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Broken conformal invariance

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The equal-time commutator algebra of the broken conformal group is studied when spontaneous breakdown of conformal invariance occurs. The nonsymmetric terms in the algebra are characterized by field-current-type identities for the divergence of the dilatation current and the dilaton field. As a consequence of this result, the generators of the group can always be redefined in such a way as to satisfy the closed Lie algebra associated with conformal invariance. The resulting expressions are interpreted on the Lagrangian level by studying the effect of a field redefinition on the dilatation current when this is only partially conserved. Our discussion is mostly classical since it neglects the existence of the anomalies. With these limitations the results are illustrated by explicit calculations for several model Lagrangians.

I. INTRODUCTION

The equal-time commutator algebras satisfied by the generators of internal symmetries have proved very useful for particle physics even when the corresponding currents are only partially conserved. Conformal invariance, regarded as an approximate symmetry of physics, differs from internal symmetries in the fact that the charges associated with conformal currents do not satisfy the Lie-algebra relations of the conformal group when the currents are not conserved. The deviations of the equal-time commutators of the broken conformal group from their symmetry values can be determined in a model-independent manner and turn out to be expressible as simple moments of the divergence of the dilatation current. By combining this result with the hypothesis of scalar-meson dominance of the trace of the energy-momentum tensor, one is naturally led to study the effect of field-current identities on the algebra of the broken conformal group.[1]

This type of questions is relevant in discussing the symmetry limit of conformal invariance when the symmetry is realized in the Goldstone manner. In the Goldstone realization of the symmetry, the breaking mechanism is induced only by a nonvanishing vacuum expectation value of a field operator and leads to a theory with a massless dilaton, massive particles, and, eventually, dimensional coupling constants. We will show that there exists a correlation among the various dimensional quantities which results necessarily in the field equation

$$\theta_{\mu\mu}(x)=c\Box B(x)\,, \tag{1.1}$$

where $\theta_{\mu\mu}(x)$ stands for the trace of the Callan-Coleman-Jackiw[2] (CCJ) tensor, c is a constant, and $B(x)$ is the "dilaton" field.

Conversely, the condition (1.1) is sufficient to guarantee the existence of an underlying theory in which conformal invariance is only spontaneously broken. This question is also examined from the Lagrangian viewpoint at the canonical level. In two previous papers[3] (hereafter referred to as I and II) we used the Euler dimensional relation to relate the trace of the energy-momentum tensor to the breaking terms in a given Lagrangian and studied the effect of a field redefinition on the Lagrangian, focusing only on the dilatation transformation. The treatment of canonical scale invariance presented in I is mostly classical. The same is true of the following discussion about the algebra of broken conformal invariance. On the quantum level our premises, and therefore Eq. (1.1), are invalidated by the existence of anomalies which destroy the naive conformal invariance of the theory as well as the equal-time commutation relations of the conformal generators. However, anomalies have been studied mostly in renormalized perturbation theory and mostly for scale invariance alone, leading to the well-known developments of renormalization-group techniques. From the algebraic viewpoint advocated in this paper, what is needed is the general structure of the anomalies which could be used in determining the modification of the equal-time commutation relations of the conformal generators. The effect of the anomalies in such a general form in unknown at present, so any discussion of the full algebra of the conformal group only makes sense on the classical level. Our discussion is a first step in the direction of a deeper understanding of the Goldstone realization of conformal symmetry on the quantum level. A clue on how to incorporate the anomalies in our scheme was implicitly offered in II, at least for the spontaneously broken theory of a massless self-interacting scalar field. For this field theory, the difficulty of ultraviolet divergences, which is the source of the anomalies, was overcome by the following prescription: The bare coupling constant of the theory goes to zero at the same time as the cutoff goes to infinity. It was shown that this procedure results in the elimination of the divergences while preserving the time independence of the dilatation generator. It should be noted that in spite of the above limiting procedure the Bethe-Salpeter amplitude for a two-particle scattering state does not vanish in the pair approximation, thus yielding a nontrivial S matrix. More important for our purposes, it was also found that, in the presence of spontaneous symmetry breaking, the *massive asymptotic* fields do not respond to scale transformation in the same way as the Heisenberg operators, but obey the dimensional transformation law previously introduced by Aurilia, Takahashi, and Umezawa.[3] The Goldstone boson transforms inhomogeneously, which is a feature of spontaneously broken scale symmetry. The dimensional transformation reduces to a scale transformation in the absence of dimensional parameters and reflects the dimensional consistency of the field equations. Thus "dimensional invariance" is exact and can never be violated. The results of I, with some modifications, can be extended to the full conformal group, and this is the main object of this paper. As for the results of II, we can only speculate that the anomalies inherent in any "conformally invariant" quantum field theory are responsible for the rearrangement of the original scale invariance into the dimensional invariance at the level of the physical Hilbert space. Should this conjecture turn out to be correct, one would have, in principle, the possibility of evaluating the deviations of the equal-time commutation relations due to the existence of anomalies. It would be interesting to interpret the results of I and II in terms of standard renormalization-group techniques and the Callan-Symanzik equations, thus establishing a link between the algebraic approach and the quantum field-theoretical approach to conformal invariance. These problems clearly require a separate investigation.

The plan of the paper is as follows. In Sec. II we study the equal-time commutator algebra of the broken conformal group. The structure of the breaking terms follows from relativistic invariance alone, without the use of canonical field commutators. Thus one avoids altogether the problem of scale-dependent renormalization constants. In Sec. III the condition (1.1) is enforced and we find that the deviations of the equal-time commutators of the broken conformal group from their symmetry values can be made to vanish by a redefinition of the 15 generators of the group. In Sec. IV we consider the effect of a field redefinition on the dilatation current in Lagrangian field theory.

Here we refer to that class of field theories for which the CCJ tensor can be constructed and for which the field virial

$$V_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi}(\delta_{\mu\nu} l - iS_{\mu\nu})\varphi \qquad (1.2)$$

is identically expressible as a total divergence.[2] This is known to be the case for all renormalizable theories. It is also the case for the conformally invariant theories of the Goldstone type considered, for instance, by Salam and Strathdee,[4] provided that one takes the Goldstone boson as a field of canonical dimensionality. This is indeed the situation envisaged in Eq. (1.1), which is shown to be necessary and sufficient for the existence of a field redefinition leading to a conformally invariant theory. In this resulting theory the symmetry is spontaneously broken. The 15 generators previously introduced, which obey the closed Lie algebra of the conformal group, correspond precisely to this spontaneously broken theory once the field redefinition is implemented.

In Sec. V several model Lagrangians are presented which illustrate the results of the previous sections.

