{int}} + I^{\text{int}}_{\text{int}}, \tag{5.1}$$

where I_0, I_{int}, I^0, and I^{int} are given by (2.2b), (2.2c), (3.10b), and (3.10c), respectively, and

$$I^{\text{int}}_{\text{int}} \equiv \sum_a \sum_i \int dt_i d^2\tau_a v_i (-\gamma_a)^{1/2} R_{ai}, \tag{5.2a}$$

$$R_{ai} = R_{ai}(T_a{}^{\mu\nu}, S_a^{\mu\nu}, v_i^\mu / v_i, x_a^\mu, z_i^\mu). \tag{5.2b}$$

The indices a, b run from 1 to S, S being the number of strings. i, j run from 1 to N, N being the number of particles. The interaction (5.2) represents a "two-body" type of interaction between strings and particles, i.e., the simplest one that can be considered. The action (5.1) is g invariant and it will be Poincaré invariant if the dependence of R_{ij}, R_{ab}, and R_{ai} on their arguments is through Poincaré-invariant quantities.

When the variational principle is applied to (5.1) and the condition $v_i^\mu v_{i\mu} = 1$ is used, one finds

$$m_i \frac{du_i^\mu}{ds^i} = \Lambda_i^\mu (V_i + U_i), \tag{5.3}$$

$$M_a \Box_a{}^2 x_{a\mu} = \mathfrak{L}_{a\mu}(\phi_a + \chi_a), \tag{5.4}$$

$$[M_a(-\gamma_a)^{1/2} x^1_{a\mu} + \mathfrak{B}^1_{a\mu}(\phi_a + \chi_a)]_{\tau_a^1 \to \overline{\alpha}_a, \overline{\beta}_a} = 0, \tag{5.5}$$

where

$$U_i \equiv \sum_a \int d^2\tau_a (-\gamma_a)^{1/2} R_{ai}, \tag{5.6}$$

$$\chi_a \equiv \sum_i \int ds_i R_{ai}. \tag{5.7}$$

Equations (5.3)–(5.7) give a complete description of the interaction between particles and strings. Note that no further restrictions on R_{ij}, R_{ab}, and R_{ai} come from the equations of motion.

To have an acceptable physical description of the system of particles and strings, one must add some conditions on R_{ai}, e.g., that when the particles and strings are far apart the interaction be zero. This condition can be satisfied by requiring that the dependence of R_{ai} on x_a and z_i be through a Green's function that produces the wanted property, e.g., $\delta((x_a - z_i)^2)$. Also, the "separability condition" mentioned above must be imposed when we are dealing with direct interactions between either particles or strings. The separability condition in these cases can be implemented in a similar way.

As before, by considering particular cases of interactions between particles and strings one can extract field theories associated with a given interaction between particles and strings. As an example we will consider the case of one particle interacting with one open string when the interaction is given by

$$R = kT^{\mu\nu} u_\mu u_\nu \delta([z(s) - x(\tau^A)]^2), \tag{5.8}$$

where k is a constant.

From (5.3)–(5.7) we obtain

$$m \frac{du_\mu}{ds} = u^\alpha u^\beta \frac{\partial}{\partial x^\mu} H_{\alpha\beta} + \frac{d}{ds}(u_\mu u^\alpha u^\beta H_{\alpha\beta} - 2u^\beta H_{\beta\mu}), \tag{5.9}$$

$$M\Box^2 x_\mu = \sqrt{-\gamma}\, h^{\alpha\beta} \frac{\partial T_{\alpha\beta}}{\partial x^\mu} - \frac{\partial}{\partial \tau^A}(2\sqrt{-\gamma}\, x^{A\rho} h_{\rho\mu} - \sqrt{-\gamma}\, T_{\alpha\beta} h^{\alpha\beta} x^A_\mu + 2\epsilon^{AB} x^\alpha_B S^\beta_\mu h_{\alpha\beta}), \tag{5.10}$$

