PHYSICAL REVIEW D VOLUME 14, NUMBER 4 15 AUGUST 1976

Center-of-mass theorem in post-Newtonian hydrodynamics

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(Received 2 March 1976)

In the post-Newtonian theory of a perfect fluid in adiabatic motion, the conservation laws of energy, momentum, and angular momentum can be obtained via Noether's theorem from the invariance properties of the Lagrangian under infinitesimal time and space translations and rotations, in complete analogy to the corresponding case of interacting particles; but, again in complete analogy, the center-of-mass theorem cannot be as directly related to the infinitesimal transformations of a group. Therefore the center-of-mass theorem had previously only been obtained by direct integration of the equations of motion. However, it has been shown recently in the case of interacting particles by Havas and Stachel that invariance of the Lagrangian under space and time translations is by itself sufficient to guarantee the existence of a center-of-mass theorem, and it is shown here that the techniques developed for that case also lead to the center-of-mass theorem for the perfect fluid.

I. INTRODUCTION AND DISCUSSION

Within a year of the creation of the general theory of relativity, Lorentz and Droste[1] succeeded in obtaining the post-Newtonian (PN) equations of motion appropriate for a number of bodies interacting gravitationally[2] from a PN Lagrangian following from an exact Lagrangian for a fluid by expansion of the metric in powers of v/c, and elimination of the gravitational field variables. Their work apparently went unnoticed, and the method was only reinvented four decades later by Plebański and Bażański,[3] who derived the PN equations of motion for a fluid in isentropic flow as well as for a number of bodies; their results can easily be extended to the adiabatic case. By the latter we mean a flow in which the entropy is constant along each streamline, but not necessarily from one streamline to another as in isentropic flow.

Similarly, the PN equations of motion for a general-relativistic fluid were obtained by direct expansion in powers of v/c of the general-relativistic field equations and equations of motion for a perfect fluid by Chandrasekhar. He also derived the PN conservation equations for energy, momentum, and angular momentum of the fluid elements, as well as integral forms of these conservation laws for a fluid body, by direct integration of the PN equations of motion.[4]

Two of us have verified that the Plebański-Bażański and Chandrasekhar formulations of the equations of motion are fully equivalent, and have derived the conservation laws for energy, momentum, and angular momentum (which also agree with those of Chandrasekhar) from standard considerations of the invariance of the Plebański-Bażański Lagrangian under time and space translations and spatial rotations, respectively.[5,6] We have also carried out the PN approximation on the Eulerian variational principle developed for relativistic hydrodynamics by Tam and O'Hanlon,[7] and verified that these are also equivalent to the Plebański-Bażański and Chandrasekhar formulations.[6,8]

This raises the question of the center-of-mass (c.m.) theorem for the fluid in PN approximation. In the past, this conservation law for a system of point particles interacting gravitationally had been derived directly from the equations of motion for the system.[9] For an elastic body it had been derived by Fock, using the approach to the equations-of-motion problem that he had first developed in 1939.[10] In Fock's method the equations of motion of the body as a whole (which he called the external equations of motion) in the PN approximation are developed by a method that requires only the use of the Newtonian equations of motion for the behavior of the internal structure of the body (which he called the internal equations of motion), without any need for a PN general-relativistic elastic theory. On the basis of his approach, Fock was able to develop the integral conservation laws for an elastic body in the PN approximation from an analysis of only the PN external equations of motion and the Newtonian internal equations of motion. His method is quite complicated, and depends upon the imposition of harmonic coordinate conditions. The resulting PN c.m. theorem for an elastic body applies, of course, *a fortiori* to a fluid body. Our result, Eq. (16), agrees with that of Fock, Eqs. (79.58) and (79.59) of Ref. 10.[11]

Yet it seems that the c.m. conservation theorem should also be derivable from Noether theorem considerations based upon symmetry properties of the PN Lagrangian for any theory for which such a Lagrangian exists, just as the Newtonian c.m. theorem was derived by Bessel-Hagen for a system of interacting particles.[12] However, even in the latter case such a derivation poses difficulties in PN order.