II. THE COMMUTATOR ALGEBRA OF THE BROKEN CONFORMAL GROUP

The 15 generators of the broken conformal group are most conveniently expressed in terms of the energy-momentum tensor $\theta_{\mu\nu}(x)$ introduced by Callan, Coleman, and Jackiw.[2] This tensor is such that

$$\begin{aligned} &\partial_\mu \theta_{\mu\nu}(x) = 0\,, \\ &\theta_{\mu\nu}(x) - \theta_{\nu\mu}(x) = 0\,, \\ &\theta_{\mu\mu}(x) = \partial_\mu D_\mu(x)\,, \end{aligned} \qquad (2.1)$$

where $D_\mu(x)$ is the current associated with dilatation transformations in space-time. One has

$$P_\mu = -\int d\sigma_\lambda(x)\theta_{\lambda\mu}(x)\,, \qquad (2.2)$$

$$M_{\mu\nu} = -\int d\sigma_\lambda(x)[x_\mu\theta_{\lambda\nu}(x) - x_\nu\theta_{\lambda\mu}(x)]\,, \qquad (2.3)$$

$$D = \int d\sigma_\lambda(x) x_\nu \theta_{\lambda\nu}(x)\,, \qquad (2.4)$$

$$K_\mu = \int d\sigma_\lambda(x)[2x_\rho x_\mu \theta_{\lambda\rho}(x) - x^2\theta_{\lambda\mu}(x)]\,, \qquad (2.5)$$

where P_μ and $M_{\mu\nu}$ are the generators of the inhomogeneous Lorentz group while D and K_μ stand for the generators of dilatations and proper conformal transformations. For any Poincaré-invariant theory with fixed dimensional parameters the above generators obey the following commutation relations:

$$[D,P_\mu]=-iP_\mu-i\int d\sigma_\mu(x)\partial_\lambda D_\lambda(x)\,, \tag{2.6}$$

$$[D,M_{\mu\nu}]=i\int d\sigma_\mu(x)x_\nu\partial_\lambda D_\lambda(x) -i\int d\sigma_\nu(x)x_\mu\partial_\lambda D_\lambda(x)\,, \tag{2.7}$$

$$[K_\lambda,P_\mu]=2i\delta_{\mu\lambda}D-2iM_{\lambda\mu}-2i\int d\sigma_\mu(x)x_\lambda\partial_\nu D_\nu(x)\,, \tag{2.8}$$

$$[K_\lambda,M_{\mu\nu}]=i\delta_{\nu\lambda}K_\mu-i\delta_{\mu\nu}K_\nu+2i\int d\sigma_\mu x_\lambda x_\nu\partial_\rho D_\rho -2i\int d\sigma_\nu x_\lambda x_\mu\partial_\rho D_\rho\,, \tag{2.9}$$

$$[D,K_\mu]=iK_\mu-i\int d\sigma_\mu(x)x^2\partial_\lambda D_\lambda(x)\,, \tag{2.10}$$

$$[K_\mu,K_\nu]=2i\int d\sigma_\mu(x)x_\nu x^2\partial_\lambda D_\lambda(x) -2i\int d\sigma_\nu(x)x_\mu x^2\partial_\lambda D_\lambda(x)\,, \tag{2.11}$$

$$[P_\lambda,P_\mu]=0\,, \tag{2.12}$$

$$[P_\lambda,M_{\mu\nu}]=i\delta_{\lambda\nu}P_\mu-i\delta_{\mu\lambda}P_\nu\,, \tag{2.13}$$

$$[M_{\mu\nu},M_{\sigma\rho}]=i(\delta_{\nu\sigma}M_{\rho\mu}+\delta_{\nu\rho}M_{\mu\sigma}+\delta_{\mu\sigma}M_{\nu\rho}+\delta_{\mu\rho}M_{\sigma\nu}). \tag{2.14}$$

In the limit of exact symmetry, i.e. when $\partial_\lambda D_\lambda(x)=0$, the above commutation relations reproduce the closed Lie algebra of the conformal group. Inspection of the above algebra shows that even in the symmetry limit the generators D and K_μ do not commute with P_4. This property together with the explicit appearance of breaking terms in the commutation relations constitute the major differences between the conformal group and internal-symmetry groups. However, the deviations of the equal-time commutators from their symmetry values are model independent and are simply expressed as moments of the divergence of the dilatation current $D_\lambda(x)$. Moreover, any breaking mechanism of conformal invariance leaves the commutators $[D,P_i]$, $[D,M_{ij}]$, $[K_\lambda,P_i]$, $[K_\lambda,M_{ij}]$, $[K_i,D]$, and $[K_i,K_j]$ unchanged. In view of the generality of the commutator algebra of the broken conformal group, it is instructive to outline the method by which we have derived it. We use a simple technique due to Takahashi[5] which enables us to calculate, in general, the commutator of the generator of a given transformation with the generators of the inhomogeneous Lorentz group without using canonical field commutators. The method involves (a) relativistic invariance in the form

$$i[\varphi(x),P_\mu]=\partial_\mu\varphi(x)\,, \tag{2.15}$$

$$i[\varphi(x),M_{\mu\nu}]=(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\varphi(x)\,, \tag{2.16}$$

where $S_{\mu\nu}$ indicates the skew-symmetric spin matrix specifying the spinor or tensor properties of the local operator $\varphi(x)$, i.e.

$$\varphi_\alpha(x)\to\varphi'_\alpha(x')=[\delta_{\alpha\beta}+\tfrac{1}{2}i(S_{\mu\nu})_{\alpha\beta}\epsilon_{\mu\nu}]\varphi_\beta(x) \tag{2.17}$$

when

$$x_\mu\to x'_\mu+\epsilon_{\mu\nu}x_\nu\quad(\epsilon_{\mu\nu}+\epsilon_{\nu\mu}=0)\,,$$

and (b) Schwinger's covariant identity[6]

$$\int d\sigma_\mu(x)\partial_\nu f(x)=\int d\sigma_\nu(x)\partial_\mu f(x) \tag{2.18}$$

which is valid for any function $f(x)$ such that

$$\vec{x}^2 f(x)\to 0$$

when

$$|\vec{x}|\to\infty\,.$$

Thus, for any tensor $F_{\lambda\tau}$, relativistic invariance implies

$$i[F_{\lambda\tau}(x),P_\mu]=\partial_\mu F_{\lambda\tau}(x) \tag{2.19}$$

and

$$i[F_{\lambda\tau}(x),M_{\mu\nu}]=(x_\mu\partial_\nu-x_\nu\partial_\mu)F_{\lambda\tau}(x) +\delta_{\mu\lambda}F_{\nu\tau}(x)-\delta_{\nu\lambda}F_{\mu\tau}(x) +\delta_{\mu\tau}F_{\lambda\nu}(x)-\delta_{\nu\tau}F_{\lambda\mu}(x). \tag{2.20}$$

