Functional measure for quantum gravity

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We derive the functional measure for quantum gravity by reducing to independent degrees of freedom in the light-cone gauge. We use the recently developed functional techniques devised to handle field theories with second-class constraints in order to analyze quantum gravity quantized along null planes, where it has been shown that all eight dependent components of the metric may be explicitly functionally eliminated, leaving only two unconstrained variables. Using our result, we confirm the result of Fradkin and Vilkovisky for the functional measure for quantum gravity, though we disagree with several other authors who have different measures.

INTRODUCTION

In the literature there is considerable confusion over the precise functional measure for quantum gravity. Previous results, based on vague invariance arguments, have all been discredited. Recent advances in the quantization of field theories with gauge invariances[1] have made possible the conclusive determination of the elusive functional measure for quantum gravity. The key to solving the problem[2] lies in observing that in the *fully reduced*, unconstrained, noncovariant Hamiltonian formulation, where all dependent variables have been explicitly integrated over, the functional measure is equal to unity. Working backwards, it is then possible to determine the functional measure that must be added into the constrained Lagrangian formalism (which is covariant and contains many dependent variables) in order to reproduce the correct measure (unity) for the reduced Hamiltonian system with only independent components. This procedure, first carried out by Faddeev and Popov,[2] gave the following result for the measure:

$$M_{\mathrm{FP}}=\prod_x\left[g^{5/2}(x)\prod_{\sigma\leq\lambda}dg^{\sigma\lambda}(x)\right]. \tag{1}$$

[In Eq. (1) $g^{\sigma\lambda}$ is the inverse of the metric tensor $g_{\mu\nu}$, and $g=\det g_{\mu\nu}$.]

A quite independent study of the problem was made by Fradkin and Vilkovisky,[3] and in their paper disagreement was found with the result of Faddeev and Popov. Fradkin and Vilkovisky obtain

$$M_{\mathrm{FV}}=\prod_x\left[g^{7/2}(x)g^{00}(x)\prod_{\sigma\leq\lambda}dg^{\sigma\lambda}(x)\right]$$
$$=\prod_x\left[g^{-3/2}(x)g^{00}(x)\prod_{\sigma\leq\lambda}dg_{\sigma\lambda}(x)\right]. \tag{2}$$

The presence of $g^{00}(x)$ may seem irksome to a naive reader (the measure looks noncovariant), but as Fradkin and Vilkovisky carefully demonstrate, one cannot judge by appearances alone. We refer the reader to their paper for details.

In this paper, we shall present a new investigation of the problem of functional measure, which is an application (and extension) of one of the recent studies of quantum gravity (reduced to independent variables) in the light-cone gauge.[4] The advantage of the light-cone gauge is that all eight redundant components of the gravitational metric can be explicitly functionally eliminated from the Lagrangian, so that we are left with an unconstrained noncovariant Hamiltonian, where we know that the functional measure is one. In the usual Coulomb gauge for gravity all redundant components cannot be explicitly eliminated from the functional integral. Seven redundant components of the metric tensor *can* be completely eliminated, but the last redundant component requires complex, formal iterative procedures. In the light-cone gauge, however, all eight redundant components can be explicitly eliminated.

We shall close the introduction with an important comment concerning Eq. (2). As the reader will note by inspecting the paper by Fradkin and Vilkovisky,[3] Eq. (2) holds only if one accepts a particular determination of certain ambiguous terms of the type [see their Eqs. (2.8), (2.9), (2.10), the discussion following those, and also the remark following their Eq. (2.21)]

$$[\partial_\mu\partial_\nu D_F(x-y)]_{x=y},$$

which consists in writing

$$[\partial_\mu\partial_\nu D_F(x-y)]_{x=y}=\left[\partial_\mu\partial_\nu\int\frac{d^4k}{k^2+i\epsilon}e^{ik\cdot(x-y)}\right]_{x\to y}, \tag{3}$$

integrating first over k_0 in the integral in Eq. (4), taking the derivatives, and *then* taking the limit $x \to y$. In the resulting expression only the term proportional to $\delta^{(4)}(0)$ is isolated.[3]