$$(M\sqrt{-\gamma}\, x^1_\mu + 2\sqrt{-\gamma}\, x^{1\beta} h_{\mu\beta} - \sqrt{-\gamma}\, T^{\alpha\beta} h_{\alpha\beta} x^1_\mu - 2x^\alpha_0 S^\beta_\mu h_{\alpha\beta})_{\tau^1 \to \overline{\alpha}, \overline{\beta}} = 0, \tag{5.11}$$

where

$$H_{\alpha\beta}(z) = k \int d^2\tau \sqrt{-\gamma}\, T_{\alpha\beta} \delta([x(\tau^A) - z]^2), \tag{5.12}$$

$$h_{\alpha\beta}(x) = k \int ds\, u_\alpha u_\beta \delta([z(s) - x]^2). \tag{5.13}$$

Note that (5.12) and (5.13) represent the gravitational field in the weak-field approximation of one string[11] and one particle,[12] respectively. Therefore the interaction (5.8) represents a gravitational interaction between one particle and one string. Also note that (5.9) is the well-known equation of motion of a particle in the presence of an external gravitational field.[13]

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APPENDIX A

In this appendix we will show that a multiplication of the equation of motion (3.23a) by x^μ_C does not give any constraint on the interaction. For simplicity we will drop the subscript a in this appendix.

From (3.21) and (3.2) one finds

$$\frac{\partial\gamma}{\partial x^{\mu}_{A}} = 2\gamma x^{A}_{\mu}. \tag{A1}$$

Using the previous identity and the definition (3.15) we have

$$x^{\mu}_{C}\Box^2 x_{\mu} = \sqrt{-\gamma}\left(\delta^{A}_{C}x^{B}_{\nu}\frac{\partial x^{\nu}_{B}}{\partial\tau^{A}} + x^{\mu}_{C}\frac{\partial x^{A}_{\mu}}{\partial\tau^{A}}\right)$$

$$= \sqrt{-\gamma}\,\frac{\partial}{\partial\tau^{B}}\delta^{B}_{C} = 0, \tag{A2}$$

in the next-to-last step the identity $x^{\mu}_{A}x^{B}_{\mu} = \delta^{B}_{A}$ has been used.

To prove that $x^{\mu}_{C}\mathfrak{L}_{\mu}\phi = 0$ we will use the form of $x^{\mu}_{C}\mathfrak{L}_{\mu}$ that one obtains from (3.16a) and (3.23a), i.e.,

$$x^{\mu}_{C}\mathfrak{L}_{\mu}\phi = x^{\mu}_{C}\left(\frac{\partial}{\partial x^{\mu}} - \frac{\partial}{\partial\tau^{A}}\frac{\partial}{\partial x^{\mu}_{A}}\right)(\sqrt{-\gamma}\,\phi)$$

$$= -\frac{\partial}{\partial\tau^{A}}\left(\sqrt{-\gamma}\,\frac{\partial\phi}{\partial x^{\mu}_{A}}x^{\mu}_{C}\right), \tag{A3}$$

where in the last step (A1) has been used.

Now from (3.16c) and (3.10d) we get

$$\frac{\partial\phi}{\partial x^{\mu}_{A}}x^{\mu}_{C} = \frac{\partial\phi}{\partial T^{\alpha\beta}}\frac{\partial T^{\alpha\beta}}{\partial x^{\mu}_{A}}x^{\mu}_{C} + \frac{\partial\phi}{\partial S^{\alpha\beta}}\frac{\partial S^{\alpha\beta}}{\partial x^{\mu}_{A}}x^{\mu}_{C}. \tag{A4}$$

Using the identities (3.18) and (3.19) after some work one finds

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\mu}_{A}}x^{\mu}_{C} = 0, \tag{A5}$$

$$\frac{\partial S^{\alpha\beta}}{\partial x^{\mu}_{A}}x^{\mu}_{C} = 0. \tag{A6}$$

Therefore from (A3)–(A6) we have

$$x^{\mu}_{C}\mathfrak{L}_{\mu}\phi = 0 \quad \text{Q.E.D.} \tag{A7}$$

The last relation has a nice geometrical meaning; it says that the "force" $\mathfrak{L}_{\mu}\phi$ is always perpendicular to the world sheet.