Assume now for later convenience $F_{\lambda\nu}=F_{\nu\lambda}$, and define the new quantities

$$Q_\mu=\int d\sigma_\lambda(x)F_{\lambda\mu}(x)\,, \tag{2.21}$$

$$Q_{\mu\nu}=\int d\sigma_\lambda(x)x_\mu F_{\lambda\nu}(x)\,, \tag{2.22}$$

$$Q_{\rho\mu\nu}=\int d\sigma_\lambda(x)x_\rho x_\mu F_{\lambda\nu}(x)\,. \tag{2.23}$$

One calculates the general commutation relations of the above quantities with the generators P_μ and $M_{\mu\nu}$ by using Eqs. (2.19)–(2.20) and the identity (2.18). We obtain

$$[Q_\mu,P_\nu]=-i\int d\sigma_\nu\partial_\lambda F_{\lambda\mu}\,, \tag{2.24}$$

$$[Q_\lambda,M_{\mu\nu}]=i\delta_{\nu\lambda}Q_\mu-i\delta_{\mu\lambda}Q_\nu+i\int d\sigma_\mu x_\nu\partial_\rho F_{\rho\lambda}-i\int d\sigma_\nu x_\mu\partial_\rho F_{\rho\lambda}\,, \tag{2.25}$$

$$[Q_{\mu\nu}, P_\rho] = i\delta_{\mu\rho}\int d\sigma_\lambda F_{\lambda\nu} - i\int d\sigma_\rho F_{\mu\nu} - i\int d\sigma_\rho x_\mu \partial_\lambda F_{\lambda\nu}\,, \tag{2.26}$$

$$\begin{aligned}[Q_{\rho\tau}, M_{\mu\nu}] = {} & i\int d\sigma_\mu x_\nu F_{\rho\tau} - i\int d\sigma_\nu x_\mu F_{\rho\tau} \\ & + i\int d\sigma_\lambda(\delta_{\nu\rho}x_\mu F_{\lambda\tau} - \delta_{\mu\rho}x_\nu F_{\lambda\tau} + \delta_{\nu\tau}x_\rho F_{\lambda\mu} - \delta_{\mu\tau}x_\rho F_{\lambda\nu}) \\ & + i\int d\sigma_\mu x_\rho x_\nu \partial_\lambda F_{\lambda\tau} - i\int d\sigma_\nu x_\rho x_\mu \partial_\lambda F_{\lambda\tau}\,,\end{aligned} \tag{2.27}$$

$$[Q_{\rho\mu\nu}, P_\tau] = i\int d\sigma_\lambda(\delta_{\tau\rho}x_\mu F_{\lambda\nu} + \delta_{\tau\mu}x_\rho F_{\lambda\nu}) - i\int d\sigma_\tau(x_\mu F_{\rho\nu} + x_\rho F_{\mu\nu} + x_\rho x_\mu \partial_\lambda F_{\lambda\nu})\,, \tag{2.28}$$

$$\begin{aligned}[Q_{\rho\sigma\tau}, M_{\mu\nu}] = {} & i\int d\sigma_\lambda(\delta_{\nu\rho}x_\sigma x_\mu F_{\lambda\tau} + \delta_{\nu\sigma}x_\rho x_\mu F_{\lambda\tau} - \delta_{\mu\rho}x_\sigma x_\nu F_{\lambda\tau} - \delta_{\mu\sigma}x_\rho x_\nu F_{\lambda\tau} - \delta_{\mu\tau}x_\rho x_\sigma F_{\lambda\nu} + \delta_{\nu\tau}x_\rho x_\sigma F_{\lambda\mu}) \\ & + i\int d\sigma_\mu(x_\sigma x_\nu F_{\rho\tau} + x_\rho x_\nu F_{\sigma\tau} + x_\rho x_\sigma x_\nu \partial_\lambda F_{\lambda\tau}) \\ & - i\int d\sigma_\nu(x_\sigma x_\mu F_{\rho\tau} + x_\rho x_\mu F_{\sigma\tau} + x_\rho x_\sigma x_\mu \partial_\lambda F_{\lambda\tau})\,.\end{aligned} \tag{2.29}$$

The algebra of the broken conformal group, with the exception of the commutators (2.10) and (2.11), follows in a straightforward manner from the above commutation relations once we identify

$$F_{\lambda\mu} = \theta_{\lambda\mu}\,,\quad Q_\mu = -P_\mu\,,\quad Q_{\mu\nu} - Q_{\nu\mu} = -M_{\mu\nu}\,,$$

$$Q_{\mu\mu} = D\,,\quad K_\mu = \delta_{\nu\rho}(2Q_{\rho\mu\nu} - Q_{\rho\nu\mu})\,.$$

This method does not apply to the calculation of the commutators (2.10) and (2.11), as they do not involve the Poincaré generators. However, they also are model-independent and can be derived from Schwinger's equal-time commutation relations among the components of the energy-momentum tensor.[7] Neglecting the presence of Schwinger terms they are

$$[\theta_{44}(x),\, \theta_{4i}(y)]_{x_0=y_0} = [\theta_{ik}(\vec{x}) + \theta_{44}(\vec{y})\delta_{ik}]\partial_k\delta(\vec{x}-\vec{y})\,, \tag{2.30}$$

$$[\theta_{4k}(x),\, \theta_{4i}(y)]_{x_0=y_0} = [\theta_{4i}(\vec{x})\partial_k + \theta_{4k}(\vec{y})\partial_i]\delta(\vec{x}-\vec{y})\,, \tag{2.31}$$

$$[\theta_{44}(x),\, \theta_{44}(y)]_{x_0=y_0} = [\theta_{4k}(\vec{x}) + \theta_{4k}(\vec{y})]\partial_k\delta(\vec{x}-\vec{y})\,, \tag{2.32}$$

and substituting directly into the expression of the commutators $[D, K_\mu]$ and $[K_\mu, K_\nu]$ one verifies the algebraic relations (2.10) and (2.11).[8] The use of the Schwinger commutators in the form (2.30)–(2.32) is supported by the fact that they lead to the same commutation relations of the broken conformal group obtained by the previous technique, and they are certainly valid for Lagrangian field theories with fields of spin $s \le 1$. These are also the field theories to which the Lagrangian formalism of Sec. IV will be applied.