If, however, one chooses the determination of Eq. (3) which consists in writing

$$k_\mu = \eta_\mu k_\eta + \sum_{i=1}^{3} \xi_\mu^i k_i \tag{4}$$

(where η_μ is a certain timelike vector of norm 1 and ξ_μ^i are three orthonormal vectors orthogonal to η^μ) and integrating first over k_η in Eq. (3), then the tensor structure multiplying $\delta^{(4)}(0)$ in Eq. (3) is $\eta_\mu \eta_\nu$. The special case $\eta_\mu = \delta_{\mu 0}$ corresponds to the previous determination.

This more general determination has the consequence of replacing Eq. (2) by

$$M_{(\eta)}^{\text{FV}} = \prod_x \left[g^{7/2}(x) \eta_\mu g^{\mu\nu}(x) \eta_\nu \prod_{\sigma\leq\lambda} dg^{\sigma\lambda}(x) \right]. \tag{5}$$

What can be said about the light-cone-type regularization of (3), which consists in integrating first over $k^- = (k^0 + k^3)/\sqrt{2}$? This is clearly the one we are interested in, since we shall study the functional measure for gravity quantized on the light cone, in which type of quantization such a regularization is a consequence of the standard expression for $D_F(x)$ (see Ref. 5),

$$D_F(x) = (2\pi)^{-3} \int d^2p \int_0^\infty \frac{d\eta}{\eta} \left[\theta(x^+) e^{-ip\cdot x} + \theta(-x^+) e^{ip\cdot x} \right]. \tag{6}$$

In Eq. (6), $p \cdot x$ stands for $p^+x^- + p^-x^+ = \eta x^- + (\vec{p}^{\,2}/2\eta)x^+$.

The reader will easily check that a term proportional to $\delta^{(4)}(0)$ (with this type of regularization) will be found only when

$$(\mu\nu) = (+\,-) \quad \text{or} \quad (\mu\nu) = (-\,+).$$

Repeating the arguments of Fradkin and Vilkovisky (for this type of regularization), one finds instead of Eqs. (2) and (5)

$$\overline{M}^{\text{FV}} = \prod_x \left[g^{7/2}(x) g^{+-}(x) \prod_{\sigma\leq\lambda} dg^{\sigma\lambda}(x) \right]. \tag{7}$$

The measure thus depends on the type of regularization adopted for the ambiguous terms, but of course *quantum gravity theory does not*, since the contributions of the measure simply cancel other contributions in the perturbation expansion proportional to $\delta^{(4)}(0)$ as long as the regularization procedure is uniform, as demonstrated by Fradkin and Vilkovisky.

I. REDUCTION OF QUANTUM GRAVITY IN THE LIGHT-CONE GAUGE TO INDEPENDENT FIELD VARIABLES

In a recent study by one of the authors (M.K.),[4] the method of path integrals was used to effect (after a $2+2$ decomposition of the metric tensor) a complete elimination of the eight dependent components. A formulation of quantum gravity quantized on the null planes in terms of only two unconstrained Lagrangian degrees of freedom was thus obtained.

For this purpose, we use the light-cone gauge condition

$$\chi_\mu = k^{\bar{l}} g^{\mu +} - \delta^{\mu +} = 0. \tag{8}$$

In Eq. (8), $k = \det(g_{ij})$ $(i,j = 1,2)$, $\bar{l}$ is an arbitrary constant, and of all the $\delta^{\mu+}$ only δ^{-+} is nonvanishing (and equal to one).

It was also demonstrated that this gauge is ghost-free. In functional integral language, this simply means that the Faddeev-Popov determinant corresponding to this choice of gauge is a constant independent of the field variables.

Let us emphasize that the gauge selection corresponding to Eq. (8) differs from most others known in the literature on quantum gravity, in that it implies a radically different quantization scheme, one along the surfaces of constant "null-plane time" $x^+ = (x^0 + x^1)/\sqrt{2}$ rather than surfaces of constant x^0.