APPENDIX B

In this appendix we will show that the end points of open strings are not slowed down by interactions. We will have interchange of momentum through the edges of the world sheets, but the tangent vectors to the edges are null vectors. In this appendix *all quantities* that appear are supposed to be *evaluated at the string end points*.

The equation of motion of the end points is

$$M\sqrt{-\gamma}\,x^{1}_{\mu} + \frac{\partial}{\partial x^{\mu}_{1}}(\sqrt{-\gamma}\,\phi) = 0. \tag{B1}$$

Let us assume that $\gamma \neq 0$ at the end points; hence we can always choose a gauge[14] where $\sqrt{-\gamma} = 1$. Thus

$$Mx^{1}_{\mu} + \frac{\partial}{\partial x^{\mu}_{1}}\phi = 0. \tag{B2}$$

From the definition of γ we have that $\gamma \neq 0$ implies $x^{\mu}_{1} \neq 0$. Hence a multiplication of (B2) by x^{μ}_{A} yields

$$M = 0. \tag{B3}$$

To obtain the above result we have used the fact that a $\gamma \neq 0$ ensures that x^{1}_{μ} is finite and also ensures that (A4)–(A5) holds. But M is a nonzero constant; therefore

$$\gamma = 0. \tag{B4}$$

In general x^{1}_{μ} and $\partial\phi/\partial x^{\mu}_{1}$ are going to be singular when $\gamma = 0$. Thus we must understand Eq. (B1) as an equation that holds in the limit, i.e.,

$$\left[M\sqrt{-\gamma}\,x^{1}_{\mu} + \frac{\partial}{\partial x^{\mu}_{1}}(\sqrt{-\gamma}\,\phi)\right]_{\tau^{1}\to\overline{\alpha},\overline{\beta}} = 0. \tag{B5}$$

Equations (3.16b) and (3.23b) must also be understood in the same way.

We have that

$$x^{\mu}_{0}x_{0\mu} \geq 0, \quad x^{\mu}_{1}x_{1\mu} \leq 0, \tag{B6}$$

where the equal sign is valid only as a limit at the end points. Therefore (B4) can be fulfilled by either one of the two possibilities

$$x^{\mu}_{0}x_{0\mu} = x^{\mu}_{0}x_{1\mu} = 0, \quad x^{\mu}_{1}x_{1\mu} \leq 0 \tag{B7}$$

$$x^{\mu}_{1}x_{1\mu} = x^{\mu}_{0}x_{1\mu} = 0, \quad x^{\mu}_{0}x_{0\mu} \geq 0. \tag{B8}$$

The first possibility, when $x^{\mu}_{1}x_{1\mu} < 0$, tells us that the boundaries of the world sheets described by an open string are tangent to null planes (planes formed with a null vector and a spacelike vector perpendicular to the first). When $x^{\mu}_{1}x_{1\mu} = 0$ the two vectors that describe the world sheet collapse at the boundaries to a single null vector. The second possibility (B8) can only be fulfilled when $x^{\mu}_{0}x_{0\mu} = 0$, because we cannot have a null vector perpendicular to a timelike one. We have not considered the possibility $x^{\mu}_{1} = 0$ at the edges, because that condition is gauge dependent, i.e., we can always do a gauge transformation and have $x^{\mu}_{1} \neq 0$.

In summary, we have that $x^{\mu}_{0}x_{0\mu} = 0$ at the edges, i.e., the end points of open strings travel with the speed of light. Note that this property of the end points does not depend on the type of interaction. Also note that the same is true for "free strings."[1,2,15]

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