III. THE MODIFIED ALGEBRA

In the following we combine the general algebra derived in the previous section with a specific mechanism of symmetry breaking. We are interested in those dynamical systems for which

$$\partial_\mu D_\mu(x) = \theta_{\mu\mu}(x) = \Box f(x)\,, \tag{3.1}$$

where $f(x)$ is a given scalar quantity. We will see shortly that a condition of the type (3.1) covers the physically interesting case of spontaneously broken conformal invariance. In such circumstances, since the energy-momentum tensor is not unique, it is natural to introduce the tensor

$$T_{\mu\nu}(x) = \theta_{\mu\nu}(x) - \tfrac{1}{3}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)f(x) \tag{3.2}$$

satisfying

$$\partial_\mu T_{\mu\nu}(x) = 0\,, \tag{3.3}$$

$$T_{\mu\nu}(x) - T_{\nu\mu}(x) = 0\,, \tag{3.4}$$

$$T_{\mu\mu}(x) = 0\,. \tag{3.5}$$

It is then a direct consequence of Schwinger's formula (2.18) that the generators of the Poincaré group are not affected by the extra term in (3.2), i.e.,

$$\begin{aligned}\tilde{P}_\mu &= -\int d\sigma_\lambda(x)T_{\lambda\mu}(x) \\ &= -\int d\sigma_\lambda(x)\theta_{\lambda\mu}(x) \\ &= P_\mu\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\tilde{M}_{\mu\nu} &= -\int d\sigma_\lambda(x)[x_\mu T_{\lambda\nu}(x) - x_\nu T_{\lambda\mu}(x)] \\ &= M_{\mu\nu}\,,\end{aligned} \tag{3.7}$$

while the generators of scale and conformal transformations are modified according to

$$\tilde{D}=\int d\sigma_\lambda(x)T_{\lambda\nu}(x)x_\nu$$
$$=D-\int d\sigma_\lambda(x)\partial_\lambda f(x) \qquad (3.8)$$

and

$$\tilde{K}_\alpha=\int d\sigma_\lambda(x)[2x_\alpha x_\nu T_{\lambda\nu}(x)-x^2T_{\lambda\alpha}(x)]$$
$$=K_\alpha+2\int d\sigma_\lambda(x)[\delta_{\lambda\alpha}f(x)-x_\alpha\partial_\lambda f(x)]\,. \qquad (3.9)$$

One also finds

$$2x_\mu\partial_\lambda\tilde{D}_\lambda(x)=\partial_\lambda\tilde{K}_{\lambda\mu}(x)=0\,, \qquad (3.10)$$

where $\tilde{D}_\lambda(x)$ and $\tilde{K}_{\lambda\mu}(x)$ are the modified dilatation and conformal currents corresponding to (3.8) and (3.9).

In terms of the new generators P_μ, $M_{\mu\nu}$, $\tilde{D}$, and $\tilde{K}_\alpha$, the equal-time commutator algebra of the broken conformal group reduces to the closed Lie algebra associated with conformal invariance,

$$[\tilde{D},P_\mu]=-iP_\mu,$$
$$[\tilde{D},M_{\mu\nu}]=0,$$
$$[\tilde{K}_\lambda,P_\mu]=2i\delta_{\mu\lambda}\tilde{D}-2iM_{\lambda\mu}\,, \qquad (3.11)$$
$$[\tilde{K}_\lambda,M_{\mu\nu}]=i\delta_{\nu\lambda}\tilde{K}_\mu-i\delta_{\mu\lambda}\tilde{K}_\nu,$$
$$[\tilde{D},\tilde{K}_\mu]=i\tilde{K}_\mu,$$
$$[\tilde{K}_\mu,\tilde{K}_\nu]=0;$$

of course, the commutator algebra of the Poincaré group is unchanged. The above result follows from the fact that the commutator algebra of Sec. II was derived solely from locality and relativistic invariance. Thus one can merely replace D, K_α, D_μ with $\tilde{D},\tilde{K}_\alpha,\tilde{D}_\mu$ in Eqs. (2.6)–(2.14) and use the relation

$$T_{\lambda\lambda}(x)=\partial_\lambda\tilde{D}_\lambda(x)=0 \qquad (3.12)$$

to obtain the Lie algebra (3.11). Of course the explicit calculation of the commutators reproduces the same result. Thus, for instance, one finds

$$[\tilde{K}_\alpha,P_\mu]=[K_\alpha,P_\mu]+2\int d\sigma_\lambda(x)[\delta_{\alpha\lambda}f(x)-x_\alpha\partial_\lambda f(x),P_\mu]$$
$$=2i\delta_{\mu\alpha}D-2iM_{\alpha\mu}-2i\int d\sigma_\mu(x)x_\alpha[\partial_\lambda D_\lambda(x)-2i\Box f(x)]-2i\delta_{\mu\alpha}\int d\sigma_\lambda(x)\partial_\lambda f(x)$$
$$=2i\delta_{\mu\alpha}\tilde{D}-2iM_{\alpha\mu}\,, \qquad (3.13)$$

and analogously for the remaining commutators of the algebra (3.11). From a physical viewpoint, the most interesting specialization of the condition (3.1) consists in setting

$$f(x)=cB(x)\,, \qquad (3.14)$$

where c is a constant and $B(x)$ is a spinless field. In the next section we will see that the field $B(x)$ can be taken as the interpolating field of the physical Goldstone boson with canonical dimensionality. The shifted field $\beta(x)=B(x)+c$ has a nonvanishing vacuum expectation value and induces the spontaneous breakdown of scale invariance. The new generators of the conformal group given by Eqs. (3.6)–(3.9) are well defined since they are expressed in terms of the physical field $B(x)$, which vanishes asymptotically.[9] The presence of a linear term in the expression for $T_{\mu\nu}(x)$ also suggests that the situation (3.14) corresponds to spontaneous breakdown of conformal invariance. This linear term leads to the matrix element

$$\langle 0|T_{\mu\nu}|B(q)\rangle=(2\pi)^{-3/2}(2q_0)^{-1/2}\tfrac{1}{3}c(q^2\delta_{\mu\nu}-q_\mu q_\nu)\,, \qquad (3.15)$$

which is a relation analogous to that defining the pion decay constant in chiral dynamics. However, the occurrence of spontaneous symmetry breakdown follows more precisely from the spectral representation

$$\langle 0|[B(x),T_{\mu\nu}(y)]|0\rangle=\int d\kappa^2[\rho_1(\kappa^2)\delta_{\mu\nu}+\rho_2(\kappa^2)\partial_\mu\partial_\nu]$$
$$\times\Delta(x-y;\kappa^2)\,. \qquad (3.16)$$

Since $T_{\mu\nu}(x)$ is conserved and traceless one finds

$$\rho_1(\kappa^2)=0,$$
$$\rho_2(\kappa^2)=\tfrac{1}{3}\bar{c}\,\delta(\kappa^2)\,, \qquad (3.17)$$

where $\bar{c}$ is an arbitrary constant.