The final result of this investigation can be described by

$$\begin{aligned} \langle f|S|i\rangle &= \int \prod_{\mu<\nu} \mathfrak{D}g^{\mu\nu} \phi_f^* \exp\left(i\int d^4x \mathcal{L}\right) \Delta_{\text{FP}} \left(\prod_x \delta_x\right) M \phi_i \\ &= \int \mathfrak{D}\alpha \mathfrak{D}\beta M \prod_x \{k^{-6\bar{l}-3} |1 + \kappa^2(\alpha^2+\beta^2)|^{-1/2}\}_x \phi_f^* \exp\left(i\int d^4x \mathcal{L}\right) \phi_i . \end{aligned} \tag{9}$$

In Eq. (9), the left-hand side is the S-matrix element for transition from the state $|i\rangle$ containing only physical gravitons to a state $|f\rangle$ of the same nature.

On the right-hand side $\alpha(x)$ and $\beta(x)$ are two field variables corresponding to two unconstrained degrees of freedom of a physical graviton, and $\mathcal{L}(\alpha,\beta)$ is the corresponding Lagrangian, which we shall display shortly. $\kappa/\sqrt{2}$ is the coupling constant of gravity.

In the intermediate expression ϕ_f and ϕ_i are the field wave functionals for the states $|f\rangle$ and $|i\rangle$; $\prod_x \delta_x$ is the δ functional corresponding to the gauge choice (8), Δ_{FP} is the corresponding Faddeev-

Popov determinant, and M is the functional measure discussed by Faddeev and Popov and Fradkin and Vilkovisky. It is left unspecified in Eq. (9).

By displaying an independent derivation of the factor in curly brackets we shall be able to determine M.

The form of $\mathcal{L}(\alpha,\beta)$ in Eq. (9) is the following:

$$\begin{aligned}\kappa^2\mathcal{L}(\alpha,\beta) = {} & \tfrac{1}{2}k^{1/2}(e^{ij}\partial_+\partial_- e_{ij}) + (2\bar{l}+\tfrac{1}{2})k^{1/2}\partial_+\partial_-\ln k \\ & + \tfrac{1}{4}k^{\bar{l}}[e^{ij}(\partial_l\partial_m e_{ij}e^{lm} - 2(\partial_i e^{jk})(\partial_k e^{li})e_{jl}] \\ & + 2e^{ij}(\partial_i k^{\bar{l}/2})\partial_j k^{\bar{l}/2} + \tfrac{1}{2}k^{-\bar{l}+1}e_{ij}M^iM^j ,\end{aligned} \tag{10}$$

$$\begin{aligned}M^j = {} & k^{\bar{l}+1}e^{ij}\frac{1}{\partial_-}[\partial_l(k^{\bar{l}}e^{lm}\partial_- e_{mi}) + \tfrac{1}{2}k^{1/2}(\partial_- e^{lm})\partial_i e_{lm} \\ & - k^{1/2}\partial_i\partial_-\ln k + \bar{l}k^{1/2}\partial_i\ln k\,\partial_j\ln k - \bar{l}k^{1/2}\partial_i\partial_-\ln k].\end{aligned} \tag{11}$$

A dimensional factor κ^2 (where $\kappa/\sqrt{2}$ is the gravitational coupling constant), usually set equal to unity by an appropriate choice of units, has been reintroduced for future convenience.

In Eq. (10) and (11), e^{ij} are given in terms of α and β by

$$\begin{aligned} e_{ij} &= [1+\kappa^2(\alpha^2+\beta^2)]^{1/2}\delta_{ij} + \kappa h_{ij} , \\ e^{ij} &= [1+\kappa^2(\alpha^2+\beta^2)]^{1/2}\delta^{ij} - \kappa h_{ij} ,\end{aligned} \tag{12}$$

where

$$h_{ij} = \begin{pmatrix}\alpha & \beta \\ \beta & -\alpha\end{pmatrix}. \tag{13}$$

and, finally, k is expressed in terms of e^{ij} and e_{ij} in the following fashion:

$$\begin{aligned} & y = \partial_-\ln k , \quad y = \sum_{m=1}^{\infty}(\bar{l}-\tfrac{1}{4})^m U_m , \\ & U_m = \frac{1}{\partial_-}\sum_{i=0}^{m=1} U_i U_{m-1-i} , \quad U_0 = \frac{1}{\partial_-}(-\tfrac{1}{2}e^{ij}\partial_-{}^2 e_{ij}) .\end{aligned} \tag{14}$$