The problem, as discussed by Havas and Stachel,[13] is basically connected with the fact that for the infinitesimal Lorentz transformation ("boost") the transformed time $t' = t + \delta t$ depends on the untransformed space (as well as time) coordinates. But then for any variational principle of the form

$$\delta \int L\, dt = 0\,, \tag{1}$$

where L depends on more than a single position vector (as it necessarily must if it is to describe interactions), it is not obvious how to define a unique dt' for all positions, as needed for Noether's theorem, which requires us to compare $\int L\, dt$ with $\int L'\, dt'$. One way out of this suggested in Ref. 13 was to consider directly a set of infinitesimal transformations

$$\delta t = \vec{\epsilon} \cdot \vec{R}/c^2, \quad \delta \vec{r}_k = \vec{\epsilon} t \tag{2}$$

associated with a c.m. vector $\vec{R}(t)$. It was then shown that any L invariant under space and time translations, i.e., which conserves energy and momentum, is also invariant under the transformations (2), provided that $\vec{R}$ is related to the total energy E and momentum $\vec{P}$ by

$$\vec{R}(t) = \int \frac{\vec{P}c^2}{E}\, dt\,, \tag{3}$$

and the corresponding conservation laws resulting from the Noether identities can be interpreted as the c.m. theorem.[14,15] Thus, once one has obtained the energy and momentum integrals for such a system, one can immediately write down a conserved c.m. vector $\vec{G}$ such that

$$\vec{G} = \vec{K} - \vec{P}t = \text{constant}, \tag{4}$$

where

$$\vec{K} \equiv \frac{E}{c^2}\, \vec{R}\,, \tag{5}$$

with

$$\frac{d\vec{K}}{dt} = \frac{E}{c^2}\, \frac{d\vec{R}}{dt} = \vec{P}\,. \tag{6}$$

Here the velocity

$$\vec{V} \equiv \frac{d\vec{R}}{dt} = \frac{\vec{P}c^2}{E}\,, \tag{7}$$

which is always defined, can be interpreted as the velocity of the c.m. point; however, to define this point explicitly requires that $\vec{R}$ (and thus $\vec{K}$) be expressible as a function of the position and velocity vectors. In general, the existence of such a vector function is not ensured; however, if the system described by the variational principle (1) represents an approximation to a Poincaré-invariant variational principle, with two-body forces, the function always exists.[16] Furthermore, it was shown in Ref. 13 that in this case it is sufficient to treat E as constant in Eq. (3), i.e., to evaluate only

$$\vec{R} = \frac{c^2}{E} \int \vec{P}(\vec{r}_k(t))\, dt \tag{8}$$

with the help of the equations of motion, and thus

$$\frac{d\vec{K}}{dt} = \frac{d}{dt}\left(\frac{E}{c^2}\, \vec{R}\right) = \vec{P} + O(v^4/c^4)\,. \tag{9}$$

For all Lagrangians discussed in Ref. 13, and indeed, as shown in Ref. 16, for all approximately relativistic Lagrangians which in the limit $c \to \infty$ reduce to the Newtonian Lagrangian

$$L = \sum_k \frac{m_k v_k{}^2}{2} + \tfrac{1}{2} \sum \sum_{i \neq k} V_{ik}(r_{ik})\,, \tag{10}$$

and for which [either exactly or at least to order $(v/c)^2$] the (two-body) interactions are symmetric in the particles; $\vec{R}$ is given by[17]

$$\vec{R} = \sum_k m_k \left[1 + \frac{1}{2}\left(\frac{v_k}{c}\right)^2\right] \vec{r}_k + \frac{1}{2c^2}\, V_k\,, \tag{11}$$

$$V_k \equiv \sum_{i \neq k} V_{ik}(r_{ik})\,,$$

where V_k is the Newtonian potential energy of the kth particle in the field of all others.

The above method involves a slight extension of Noether's theorem to include infinitesimal transformation which are functionals [such as Eqs. (2) with (3)]. Another method, which also involves a slight extension of Noether's theorem, has been developed by two of us recently.[18] It requires consideration of transformations in which the $L\, dt$ of Eq. (1) is not restricted to be invariant up to a divergence, but where it has this property only modulo the equations of motion.[19] With this extension, and an interpretation of the action of the Lorentz transformation on the coordinates for which $\delta t = 0$, so that $t' = t$, as discussed in Ref. 14, all action integrals (1) considered in Refs. 13 and 16 are indeed invariant to order $(v/c)^2$ under the full Poincaré group. These questions will be discussed in detail elsewhere.[18] Both methods, of course, must lead to the same c.m. theorem.