Thus,

$$\langle 0|[B(x),T_{\mu\nu}(y)]|0\rangle=\tfrac{1}{3}\bar{c}\,\partial_\mu\partial_\nu D(x-y)\,, \qquad (3.18)$$

which signals the existence of a massless boson so long as $\bar{c}\neq 0$. But from (3.18) and the definition

(3.8) we obtain

$$\tilde{c} = \langle 0 | [B(x), \tilde{D}] | 0 \rangle$$
$$= -c \int d\sigma_\lambda(y) \langle 0 | [B(x), \partial_\lambda B(y)] | 0 \rangle$$
$$= ic$$
$$\neq 0. \tag{3.19}$$

Thus the symmetry corresponding to the Lie algebra (3.11) is spontaneously broken. We can summarize the argument of this section by stating that whenever the breaking mechanism of conformal invariance is such that the condition (3.1) is satisfied, it is possible to "renormalize" the theory in such a way as to absorb the effect of the deviations of the commutator algebra from its symmetric form into the expression of the generators of the conformal group. The two sets of generators P_μ, $M_{\mu\nu}$, D, K_α, and $P_\mu, M_{\mu\nu}$, $\tilde{D}$, $\tilde{K}_\alpha$ satisfy the commutator algebra (2.1)–(2.14) and (3.11), respectively. In the former case the breakdown of conformal invariance is explicitly represented by the equation

$$\theta_{\mu\mu}(x) = \partial_\mu D_\mu(x) = c\Box B(x), \tag{3.20}$$

while in the latter case one has

$$T_{\mu\mu}(x) = \partial_\mu \tilde{D}_\mu(x) = 0; \tag{3.21}$$

but

$$\langle 0 | [B(x), \tilde{D}] | 0 \rangle \neq 0, \tag{3.22}$$

so that conformal invariance is only spontaneously broken.

The condition (3.20) is precisely a remainder of this underlying symmetry in the theory in which dimensional parameters appear explicitly. Indeed, in the following section we will see that the various breaking terms in a canonical conformally invariant Lagrangian—arising because of the asymmetry of the vacuum—conspire in such a way as to lead to the field equation (3.20) involving the Goldstone boson and the divergence of the dilatation current.

IV. LAGRANGIAN FORMALISM

On the Lagrangian level, it is usually assumed that only mass terms break conformal invariance. In what follows we relax this restriction and consider the contribution to the trace of the energy-momentum tensor due to all possible dimensional parameters g_i appearing in the Lagrangian.

Thus we write

$$\mathcal{L} = \mathcal{L}(\vec{\varphi}, \partial_\mu \vec{\varphi}, g^i), \tag{4.1}$$

where we have concisely assembled all the fields in the theory into a vector $\vec{\varphi}$. The dimensions are set as follows:

$$[\varphi^i] = L^{-l^i}, \quad [g^i] = L^{-\lambda^i}.$$

Then we define the *scale deficiency*[3,10]

$$\rho(x) = -\sum_i \lambda^i g^i \frac{\partial \mathcal{L}(x)}{\partial g^i} \tag{4.2}$$

and use Euler's dimensional relation

$$4\mathcal{L} - l\frac{\partial \mathcal{L}}{\partial \vec{\varphi}} \cdot \vec{\varphi} - (l+1)\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\varphi}} \cdot \partial_\mu \vec{\varphi} + \rho = 0 \tag{4.3}$$

to derive

$$\theta_{\mu\mu}(x) = \rho(x) \tag{4.4}$$

since[2,3]

$$\partial_\mu D_\mu(x) = (l+1)\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\varphi}} \cdot \partial_\mu \vec{\varphi} + l\frac{\partial \mathcal{L}}{\partial \vec{\varphi}} \cdot \vec{\varphi} - 4\mathcal{L}. \tag{4.5}$$

Thus, according to the scheme of the previous section, whenever the dynamics of the system (4.1) is such that

$$-\sum_i \lambda^i g^i \frac{\partial \mathcal{L}(x)}{\partial g^i} = \Box f(x) \tag{4.6}$$

we construct the new tensor

$$T_{\mu\nu}(x) = \theta_{\mu\nu}(x) - \tfrac{1}{3}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu) f(x), \tag{4.7}$$

which is conserved, symmetric and traceless.

We emphasize that in order to satisfy the relation (4.6) it is essential to take into account not only the contributions of masses, but those of all dimensional parameters in the Lagrangian (4.1). Moreover, in writing $T_{\mu\nu}(x)$ in terms of $\theta_{\mu\nu}(x)$ as in Eq. (4.7), we have implicitly assumed that the field virial (1.2) is identically equal to a total divergence.[2] It is known that the virial condition is necessary for conformal invariance and is sufficient for the explicit construction of $\theta_{\mu\nu}(x)$ itself.[2] In the following we shall restrict our considerations to theories involving fields of spin $s \leq 1$, without derivative couplings and for which the kinetic term is of the standard form. For such theories the virial condition holds true as an identity. On the other hand, the condition (4.6) is of dynamical nature since it is strictly a consequence of the field equations. The property

$$T_{\mu\mu}(x) = 0 \tag{4.8}$$

cannot be interpreted as a necessary and sufficient condition for conformal invariance since the scale deficiency, i.e. the trace of the "true" energy-momentum tensor $\theta_{\mu\nu}(x)$, is nonvanishing. Nevertheless, in view of the properties of $T_{\mu\nu}(x)$ shown in Sec. III, it is meaningful to inquire under what conditions a given Lagrangian can be redefined so that $T_{\mu\nu}(x)$ can be interpreted as the CCJ tensor of the new theory. Therefore, consider the operation

R consisting of a field redefinition

$$R:\ \vec{\psi}(x) = \vec{\psi}[\vec{\varphi}(x), g^i] \tag{4.9}$$

on the system (4.1), so that

$$R[\mathcal{L}(x)] = \mathcal{L}(\vec{\psi}(x), \partial_\mu \vec{\psi}(x), \eta^i). \tag{4.10}$$

Correspondingly we set

$$[\psi^i] = L^{-\kappa^i}, \quad [\eta^i] = L^{-t^i}.$$

We require that the field redefinition is such that the virial condition is identically satisfied for the redefined Lagrangian as well. Thus, the CCJ tensor $\theta'_{\mu\nu}(x)$ exists and is such that

$$\theta'_{\mu\mu}(x) = \rho'(x) = -\sum_i t^i \eta^i \frac{\partial \mathcal{L}(x)}{\partial \eta^i} \tag{4.11}$$

and the usual connection of $\theta'_{\mu\nu}(x)$ with scale and conformal transformations is established for the new Lagrangian as well. We recall from I that the canonical energy-momentum tensor $T^c_{\mu\nu}(x)$ is invariant under the field redefinition, i.e.,

$$R[T^c_{\mu\nu}(x)] = T^{c\prime}_{\mu\nu}(x), \tag{4.12}$$

where the primed quantity is calculated from the redefined Lagrangian. However,

$$R[\rho(x)] \neq \rho'(x) \tag{4.13}$$

and more generally

$$R[\theta_{\mu\nu}(x)] \neq \theta'_{\mu\nu}(x). \tag{4.14}$$

The Euler relation is applicable to both theories and gives

$$\rho(x) - l\frac{\partial \mathcal{L}}{\partial \vec{\varphi}} \cdot \vec{\varphi} - l\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\varphi}} \cdot \partial_\mu \vec{\varphi} = T^c_{\mu\mu}(x), \tag{4.15}$$

$$\rho'(x) - k\frac{\partial \mathcal{L}}{\partial \vec{\psi}} \cdot \vec{\psi} - k\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\psi}} \cdot \partial_\mu \vec{\psi} = T^{c\prime}_{\mu\mu}(x). \tag{4.16}$$