II. THE HAMILTONIAN PATH INTEGRAL FOR CONSTRAINED SYSTEMS WITH SECOND-CLASS CONSTRAINTS

While the Lagrangian exhibited in Eq. (10) contains only two independent field variables, $\alpha(x)$ and $\beta(x)$, we shall see shortly that the Hamiltonian formulation corresponding to that Lagrangian contains constraints. This may seem bizarre to those readers who have sometimes encountered the statement that "constraints in the Hamiltonian formalism correspond to a gauge invariance of the theory" [and in the transition to the reduced-variable description the original gauge invariance of quantum gravity has disappeared (after all, the gauge has been fixed before that transition)].

However, the above statement is only partially true—it applies only to the so-called first-class constraints, which are those whose standard Poisson brackets with each other vanish upon application of the constraint equations.[6]

Second-class constraints[6] (which can be defined as those that are not first-class) can also appear in certain field theories, and they *do not* correspond to a gauge invariance.

A notable example are field theories rewritten in terms of null-plane variables. All these contain second-class constraints.[7]

The functional measure for constrained Hamiltonian systems containing only first-class constraints was found by Faddeev.[8] For the case of systems containing both first- and second-class constraints, the measure was evaluated (extending Faddeev's method) by one of the present authors (P.S.),[9] and independently by Yabuki.[10] We shall need only a special case of their result, namely the one holding for theories containing *only* second-class constraints, which can be formulated as follows:

$$\begin{aligned}\langle f|S|i\rangle = \int & \prod_k \mathcal{D}\phi_k\mathcal{D}\Pi_k \prod_{x,a}\delta(\theta^a(x)) \prod_t |\det_{(\vec{x}a,\vec{y}b)}\{\theta_a(\vec{x}),\theta_b(\vec{y})\}|^{1/2} \\ & \times \phi_f^* \exp\left\{ i\int d^4x\left[\sum_l \Pi_l(x)\dot{\phi}_l(x) - \mathcal{H}(x)\right]\right\}\phi_i .\end{aligned} \tag{15}$$

ϕ_f and ϕ_i are final and initial field wave functionals; $\{\ ,\ \}$ denotes a standard equal-time Poisson bracket, $\theta^a(\vec{x})$ are the second-class constaints, ϕ_k's are the set of fields appearing in the theory, Π_k are their conjugate momenta, $\mathcal{H}$ is the Hamiltonian, and the square root of the functional determinant is precisely the above-mentioned functional measure.

III. FUNCTIONAL MEASURE FOR GRAVITY IN THE LIGHT-CONE GAUGE REDUCED TO INDEPENDENT VARIABLES

We are now well equipped to proceed with the calculation of the appropriate functional measure corresponding to the Lagrangian (10).

The first step will consist in rewriting the relevant portions of that Lagrangian directly in terms of the variables α, β. We shall call $\alpha = q_1$ and $\beta = q_2$.

It is a simple exercise to show that

$$\kappa^2 \mathcal{L}(q) = \kappa^2 k^{1/2} M_{mm}(\partial_+ q_m)(\partial_- q_m) + (2l + \tfrac{1}{2}) k^{1/2} \partial_+ \partial_- \ln k + V(q) , \quad (16)$$

where

$$M_{mm} = \delta_{mm} - \kappa^2 \frac{q_m q_n}{1 + \kappa^2 q^2} , \quad q^2 = q_k q_k .$$

In Eq. (16), $V(q)$ does not contain "time" derivatives (i.e., derivatives with respect to x^+) of the q's, and k is still given by the chain of equations (14), with the last equation in the chain now rewritten as

$$U_0 = -\kappa^2 \frac{1}{\partial_-} [(\partial_- q_n) M_{mn} (\partial_- q_m)] . \quad (17)$$

For the purpose of transition to the Hamiltonian formalism, the system will be taken to "evolve" along the direction of varying x^+, i.e., the role of the time variable will be played by the null-plane time x^+.

