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Minimum-uncertainty states and pseudoclassical dynamics

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All the minimum-uncertainty states $|z(\alpha)\rangle$ are built as the eigenstates of generalized annihilation operators $a(\alpha)$. We obtain the most general Hamiltonian which verifies the following property: Given an initial minimum-uncertainty state $|z(\alpha)\rangle$ at $t=0$, the z' which corresponds to the minimum-uncertainty state that maximizes $|\langle z'(\alpha)|z(\alpha),t\rangle|^2$ follows the classical trajectory in phase space. Finally, we comment on its relationship with the most general Hamiltonian which preserves minimality.

I. INTRODUCTION

Recently[1] an appealing algorithm for expressing the nonrelativistic quantum mechanics of a spinless particle in pseudoclassical form was proposed. It naturally makes use of minimum-uncertainty states (MUS's). But in this scheme it is not demanded that minimality be conserved. One is interested rather in seeing whether the MUS that "best" describes the time-evolving state of an initial MUS exactly verifies the classical equations of motion (in the sense defined in Sec. III). It should be emphasized that his approach does not deal with any kind of classical limit or the so-called semiclassical approximations. Therefore a strictly classical interpretation is not in principle entailed. This means that the present approach is independent of whether a classical calculation will in fact provide a meaningful description of the motion.

The aim of this paper is to seek the most general class of Hamiltonians that allow the description of the dynamics in terms of such a pseudoclassical way. We follow this program making use of the very useful technique of annihilation operators and their eigenstates, which in recent years have successfully been applied in many fields of physics such as nonlinearity[2] and adiabatic invariants for both classical and quantum systems.[3]

The main body of this paper is divided into three major sections as follows. In Sec. II we define for each Δ_p (momentum uncertainty) the family of MUS's as eigenstates of a generalized annihilation operator; we reobtain the most general Hamiltonian which conserves minimality.[4-7] Our main results are contained in Sec. III, where we find the necessary and sufficient condition in order for a system to verify the pseudoclassical evolution. In Sec. IV we carefully comment on our result being a generalization of the Hamiltonian considered in Sec. II.

II. MINIMUM-UNCERTAINTY STATES

We define a family of non-Hermitian operators depending on a real, positive parameter α, $a(\alpha)$, in the following way:

$$a(\alpha)=\alpha q+\frac{i}{2\alpha}p\,. \tag{2.1}$$

If α takes on the value $(m\omega/2)^{1/2}$, then a and $a^\dagger$ are the usual ladder operators for a harmonic oscillator of mass m and angular frequency ω. From the definition it follows that

$$[a(\alpha),a^\dagger(\alpha)]=1\,, \tag{2.2}$$

where $a^\dagger$ denotes the adjoint of a, and the square brackets denote the commutator.

For each value of α we construct the normalized eigenstates of $a(\alpha)$ in an analogous way to the harmonic-oscillator coherent states[8]

$$|z(\alpha)\rangle=\exp[-\tfrac{1}{2}|z(\alpha)|^2]\sum_{n=0}^{\infty}\frac{z^n(\alpha)}{\sqrt{n!}}|n\rangle\,, \tag{2.3}$$

where $|n\rangle_\alpha$ is the eigenstate of the number operator $a^\dagger(\alpha)a(\alpha)$ with eigenvalue n (n is a non-negative integer). Note that n is independent of α, owing to (2.2), although naturally the state $|n\rangle_\alpha$ depends on α because the number operators $a^\dagger(\alpha)a(\alpha)$ and $a^\dagger(\beta)a(\beta)$ do not commute (and they cannot have a complete set of simultaneous eigenstates). The set of numbers $z(\alpha)$ spans the entire complex plane.

Since a is non-Hermitian, its eigenvalues are in general complex; it is easily verified that

$$z(\alpha)=\alpha\langle q\rangle+\frac{i}{2\alpha}\langle p\rangle\,, \tag{2.4}$$

where the angular brackets denote the expected value in the corresponding eigenstate $|z(\alpha)\rangle$.

The main properties of these states are the following:

(i) Nonorthogonality:

$$\langle z_1(\alpha)|z_2(\alpha)\rangle = \exp\{-\tfrac{1}{2}[|z_1(\alpha)|^2+|z_2(\alpha)|^2] + z_1^*(\alpha)z_2(\alpha)\}. \quad (2.5)$$

This overlapping goes to zero when $|z_1(\alpha)-z_2(\alpha)| \gg 1$.