In writing Eqs. (4.15), (4.16) we have used the standard definition of the canonical momentum-energy tensor. Using the equation of motion and the invariance of $T^c_{\mu\nu}(x)$ we obtain the general relationship between the two theories

$$\rho - \partial_\mu\left(l\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\varphi}} \cdot \vec{\varphi}\right) = \rho' - \partial_\mu\left(k\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\psi}} \cdot \vec{\psi}\right). \tag{4.17}$$

If the scale deficiency $\rho'(x)$, calculated from the redefined Lagrangian, turns out to be vanishing, then necessarily

$$\rho(x) = \partial_\mu\left(l\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\varphi}} \cdot \vec{\varphi} - k\frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\psi}} \cdot \vec{\psi}\right). \tag{4.18}$$

This agrees with the expression of $\rho(x)$ given in I.

Thus, when the redefined Lagrangian is conformally invariant $\rho(x)$ must be of the form

$$\rho(x) = \partial_\mu \chi_\mu(x) \tag{4.19}$$

where $\chi_\mu(x)$ is a local vector quantity. The more restrictive form

$$\rho(x) = \Box f(x) \tag{4.20}$$

follows from the requirements that the tensor $T_{\mu\nu}(x)$ be symmetric since this property is essential in deriving the closed Lie algebra of the conformal group in terms of the modified generators introduced in Sec III. Moreover, the condition (4.19) is necessary but not sufficient to guarantee the existence of a field redefinition leading to a conformally invariant Lagrangian. However, as one might expect, the further restriction

$$\rho(x) = c\Box B(x) \tag{4.21}$$

provides a necessary and sufficient condition for the field redefinition

$$\beta(x) = B(x) + c \tag{4.22}$$

to lead to a conformally invariant Lagrangian for which $\rho'(x) = 0$. In Sec. III we have observed that the symmetry underlying the closed Lie algebra (3.11) is spontaneously broken in the sense that

$$i\langle 0|[B(x), \tilde{D}]|0\rangle = -c, \tag{4.23}$$

which also means that the scale dimension of $B(x)$ under the action of $\tilde{D}$ is not well defined. This is a a general feature of spontaneously broken scale invariance. However, the shifted field $\beta(x)$ responds to the action of $\tilde{D}$ according to

$$i[\beta(x), \tilde{D}] = -(x\cdot\partial + 1)\beta(x) \tag{4.24}$$

and has therefore the canonical dimension usually assigned to Bose fields. Indicating by R_c the change of variable (4.22), we know that

$$R_c[\theta_{\mu\nu}(x)] \neq \theta'_{\mu\nu}(x). \tag{4.25}$$

However, we have

$$R_c[T_{\mu\nu}(x)] = \theta'_{\mu\nu}(x) \tag{4.26}$$

with

$$\theta'_{\mu\mu}(x) = 0. \tag{4.27}$$

This follows from the definition

$$T_{\mu\nu}(x) = \theta_{\mu\nu}(x) - \tfrac{1}{3}c(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B(x) \tag{4.28}$$

with the explicit expression of $\theta_{\mu\nu}(x)$, i.e.,

$$T_{\mu\nu}(x) = T^s_{\mu\nu}(x) - \sum_{\text{spin-0 fields} \neq B(x)} \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\vec{\varphi}^2(x) - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B^2(x) - \tfrac{1}{3}c(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B(x), \tag{4.29}$$

where $T^s_{\mu\nu}(x)$ is the symmetric canonical energy-momentum tensor. The only terms affected by the change of variable (4.22) are the last two terms on the right-hand side of (4.29), and one finds

$$T_{\mu\nu}(x)=T^{s}_{\mu\nu}(x)-\sum_{\text{spin-0 fields}}\tfrac{1}{6}(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\vec{\varphi}^2(x)\,, \tag{4.30}$$

where the sum includes now the field $\beta(x)$ itself. This is indeed the CCJ energy-momentum tensor of the redefined Lagrangian. Since $T_{\mu\nu}(x)$ is traceless, the new theory is conformally invariant. Conversely, suppose that under R_c a given Lagrangian becomes conformally invariant, i.e.,

$$\theta'_{\mu\mu}(x)=0\,. \tag{4.31}$$

Then setting $\beta(x)=B(x)+c$ in $\theta'_{\mu\nu}(x)$ one finds necessarily

$$\theta_{\mu\nu}(x)-\theta'_{\mu\nu}(x)=\tfrac{1}{3}c(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)B(x) \tag{4.32}$$

and therefore

$$\rho(x)=c\,\Box B(x)\,.$$

In the spontaneously broken theory in which the generators $\tilde{D}$ and $\tilde{K}_\alpha$ are defined the field $\beta(x)$ possesses canonical dimensionality and has a nonvanishing vacuum expectation value. The field $\beta(x)$ is related to the Goldstone field σ introduced by Salam and Strathdee.[4] The σ field has anomalous transformation properties under scale and conformal transformations,

$$\begin{aligned}\delta\sigma(x)&=-c-x\cdot\partial\sigma(x)\,,\\ \delta_\mu\sigma(x)&=-(2x_\mu x_\lambda-x^2\delta_{\mu\lambda})\partial_\lambda\sigma(x)-2cx_\mu\,.\end{aligned} \tag{4.33}$$