One can now transcribe the action corresponding to Eq. (16) as

$$S = \int d^4x [\overline{\mathcal{L}}(q)]_x , \quad (18)$$

$$\kappa^2 \overline{\mathcal{L}}(q) \big|_x = \kappa^2 k^{1/2} M_{mn} (\partial_+ q_n)(\partial_- q_n) \big|_x + (2l + \tfrac{1}{2})(\partial_+ q_s) \big|_x \int dz^- \frac{\delta(\partial_- \ln k)|_{z^-, x^+, \vec{x}}}{\delta q_s(x^-, x^+, \vec{x})} \times k^{1/2}(z^-, x^+, \vec{x}) + V(q) . \quad (19)$$

The functional derivative in Eq. (19) is an ordinary local derivative as far as the dependence of q_s on x^+ and $\vec{x}$ is concerned. It is a genuine functional derivative only as far as the dependence of q_s on x^- goes. E.g.,

$$\frac{\delta q_m(y^-, x^+, \vec{x})}{\delta q_s(x^-, x^+, \vec{x})} = \delta_{sm} \delta(y^- - x^-) .$$

To go over to the Hamiltonian formalism, one must derive the conjugate momenta. One obviously finds

$$p_m(x) = \frac{\partial \overline{\mathcal{L}}}{\partial(\partial_+ q_m)} \bigg|_x = k^{1/2} M_{mn} (\partial_- q_n) \big|_x + (2l + \tfrac{1}{2}) \frac{1}{\kappa^2} \left[\int dz^- \frac{\delta(\partial_- \ln k)_{z^-}}{\delta q_m(x^-)} k^{1/2}(z^-) \right]_{x^+, \vec{x}} . \quad (20)$$

Equations (20) represent constraints, since they do not contain "time" derivatives.

Our constraints are therefore

$$\theta_m(x) = p_m(x) - \Lambda_m(x) , \quad (21)$$

$$\Lambda_m(x) = [k^{1/2} M_{mn} \partial_- q_n]_x + (2l + \tfrac{1}{2}) \frac{1}{\kappa^2} \left[\int dz^- \frac{\delta(\partial_- \ln k)_{z^-}}{\delta q_m(x^-)} k^{1/2}(z^-) \right]_{x^+, \vec{x}} . \quad (22)$$

One then derives the null-plane Hamiltonian:

$$H = \int dx^- d\vec{x} [p_m \partial_+ q_m - \overline{\mathcal{L}}] = -\int V(q) dx^- d\vec{x} . \quad (23)$$

In the second step, the constraints (21) were used.

These are no secondary constraints generated from the constraints appearing in Eq. (21) by application of the equations of motion. The proof is simple. According to Dirac,[6] the equation of motion for a quantity f is given by

$$\dot{f} = \{f, H\} + \lambda_m \{f, \theta_m\} . \quad (24)$$

The symbol $\{\ ,\ \}$ denotes standard Poisson brackets, λ_m are a set of *a priori* arbitrary) multipliers, and θ_m are the constraints. In our case the contraints are labeled by a set of continuous indices $x^-, \vec{x}$ [we are dealing with a *field* (i.e., *continuum*) theory at fixed x^+], and by a discrete index m $(m = 1, 2)$.

In our case, therefore, Eq. (24) reads (if one takes f to be the constraints themselves)

$$\partial_+ \theta_s(x^+, x^-, \vec{x}) = \{\theta_s(x^+, x^-, \vec{x}), H(x^+)\} + \int dy^- d\vec{y} \lambda_m(x^+, y^-, \vec{y}) \{\theta_s(x^+, x^-, \vec{x}), \theta_m(x^+, y^-, \vec{y})\} . \quad (25)$$

An explicit evaluation in the rest of this section will show that for our theory

$$\det\{\theta_s(x), \theta_m(y)\} \big|_{x^+ = y^+} \neq 0 , \quad (26)$$

and thus Eq. (25) simply determines the multipliers and generates no new constraints. Equation (26) is also Dirac's criterion,[6] which, when fullfilled, implies that all the constraints $\theta_s(x)$ (and any linear combination of those) are second class.