(ii) Resolution of unity:

$$\pi^{-1}\int d^2z(\alpha)\,|z(\alpha)\rangle\langle z(\alpha)| = 1. \quad (2.6)$$

(iii) Overcompleteness, in the sense that if we have any convergent sequence of complex numbers $z_n \to z_0$, the MUS's $|z_n(\alpha)\rangle$ themselves form a complete set.[9] This property was already noted by Von Neumann in the case of harmonic-oscillator coherent states.

(iv) Generation of $|z(\alpha)\rangle$ through displacement operators[10]

$$|z(\alpha)\rangle = D_\alpha[z(\alpha)]\,|0\rangle, \quad (2.7)$$

where D_α is a displacement operator

$$D_\alpha(z) = \exp[z a^\dagger(\alpha) - z^* a(\alpha)]. \quad (2.8)$$

It is unitary,

$$D_\alpha^\dagger(z) = D_\alpha(-z) = D_\alpha^{-1}(z), \quad (2.9)$$

and has the following multiplication law:

$$D_\alpha(z_2)D_\alpha(z_1) = D_\alpha(z_2+z_1)\exp[\tfrac{1}{2}(z_2z_1^* - z_2^*z_1)]. \quad (2.10)$$

Equivalently, Eq. (2.7) can be written as

$$|z(\alpha)\rangle = \exp[-\tfrac{1}{2}|z(\alpha)|^2 + z^*(\alpha)a^\dagger(\alpha)]\,|0\rangle \quad (2.11)$$

owing to the Baker-Hausdorff theorem.

(v) $|z(\alpha)\rangle$ are the most general MUS's.[11] In fact it is easily seen that

$$\Delta_q = (2\alpha)^{-1} \quad (2.12a)$$

and

$$\Delta_p = \alpha. \quad (2.12b)$$

The $|z(\alpha)\rangle$ states differ from the harmonic-oscillator coherent states only by an obvious scale transformation[12] (see also the end of this section), but their physical interpretation as harmonic-oscillator states is rather different for $\alpha \neq (m\omega/2)^{1/2}$.

We now ask the following question: which is the most general Hamiltonian H_1 which preserves a MUS as such? Note that the arbitrary Δ_p of the initial MUS defines the α with which we shall work. In exactly the same way as in the case of harmonic-oscillator coherent states,[4,5] it can easily be shown that a necessary and sufficient condition for the initial MUS $|z(\alpha)\rangle$ to remain a MUS is

$$\frac{d}{dt}a(\alpha;t) = f[a(\alpha;t),t], \quad (2.13)$$

i.e., that the time derivative of a does not involve a functional dependence on $a^\dagger$. Furthermore, since H_1 is Hermitian, one concludes that H_1 must be of the form

$$H_1(\alpha) = \omega(t)a^\dagger(\alpha;t)a(\alpha;t) + f(t)a^\dagger(\alpha;t) + f^*(t)a(\alpha;t) + v(t), \quad (2.14)$$

where $\omega(t)$ and $v(t)$ are arbitrary real functions and $f(t)$ is an arbitrary function. We shall now comment on this result:

(1) Using the relation

$$a(\alpha) = c(\alpha,\beta)a(\beta) + s(\alpha,\beta)a^\dagger(\beta), \quad (2.15)$$

where

$$c(\alpha,\beta) = \frac{\beta^2+\alpha^2}{2\alpha\beta}, \qquad s(\alpha,\beta) = \frac{\beta^2-\alpha^2}{2\alpha\beta}, \quad (2.16)$$

as is immediate from the definition in Eq. (2.1), we may express H_1 in terms of $a(\beta)$ and its adjoint

$$H_1(\beta) = \omega(t)C(\alpha,\beta)a^\dagger(\beta)a(\beta) + \tfrac{1}{2}\omega(t)S(\alpha,\beta)[a^{\dagger 2}(\beta)+a^2(\beta)] + g(\alpha,\beta,t)a^\dagger(\beta) + g^*(\alpha,\beta,t)a(\beta) + b(t), \quad (2.17)$$

where $C(\alpha,\beta) = c^2+s^2$, $S(\alpha,\beta) = 2cs$ (in the same way as for c and s, C and S can be considered as the cosh and sinh of an arbitrary real positive function), $g(\alpha,\beta,t) = f(t)c + f^*(t)s$, and $b(t) = \omega(t) + v(t)$.