Thus, if we define

$$B(x)=c\left[\exp\left(\frac{\sigma(x)}{c}\right)-1\right] \tag{4.34}$$

we can take $B(x)$ as the interpolating field for the Goldstone boson in the theory for which the basic condition

$$\rho(x)=c\,\Box B(x) \tag{4.35}$$

holds true. The effective substitution (4.34) in the Lagrangian density will lead to a spontaneously broken theory in which dimensional parameters may explicitly appear, owing to the anomalous behavior of the σ field under scale and conformal transformations. In the following section we will give some explicit examples of this mechanism. We can summarize the results of this section as follows. We have given an interpretation, in the framework of Lagrangian field theory, of the tensor $T_{\mu\nu}(x)$ introduced in Sec. III. The basic property

$$\theta_{\mu\mu}(x)=\rho(x) \tag{4.36}$$

when (4.6) is verified becomes the traceless condition

$$T_{\mu\mu}(x)=0\,. \tag{4.37}$$

Equations (4.36) and (4.37) are nothing but different ways of writing the Euler dimensional relation (4.3). Thus the algebra of the broken conformal group and the closed Lie algebra derived in Sec. II reflect first of all the dimensional consistency of the theory. However, while Eq. (4.36) alone specifies the breaking mechanism of conformal invariance, Eq. (4.37) instructs us on how to absorb the breaking terms into a redefinition of the generators of the conformal group. Under the more restrictive condition (4.35), Eq. (4.37) also tells that the breaking terms in a given Lagrangian are "spurious" and can be gauged away by shifting the field variables. The constant c, analogous to the pion decay constant of chiral dynamics, signals the spontaneous breakdown of conformal invariance in a world which is otherwise dimensionless. The condition

$$\rho(x)=c\,\Box B(x)$$

has been shown to be necessary and sufficient in order for the field redefinition

$$\beta(x)=B(x)+c$$

to lead to a conformally invariant Lagrangian. The CCJ tensor associated with the new Lagrangian is precisely the tensor $T_{\mu\nu}(x)$ expressed in terms of the new variables.

The following section is devoted to illustrate the above considerations for several Lagrangian field theories.

V. EXAMPLES

It is possible to include massive particles and dimensional coupling constants in a scale-invariant theory by treating scale invariance as a spontaneously broken symmetry.[4] This is achieved by introducing a scalar field $\sigma(x)$ with the anomalous transformation laws (4.33) so that the quantity $c\,\exp[\sigma(x)/c]$ has canonical dimensionality and a nonvanishing vacuum expectation value. Thus the conformally invariant description of a massive scalar field is

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}(\partial_\mu\sigma)^2e^{2\sigma/c}-\tfrac{1}{2}m^2\varphi^2e^{2\sigma/c}\,, \tag{5.1}$$

with $\sigma(x)$ the dilaton field.

In order to apply the formalism of the previous section to the above Lagrangian we introduce the interpolating field for the Goldstone boson

$$B(x)=c\,\exp[\sigma(x)/c-1] \tag{5.2}$$

and rewrite (5.1) as

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}m^2\varphi^2 - (m^2/c)\varphi^2 B - \tfrac{1}{2}(m^2/c^2)\varphi^2 B^2 . \quad (5.3)$$

This model Lagrangian was considered in I, and one finds at once

$$\rho(x) = -\sum_i \lambda^i g^i \frac{\partial \mathcal{L}}{\partial g^i} = c\,\Box B(x) \quad (5.4)$$

because of the field equations. Clearly, the form (5.4) of the scale deficiency is a remainder of the underlying invariance of the original theory. Conformal invariance is manifest by rewriting (5.3) as

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{1}{2}(\partial_\mu \beta)^2 - \tfrac{1}{2}(m^2/c^2)\beta^2\varphi^2, \quad (5.5)$$

where

$$\beta(x) = B(x) + c . \quad (5.6)$$

The scale deficiency of the system (5.5) vanishes:

$$\rho'(x) = 0 . \quad (5.7)$$

The CCJ tensors of the two Lagrangians (5.3) and (5.5) are given

$$\theta_{\mu\nu}(x) = T^c_{\mu\nu}(x) - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\varphi^2 - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B^2 , \quad (5.8)$$

$$\theta'_{\mu\nu}(x) = T^{c\prime}_{\mu\nu}(x) - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\varphi^2 - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\beta^2 , \quad (5.9)$$

and the values of their traces are consistently given by (5.4) and (5.7). In agreement with the general discussion of the previous section, one finds that

$$R_c[T^c_{\mu\nu}(x)] = T^{c\prime}_{\mu\nu}(x) \quad (5.10)$$

and in particular

$$T^c_{\mu\mu}(x) = \tfrac{1}{2}\Box\varphi^2 + \tfrac{1}{2}\Box B^2 + m^2\varphi^2 + (m^2/c)\varphi^2 B \to \tfrac{1}{2}\Box\varphi^2 + \tfrac{1}{2}\Box\beta^2 = T^{c\prime}_{\mu\mu}(x) . \quad (5.11)$$

However,

$$R_c[\theta_{\mu\nu}(x)] \neq \theta'_{\mu\nu}(x) . \quad (5.12)$$

Rather, one easily checks that

$$R_c[T_{\mu\nu}(x)] = \theta'_{\mu\nu}(x) , \quad (5.13)$$

where

$$T_{\mu\nu}(x) = \theta_{\mu\nu}(x) - \tfrac{1}{3}c(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B(x) . \quad (5.14)$$

Thus we can also take the tensor $T_{\mu\nu}(x)$, expressed in terms of the σ field via Eq. (5.2), as the correct energy-momentum tensor of the original Lagrangian (5.1). The generators P_μ, $M_{\mu\nu}$, $\tilde{D}$, and $\tilde{K}_\alpha$ are given by Eqs. (3.6)–(3.9) with $f(x) = cB(x)$ and do satisfy the Lie algebra of the conformal group. As anticipated in Sec. III, one also finds

$$\langle 0|[B(x), \tilde{D}]|0\rangle = ic \neq 0 , \quad (5.15)$$

which indicates the spontaneous nature of the symmetry violation. It is clear that all conformally invariant Lagrangian field theories constructed by the Salam-Strathdee method will meet our condition (4.35) by performing the field redefinition (4.34). Thus the Lagrangian of a free massive fermion field can be made conformally invariant by writing

$$\mathcal{L} = -\bar{\psi}\gamma\cdot\partial\psi - m\bar{\psi}\psi e^{g\sigma/m} - \tfrac{1}{2}(\partial_\mu\sigma)^2 e^{2(g/m)\sigma} \quad (5.16)$$

with g an arbitrary constant.

The definition

$$B(x) = (m/g)[e^{(g/m)\sigma} - 1] = \beta(x) - m/g \quad (5.17)$$

leads to the simple σ-model-type Lagrangian with Yukawa coupling

$$\mathcal{L} = -\bar{\psi}\gamma\cdot\partial\psi - m\bar{\psi}\psi - \tfrac{1}{2}(\partial_\mu B)^2 - g\bar{\psi}\psi B . \quad (5.18)$$

This model has been studied in Refs. (11 and 3), where details can be found. We limit ourselves here to observing that the scale deficiency is simply given by

$$\rho(x) = m\bar{\psi}\psi = (m/g)\Box B(x) , \quad (5.19)$$

where $B(x)$ is the Goldstone boson.