After some rather straightforward algebra, one finds, using Eqs. (21) and (22),

$$\{\theta_m(x),\theta_l(y)\}|_{x^+=y^+}=\delta(\vec{x}-\vec{y})\Big\{-2k^{1/2}(x^-)M_{ml}(x^-)\frac{\partial}{\partial x^-}\delta(x^--y^-)-\tfrac{1}{2}k^{-1/2}(x^-)[\partial_-^x k(x^-)]M_{ml}(x^-)\delta(x^--y^-)$$
$$+\left[\frac{2\kappa^2 q_m k^{1/2}}{1+\kappa^2q^2}M_{kl}(\partial_- q_k)\right]_{x^-}\delta(x^--y^-)$$
$$+\frac{\delta k^{1/2}(y^-)}{\delta q_m(x^-)}M_{lk}(y^-)\partial_-^y q_k(y^-)-\frac{\delta k^{1/2}(x^-)}{\delta q_l(y^-)}M_{ms}(x^-)\partial_-^x q_s(x^-)$$
$$-\frac{1}{\kappa^2}(2l+\tfrac{1}{2})\int dz^-\left[\frac{\delta(\partial_-\ln k)|_{z^-}}{\delta q_m(x^-)}\frac{\delta k^{1/2}(z^-)}{\delta q_l(y^-)}-\frac{\delta(\partial_-\ln k)|_{z^-}}{\delta q_l(y^-)}\frac{\delta k^{1/2}(z^-)}{\delta q_m(x^-)}\right]\Big\}\Big|_x . \quad (27)$$

Using standard properties of determinants, we can see that

$$\prod_{x^+}\det\{\theta_m(x),\theta_l(y)\}|_{x^+=y^+}=\prod_x[k(x)\det M_{ml}(x)]S, \quad (28)$$

$$S=\prod_{x^+,\vec{x}}\det(x^-,l;y^-,s)\Big\{\frac{\partial}{\partial x^-}\delta_{ls}\delta(x^--y^-)$$
$$+\frac{1}{4}\frac{k'(x)}{k(x)}\delta_{ls}\delta(x^--y^-)-\kappa^2(qM_{sk}q_k')_x\delta(x^--y^-)$$
$$+\frac{1}{4}\frac{\delta\ln k(x)}{\delta q_s(y)}q_l'(x)-\frac{1}{4}\frac{k^{1/2}(y)}{k^{1/2}(x)}(M^{-1})_{lm}(x)M_{sk}(y)q_k'(y)\frac{\delta\ln k(y)}{\delta q_m(x)}$$
$$+\frac{1}{2\kappa^2k^{1/2}(x)}(M^{-1})_{sm}(x)(2l+\tfrac{1}{2})\int dz^-\left[\frac{\delta(\partial_-\ln k)_{z^-}}{\delta q_m(x^-)}\frac{\delta k^{1/2}(z^-)}{\delta q_l(y^-)}-\frac{\delta(\partial_-\ln k)_{z^-}}{\delta q_l(y^-)}\frac{\delta k^{1/2}(z^-)}{\delta q_m(x^-)}\right]\Big\}\Big|_{x^+,\vec{x}} . \quad (29)$$

Primes in Eq. (29) denote derivatives with respect to x^-.