(2) In the special case $\alpha = (m\omega/2)^{1/2}$, one obtains the most general Hamiltonian which preserves the oscillator coherent states.[4,5]

(3) This result can also be deduced from Ref. 6, where it is shown that the "coherent states" (our MUS's) are unitarily equivalent to the "minimum-uncertainty packets" [eigenfunctions of an operator $S = (2\mu)^{-1/2}(x + i\mu p)$ with arbitrary, real μ, i.e., of $a(\alpha = +(2\mu)^{-1/2})$]. Equation (2.17) is obtained by applying a unitary operator U to the harmonic-oscillator–coherent-state-preserving Hamiltonian [i.e., Eq. (2.14) for $\alpha = (m\omega/2)^{1/2}$]. But Eq. (2.14) is just Eq. (2.17) in terms of $a(\alpha)$. Q. E. D.

Finally, it should be emphasized that in our scheme, MUS's are by definition the eigenstates of annihilation operators $a(\alpha)$. For each α, the set of all $|z(\alpha)\rangle$ is overcomplete. The change of α to β can be said to correspond to a change of "basis" in the same (MUS) representation. This change is performed through the unitary operator[6]

$$U(\alpha\to\beta)=\exp\{\tfrac{1}{2}r\,[a^2(\alpha)+a^{\dagger 2}(\alpha)]\}, \tag{2.18}$$

where

$$r=\cosh^{-1}(\beta^2+\alpha^2)/2\alpha\beta, \tag{2.19}$$

so that

$$a(\beta)=U(\alpha\to\beta)\,a(\alpha)\,U^\dagger(\alpha\to\beta). \tag{2.20}$$

Equations (2.20), (2.15), and (2.2) express the fact that operator U corresponds to a canonical transformation.[13]

III. PSEUDOCLASSICAL DYNAMICS IN QUANTUM MECHANICS

In this section we consider the following problem: Given an initial MUS $|z(\alpha)\rangle$ at $t=0$, we let it evolve in time under the action of an arbitrary Hamiltonian. We find the MUS $|z'(\alpha)\rangle$, which has maximum overlapping with our evolving state at time $t>0$. The basic question is: Which is the most general Hamiltonian H_2 for which z' describes the classical trajectory in phase space? In other words, we ask: When will $\alpha^{-1}\mathrm{Re}(z')$ and $2\alpha\,\mathrm{Im}(z')$ describe the classical trajectories of q and p [see Eq. (2.4)]?

Given the initial condition $|\psi(t=0)\rangle=|z(\alpha)\rangle$, the formal solution to the Schrödinger equation is (taking $\hbar=1$)

$$|\psi(t)\rangle=e^{-iHt}|z(\alpha)\rangle\equiv|z(\alpha),t\rangle. \tag{3.1}$$

We now define the phase-space (quasi) probability $|\langle z'(\alpha)|z(\alpha),t\rangle|^2$. Note that this approach does not lead to a true joint probability distribution.[14] The physical significance of $|\langle z'(\alpha)|z(\alpha),t\rangle|^2$ is the quantum probability that the particle would be found, if measured, in the MUS $|z'(\alpha)\rangle$. We emphasize that it is not the (nonexistent) probability density of finding the particle at position $q=\alpha^{-1}\mathrm{Re}(z')$ with momentum $p=2\alpha\,\mathrm{Im}(z')$, but it is the most "classical" possible quantum description of the system. For this reason, the MUS representation is called a "classical particle" quantum description of the system.[15]

It is evident that in the particular case when $|z(\alpha),t\rangle$ propagates as a MUS, then the maximum corresponds to $|z'(\alpha)\rangle=|z(\alpha),t\rangle$. This implies that $H_1(\alpha)$ given in Eq. (2.14) will be a particular case of our general solution.

The distribution $|\langle z'(\alpha)|z(\alpha),t\rangle|^2$ is a function of two real independent variables $q'(\alpha)$ and $p'(\alpha)$, and is defined by the real parameters $q(\alpha)$ and $p(\alpha)$ (initial condition) and by the Hamiltonian which gives the dynamics. In order to find $|z'(\alpha)\rangle$ we must maximize with respect to $q'(\alpha)$ and $p'(\alpha)$.

From now on the α dependence will be understood, without our having to write it down. Taking into account the relations

$$\frac{\partial}{\partial q}=\alpha\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^*}\right), \tag{3.2a}$$

$$\frac{\partial}{\partial p}=\frac{i}{2\alpha}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right), \tag{3.2b}$$

we impose the condition that the first derivative be equal to zero:

$$\begin{aligned}0&=\frac{\partial}{\partial q'}|\langle z'|z,t\rangle|^2\\&=\alpha[-(z'+z'^*)|\langle z'|z,t\rangle|^2\\&\quad+\langle z'|a|z,t\rangle\langle z,t|z'\rangle\\&\quad+\langle z'|z,t\rangle\langle z,t|a^\dagger|z'\rangle],\end{aligned} \tag{3.3a}$$

$$\begin{aligned}0&=\frac{\partial}{\partial p'}|\langle z'|z,t\rangle|^2\\&=\frac{i}{2\alpha}[-(z'^*-z')|\langle z'|z,t\rangle|^2\\&\quad+\langle z'|a|z,t\rangle\langle z,t|z'\rangle\\&\quad+\langle z'|z,t\rangle\langle z,t|a^\dagger|z'\rangle].\end{aligned} \tag{3.3b}$$