The tensor $T_{\mu\nu}(x)$ is

$$T_{\mu\nu}(x) = -\tfrac{1}{4}\bar{\psi}(x)\gamma_\mu(\partial_\nu - \overleftarrow{\partial}_\nu)\psi(x) - \tfrac{1}{4}\bar{\psi}(x)\gamma_\nu(\partial_\mu - \overleftarrow{\partial}_\mu)\psi(x) - \partial_\mu B(x)\partial_\nu B(x) + \tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda B(x))^2 - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B^2(x) - \tfrac{1}{3}(m/g)(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)B(x) , \quad (5.20)$$

in terms of which it is straightforward to construct the generators P_μ, $M_{\mu\nu}$, $\tilde{D}$, and $\tilde{K}_\alpha$ satisfying the commutator algebra of the conformal group. The tensor $T_{\mu\nu}(x)$ is readily interpreted as the proper CCJ tensor of the invariant Lagrangian (5.16), or, equivalently,

$$\mathcal{L} = -\bar{\psi}\gamma\cdot\partial\psi - \tfrac{1}{2}(\partial_\mu\beta)^2 - g\bar{\psi}\psi\beta \quad (5.21)$$

with

$$\langle 0|\beta(x)|0\rangle = m/g . \quad (5.22)$$

Although the symmetry is spontaneously broken, the generators P_μ, $M_{\mu\nu}$, $\tilde{D}$, and $\tilde{K}_\alpha$ are well defined since they are expressed in terms of the physical field $B(x)$, which vanishes asymptotically.[9,11]

One also finds, by explicit calculation, that

$$\langle 0|[B(x),\tilde{D}]|0\rangle = im/g,$$

which is in agreement with the spontaneous nature of the symmetry violation.[11] We emphasize again that the special form (5.19) of $\rho(x)$ is strictly a consequence of the original invariance of the theory. As our next example we quote a four-parameter class of Lagrangians proposed by Chang and Freund[12] in their study of field-current identities for scalar mesons and the divergence of the dilatation current. The Lagrangian is

$$\mathcal{L} = \frac{c^2(b+\varphi)^2}{1-c^2(b+\varphi)^2}\,\tfrac{1}{2}(\partial_\mu\varphi)^2 + d\left[-\frac{1}{c^2}+(b+\varphi)^2\right]^2 - \frac{a}{8}(b+\varphi)[1+c^2(b+\varphi)^2] + \frac{ab}{4} + \frac{a}{16c}[1-c^2(b+\varphi)^2]^2 \ln\left|\frac{1+c(b+\varphi)}{1-c(b+\varphi)}\right| \tag{5.23}$$

and is such that

$$\theta_{\mu\mu}(x) = a\varphi + b\Box\varphi \tag{5.24}$$

as a consequence of the field equations. This Lagrangian is a nonpolynomial one and the kinetic term is not of the standard form. However, the virial condition is seen to be identically satisfied by expanding the kinetic term in power series. In this sense, the system (5.23) belongs to the class of Lagrangians considered in the previous section. The relation (5.24) represents the partial conservation of the dilatation current, which, according to the conclusion of the previous section, is now a combined effect of intrinsic and spontaneous breakdown of scale invariance. Indeed, the two limiting cases

$$(b=0;\ c=\infty)$$

and

$$(a=0;\ c=\infty)$$

give

$$\mathcal{L}(b=0;c=\infty) = -\tfrac{1}{2}(\partial_\mu\varphi)^2 - \tfrac{1}{3}a\varphi + d\varphi^4 \tag{5.25}$$

with

$$\theta_{\mu\mu}(b=0) = a\varphi(x), \tag{5.26}$$

and

$$\mathcal{L}(a=0;c=\infty) = -\tfrac{1}{2}(\partial_\mu\varphi)^2 + d(b+\varphi)^4 \tag{5.27}$$

with

$$\theta_{\mu\mu}(a=0) = b\Box\varphi(x). \tag{5.28}$$

Thus, the a term in $\theta_{\mu\mu}(x)$ represents the partial conservation of dilatation current as obtained from the standard σ-type model Lagrangian. Instead, the b term in $\theta_{\mu\mu}(x)$ signals the spontaneous nature of the symmetry violation in the conformally invariant theory of a massless scalar field with quartic self-interaction,

$$\mathcal{L}(\psi,\partial_\mu\psi) = -\tfrac{1}{2}(\partial_\mu\psi)^2 + d\psi^4. \tag{5.29}$$

The breaking terms arising by shifting the field ψ according to

$$\psi(x) = \varphi(x) + b \tag{5.30}$$

"remember" the original invariance of the theory and are so correlated as to produce the field equation (5.28). Indeed it is readily checked that the general Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 - g\varphi^3 - f\varphi^4 - k\varphi - h \tag{5.31}$$

will lead to the condition (5.28) if and only if

$$\frac{m}{3g} = \frac{g}{4f} = \frac{3k}{m^2} = \frac{4h}{k} \equiv b\,. \tag{5.32}$$

When such correlation exists, the shift (5.30) changes trivially the Lagrangian (5.31) into (5.29). The above considerations can be easily extended to the general σ-model Lagrangian, inclusive of the pion and nucleon fields. Thus the conformally invariant Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\vec{\pi})^2 - \tfrac{1}{2}(\partial_\mu\beta)^2 - \bar{N}[\gamma\cdot\partial + f(\beta + i\tau\cdot\vec{\pi}\gamma_5)]N - \lambda(\vec{\pi}^2+\beta^2)^2 \tag{5.33}$$

is obtained from

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\vec{\pi})^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \lambda(\vec{\pi}^2+B^2)^2 - \tfrac{1}{2}m^2\vec{\pi}^2 - \tfrac{1}{2}\mu^2B^2 - gB(\vec{\pi}^2+B^2) - \bar{N}[\gamma\cdot\partial + M + f(B+i\tau\cdot\vec{\pi}\gamma_5)]N - kB - h \tag{5.34}$$

by the field redefinition

$$B(x) = \beta(x) - c \tag{5.35}$$

provided that

$$\frac{3k}{\mu^2}=\frac{M}{f}=\frac{g}{4\lambda}=\frac{\mu^2}{3g}=\frac{m^2}{g}=\frac{4h}{k}=c\,, \tag{5.36}$$

and one finds that because of this correlation

$$\theta_{\mu\mu}(x)=c\Box B(x)\,. \tag{5.37}$$

The commutator algebra of the broken conformal group associated with the system (5.34) is immediately obtained from the general results of Sec. III. The modified generators obtained from the new tensor

$$T_{\mu\nu}(x)=\theta_{\mu\nu}(x)-\tfrac{1}{3}c(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)B(x)$$

do satisfy the closed Lie algebra of the conformal group. Thus the emergence of dimensional parameters in (5.34) is consistent with original invariance of the Lagrangian (5.33). This has also been shown in II for the φ^4 theory by means of explicit dynamical calculations.

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