Let us now go back to Eq. (7). In the light-cone gauge,[4] using $g=k^{2\bar{l}+1}$ and $g^{+-}=k^{-\bar{l}}$, one finds for M in this equation

$$\bar{M}_{FV}=\prod_x\left[k^{6\bar{l}+7/2}(x)\prod_{\mu\le\nu}dg^{\mu\nu}(x)\right]. \quad (30)$$

On the other hand, the measure of Faddeev and Popov [Eq. (2)] gives in this gauge

$$M_{FP}=\prod_x\left[k^{5\bar{l}+5/2}(x)\prod_{\mu\le\nu}dg^{\mu\nu}(x)\right]. \quad (31)$$

Equation (28), combined with the use of

$$\det M_{ml}=\frac{1}{1+\kappa^2q^2}, \quad (32)$$

yields

$$\prod_{x^+}|\det\{\theta_m(x),\theta_l(y)\}|^{1/2}|_{x^+=y^+}$$
$$=\prod_x\left[k^{1/2}\frac{1}{(1+\kappa^2q^2)^{1/2}}\right]_x|S|^{1/2}. \quad (33)$$

This must equal the measure in Eq. (9), which produces the equation

$$M=\prod_x[k^{6\bar{l}+7/2}(x)]\,|S|^{1/2}. \quad (34)$$

We have been able to calculate S only by expanding it in powers of κ [up to (and including) terms of order κ^2] and have found

$$|s|=\left|\det\frac{\partial}{\partial x^-}\right|[1+O(\kappa^6)].$$

To this order, this agrees with Eq. (30) and disagrees with Eq. (31) [note that $k=1+O(\kappa^2)$], and therefore settles the controversy in favor of Fradkin and Vilkovisky.

To spare the reader from the unenlightening intricacies of the calculation which includes terms of order κ^4, we shall limit our presentation to the discussion of the evaluation of S up to (and including) terms of order κ^2. The (cumbersome) evaluation of the $O(\kappa^4)$ terms was accomplished by a straightforward extension of the techniques used to perform the calculations of the terms of order κ^2.

Throughout, we shall use

$$\left[\left(\frac{1}{\partial_-}\right)^m\right]_{xy}=\frac{1}{(n-1)!}(x-y)^{n-1}\tfrac{1}{2}\epsilon(x-y), \quad n=1,2,3. \quad (35)$$

This is the standard determination made in light-cone quantization (see, e.g., the paper by Cornwall and Jackiw,[11] especially the formula at the end of their Sec. II).

Extracting an infinite constant factor $\det(\partial/\partial x^-)$ we obtain for S, to $O(\kappa^2)$,

$$\det(1+\kappa^2 Q)=1+\kappa^2\,\mathrm{Tr}Q\,. \tag{36}$$

In the calculation to $O(\kappa^4)$, we used

$$\det(1+\kappa^2Q_1+\kappa^4Q_2)=\exp\mathrm{Tr}\ln(1+\kappa^2Q_1+\kappa^4Q_2)$$
$$=\exp\mathrm{Tr}(\kappa^2Q_1+\kappa^4Q_2-\tfrac{1}{2}\kappa^4Q_1{}^2)\,.$$

In Eq. (36),

$$\mathrm{Tr}Q=\int dx\,Q_{ll}(x)\,, \tag{37}$$

and

$$Q_{ls}(x,z)=\frac{1}{\kappa^2}\int\left(\frac{1}{\partial_-}\right)_{yz}dy\left[\frac{1}{4}\frac{k'(x)}{k(x)}\delta_{ls}\delta(x-y)-\kappa^2(q_lM_{sk}q_k')_x\delta(x-y)+\frac{1}{4}\frac{\delta\ln k(x)}{\delta q_s(y)}q_l'(x)-\frac{1}{4}\frac{\delta\ln k(y)}{\delta q_l(x)}q_s'(y)\right.$$
$$\left.+\frac{1}{2\kappa^2}(2l+\tfrac{1}{2})\int dz'\left(\frac{\delta(\partial_-\ln k)_{z'}}{\delta q_m(x)}\frac{\delta k^{1/2}(z')}{\delta q_l(y)}-\frac{\delta(\partial_-\ln k)_{z'}}{\delta q_l(y)}\frac{\delta k^{1/2}(z')}{\delta q_m(x)}\right)\right], \tag{38}$$

obtained from Eq. (29) by keeping only $O(\kappa^2)$ terms and relabeling, e.g., $z^-\to z$.