Adding and subtracting these two equations we obtain the simpler ones:

$$\langle z'|a|z,t\rangle=z'\langle z'|z,t\rangle, \tag{3.4a}$$

$$\langle z,t|a^\dagger|z'\rangle=z'^*\langle z,t|z'\rangle. \tag{3.4b}$$

Observe that Eqs. (3.4a) and (3.4b) are complex conjugates of each other.

In the particular case when $|z,t\rangle$ is an MUS ($|z'\rangle$ will then be equal to $|z,t\rangle$), Eqs. (3.4) are trivially verified.

Note that Eq. (3.4) is valid for all t, but that the solution z', as well as the value of the maximum probability $|\langle z'|z,t\rangle|^2$, both depend on time. If z' and z'' are the solutions corresponding to times t and $t-t_0$ $(0\le t_0<t)$, the values of the maximum probabilities at t and $t-t_0$, which can be written as

$$|\langle z'|z,t\rangle|^2=|\langle z',-t_0|z,t-t_0\rangle|^2$$

and

$$|\langle z''|z,t-t_0\rangle|^2=|\langle z'',t_0|z,t\rangle|^2,$$

are in general different. Only in the case when $|z,t\rangle$ propagates as an MUS will the probabilities be equal (in fact, time independent and equal to unity); but still in this case, z' will depend on time, with the initial condition $z'(t=0)=z$. This time dependence of z' is the crux of the problem, because our aim is to find the most general Hamiltonian for which the trajectory of z' is formally equal to the corresponding classical one. In particular, for very small t we have from Eq. (3.4a)

$$\langle z+At+O(t^2)|\, a\,[1-iHt+O(t^2)]\,|z\rangle=[z+At+O(t^2)]\langle z+At+O(t^2)|\,1-iHt+O(t^2)\,|z\rangle\,, \tag{3.5}$$

where we have written

$$z'(t)=z'(0)+At+O(t^2)$$
$$=z+At+O(t^2)\,. \tag{3.6}$$

Up to first order in t, from (3.5) one finds that

$$A=-i\langle z|\,[a,H]\,|z\rangle\,. \tag{3.7}$$

Substituting back into (3.6), we get

$$z'=\langle z|\,a-it[a,H]\,|z\rangle+O(t^2)\,. \tag{3.8}$$

This means that z' is equal to $\langle z|a(t)|z\rangle$ up to first order in t.

Our objective now is to find what Hamiltonians verify

$$q'(t)=q_{\text{clas}}(t)\,,\qquad p'(t)=p_{\text{clas}}(t)\,, \tag{3.9}$$

that is, to first order in t

$$\langle z,t|\,q\,|z,t\rangle=q_{\text{clas}}(t)\,,\qquad \langle z,t|\,p\,|z,t\rangle=p_{\text{clas}}(t)\,, \tag{3.10}$$

where $q_{\text{clas}}(t)$ and $p_{\text{clas}}(t)$ stand for the classical trajectories. From Eq. (3.9) and from Ehrenfest's theorem[16] we deduce that if q', p' must satisfy the same equations of motion as $q_{\text{clas}}, p_{\text{clas}}$, then the Hamiltonian must be, at most, a quadratic polynomial in q and p (with no crossed terms),

$$H_2=\frac{1}{2m}p^2+c_1p+c_2q^2+c_3q+c_4\,, \tag{3.11}$$

where c_i ($i=1$ to 4) must be real, owing to the Hermiticity of H_2. Also c_i ($i=1$ to 4) can be arbitrary (real) functions of time.

Equation (3.11) expresses a necessary condition, that is, the Hamiltonian we are looking for cannot be more general than (3.11). But it also is a sufficient condition, as has been already explicitly demonstrated,[1] except for the term linear in p. But this case is implicitly also included, by completing the square, i.e.,

$$\frac{p^2}{2m}+c_1p=\frac{p'^2}{2m}-\frac{mc_1^{\,2}}{2}\,,$$

with $p'=p+mc_1$.

It should be noted that for the Hamiltonian in (3.11), as both $\langle z,t|\,a\,|z,t\rangle$ (owing to Ehrenfest's theorem) and z' (see discussion above) follow the classical trajectory, we can conclude that

$$z'=\langle z,t|\,a\,|z,t\rangle=\langle z|\,a(t)\,|z\rangle \tag{3.12}$$

to all orders in t (for H_2).

Our result, Eq. (3.11), corresponds to the following physical systems[17]:

(i) free particle ($c_i=0$ for all i),

(ii) harmonic oscillator (only $c_2\neq 0$),

(iii) particle in an external uniform field (e.g., electrical or gravitational; only $c_3\neq 0$)

(iv) harmonic oscillator in an external uniform field (only $c_2, c_3\neq 0$),

(v) $c_1\neq 0$ corresponds in cases (i) to (iv) to a (nonrelativistic) moving frame of reference,

(vi) $c_4\neq 0$ corresponds in cases (i) to (v) to a shift in the energy scale.

Observe that the lack of a crossed term in q and p means that the case of a particle (or an oscillator) in a uniform magnetic field[18] is not included.

If we express H_2 in terms of $a(\alpha)$ and its adjoint $a^\dagger(\alpha)$, we must distinguish two cases in Eq. (3.11):

(a) $c_2\neq 0$:

$$H_2(\alpha)=\omega C a^\dagger(\alpha)a(\alpha)+\tfrac{1}{2}\omega S[a^{\dagger 2}(\alpha)+a^2(\alpha)]+ga^\dagger(\alpha)+g^*a(\alpha)+b\,, \tag{3.13a}$$

where

$$\omega=(2c_2/m)^{1/2}\,,$$
$$C=(m/2c_2)^{1/2}\,\frac{c_2m+2\alpha^4}{2\alpha^2m}\,,$$
$$S=(c^2-1)^{1/2}\,,$$
$$g=(c_3/2\alpha)+i\alpha c_1\,,$$
$$b=c_4+\alpha^2(2m)^{-1}+c_2(2\alpha)^{-2}\,.$$

(b) $c_2=0$:

$$H_2(\alpha)=\frac{\alpha^2}{m}a^\dagger(\alpha)a(\alpha)-\frac{\alpha^2}{2m}[a^{\dagger 2}(\alpha)+a^2(\alpha)]+ga^\dagger(\alpha)+g^*a(\alpha)+b \tag{3.13b}$$

Our result, Eq. (3.11), generalizes the Hamiltonian H_1 of Eq. (2.14), which we had expressed in a very similar way to $H_2(\alpha)$ for $c_2\neq 0$ in Eq. (2.17) in terms of an arbitrary β ($\beta\neq\alpha$). The generalization is evident: MUS-preserving Hamiltonians are a particular case of (3.13a) (in fact, for $S=0$, i.e., for $c_2=2\alpha^4m^{-1}$). For instance, the free particle Hamiltonian [case (i), corresponding to (3.13b) with $g=0$] does not conserve MUS's, but it does verify that the maxima of the distribution $|\langle z'|z,t\rangle|^2$ follow the classical trajectory.

It should be emphasized that the derivation of our result, Eq. (3.11), does not involve any approximations, and that it will therefore be true independent of the fact that the classical description of the trajectory between measurements could be meaningful or not.[1] Also, in this scheme we are not involved in classical limits.[19]

IV. FINAL DISCUSSION

The general quadratic Hamiltonian in q and p with no crossed terms exhausts the class of Hamiltonians which make the MUS's follow pseudoclassical trajectories.

In spite of the formal similarity between $H_1(\alpha)$ in Eq. (2.14) and $H_2(\alpha)$ in Eq. (3.13a), their respective physical interpretations are rather different.

In Sec. II we saw that H_1, as given in Eq. (2.14) in terms of $a(\alpha)$ or in Eq. (2.17) in terms of $a(\beta)$ (arbitrary $\beta \neq \alpha$), conserved the minimal character of an initial MUS $|z(\alpha), t=0\rangle$.

In Sec. III we showed that, given an initial MUS $|z(\alpha), t=0\rangle$, $H_2(\alpha)$ as expressed in Eq. (3.13a) verified the pseudoclassical evolution condition for $z'(\alpha)$. The similarity is due to the fact that there exists a β [in fact $\beta = (mc_2/2)^{1/4}$] such that $H_2(\alpha) = H_1(\beta)$, that is, $H_2(\alpha)$ for $c_2 > 0$ preserves the minimality of the states $|z(\beta)\rangle$ (β fixed and independent of α).

It is clear that the family of all $H_1(\beta)$ (varying β) is a subset of the Hamiltonians $H_2(\alpha)$ for any α: It is the one obtained when c_2 spans all the positive nonzero values.

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