Note that Q_s itself is of order κ^0.

Observing that

$$\int dx\frac{k'(x)}{k(x)}=\int dx\frac{\partial}{\partial x}\ln k=0\,,$$
$$\int dx q_lM_{lk}q_k'=\int dx\frac{q\cdot q'}{1+\kappa^2q^2}=\tfrac{1}{2}\int dx\partial_x\ln(1+\kappa^2q^2)=0\,, \tag{39}$$

we find

$$\mathrm{Tr}Q=\frac{1}{\kappa^2}\int dxdy\left(\frac{1}{\partial_-}\right)_{yx}\left[\frac{1}{2}\frac{\delta\ln k(x)}{\delta q_l(y)}q_l'(x)+\frac{1}{\kappa^2}(2l+\tfrac{1}{2})\int dz\frac{\delta(\partial_-\ln k)_z}{\delta q_l(x)}\frac{\delta}{\delta q_l(y)}k^{1/2}(z)\right], \tag{40}$$

where we have used the antisymmetry of $(1/\partial_-)_{yx}$ which follows from Eq. (35). To order κ^2, we have

$$\frac{\delta}{\delta q_l(y)}k^{1/2}(z)=\frac{1}{2}\frac{\delta\ln k(z)}{\delta q_l(y)}$$
$$=\kappa^2\frac{\partial}{\partial y}\left[\left(\frac{1}{\partial_-}\right)^2_{zy}q_l'(y)\right]. \tag{41}$$

Using Eq. (41) and similar identities for $\delta\ln k(x)/\delta q_l(y)$ and $\delta(\partial_-\ln k)_z/\delta q_m(x)$, we find

$$\mathrm{Tr}Q=\int dxdy\left(\frac{1}{\partial_-}\right)_{yx}\left\{\frac{\partial}{\partial y}\left[\left(\frac{1}{\partial_-}\right)^2_{xy}q_l'(y)\right]\right\}q_l'(x)$$
$$+2(2l+\tfrac{1}{2})\int dxdydz\left(\frac{1}{\partial_-}\right)_{yx}\frac{\partial}{\partial y}\left[\left(\frac{1}{\partial_-}\right)^2_{zy}q_l'(y)\right]$$
$$\times\frac{\partial}{\partial x}\left[\left(\frac{1}{\partial_-}\right)_{zx}q_l'(x)\right]. \tag{42}$$

A partial integration in y in both terms and an integration in z in the second term produces the expression

$$\mathrm{Tr}Q=-\int dx\,q_l'(x)q'(x)\left(\frac{1}{\partial_-}\right)^2_{xx}$$
$$-2(2l+\tfrac{1}{2})\int dx\,q_l'(x)\frac{\partial}{\partial x}\left[\left(\frac{1}{\partial_-}\right)^3_{xx}q_l'(x)\right]. \tag{43}$$

But in view of Eq. (35),

$$\left(\frac{1}{\partial_-}\right)^k_{xx}=0,\quad k=2,3$$

so that $\mathrm{Tr}Q$ vanishes. This completes our promised presentation of the proof that

$$|S|=\det\left|\frac{\partial}{\partial x^-}\right|$$

to order κ^2.

SUMMARY

We have presented a method of calculating the functional measure for quantum gravity reduced to independent variables in the light-cone gauge, studied recently by Aragone and Chela Flores, Kaku, and Scherk and Schwarz.[4] This method is based on recent results by one of the present authors (P.S.)[9] (independently found also by Yabuki[10]) for the functional measure in the Hamiltonian-type path integral for an arbitrary Hamiltonian system containing second-class constraints.

Our result was then used to extract information about the functional measure for quantum gravity in its standard formulation. The resulting expression was found to agree with the result of Fradkin and Vilkovisky.[3] It therefore disagrees with the result of Faddeev and Popov.[2]

For objections to the result of Faddeev and Popov, as well as a discussion of the relevance of the problem of measure, we refer the reader to Ref. 3.

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