II. THE POST-NEWTONIAN FLUID

The action principle for the PN fluid poses the same problems as those for the various action principles for approximately relativistic particle systems. It still has the form (1), where L, as given in Ref. 3, is

$$L=-\frac{c^4}{G}\int \mu\, d^3x+\frac{1}{2}\frac{c^2}{G}\int \mu v^2 d^3x+\frac{1}{8G}\int \mu v^4 d^3x+\frac{c^2}{G}\int\int\frac{\mu\mu'}{|\vec{x}-\vec{x}'|}\left(\tfrac{3}{4}v^2+\tfrac{3}{4}v'^2-2v^m v'^m\right)d^3x\, d^3x'$$
$$-\frac{1}{4}\frac{c^2}{G}\int\int \mu\mu'|\vec{x}-\vec{x}'|_{,km'}v^k v'^m d^3x\, d^3x'+\frac{1}{2}\frac{c^4}{G}\int\int\frac{\mu\mu'}{|\vec{x}-\vec{x}'|}d^3x\, d^3x'$$
$$-\frac{1}{6}\frac{c^4}{G}\int\int\int \mu\mu'\mu''\left(\frac{1}{|\vec{x}-\vec{x}'||\vec{x}-\vec{x}''|}+\frac{1}{|\vec{x}-\vec{x}'||\vec{x}'-\vec{x}''|}+\frac{1}{|\vec{x}-\vec{x}''||\vec{x}''-\vec{x}'|}\right)d^3x\, d^{3\prime}\, d^3x''$$
$$-\frac{c^2}{G}\int \mu\left(1-\frac{1}{2}\frac{v^2}{c^2}-\int\frac{\mu'}{|\vec{x}-\vec{x}'|}d^3x'\right)\int_0^p\frac{dp}{\rho}\,d^3x+\int p\left(1+2\int\frac{\mu'}{|\vec{x}-\vec{x}'|}d^3x'\right)d^3x\,, \tag{12}$$

where ρ is the conserved Newtonian matter density, a comma denotes partial differentiation, and

$$\frac{c^2\mu}{G}\equiv\rho^*=\rho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2+3G\int\frac{\rho'}{|\vec{x}-\vec{x}'|}d^3x'\right)\right]$$

is the conserved post-Newtonian matter density. $v^a(\vec{x})$ represents the local three-velocity of the fluid and $p(\vec{x})$ its pressure. The (approximately) conserved energy and momentum following from the invariance of (12) under infinitesimal translations in time and space as given by Pascoe and Stachel are[5-7]

$$E=\frac{1}{2}\int\rho v^2 d^3x-\frac{1}{2}G\int\int\frac{\rho\rho'}{|\vec{x}-\vec{x}'|}d^3x'\, d^3x+\int\left(\rho\int_0^p\frac{dp}{\rho}-p\right)d^3x$$
$$+\frac{1}{c^2}\left[\frac{5}{8}\int\rho v^4 d^3x+G\int\int\frac{\rho\rho'}{|\vec{x}-\vec{x}'|}\left(\tfrac{5}{2}v^m v^m-2v^m v'^m\right)d^3x'\, d^3x-\frac{1}{4}G\int\int\rho\rho'|\vec{x}-\vec{x}'|_{,rm'}v^r v'^m d^3x' d^3x\right.$$
$$-\frac{5}{2}G^2\int\int\int\frac{\rho\rho'\rho''}{|\vec{x}-\vec{x}'||\vec{x}-\vec{x}''|}d^3x''\, d^3x'\, d^3x+\int\left(\rho v^2+2G\int\frac{\rho\rho'}{|\vec{x}-\vec{x}'|}d^3x'\right)\left(\int_0^p\frac{dp}{\rho}\right)d^3x$$
$$\left.-2G\int\int\frac{p\rho'}{|\vec{x}-\vec{x}'|}d^3x'\, d^3x\right]\,, \tag{13}$$

$$\vec{P}=\int\vec{\pi}\, d^3x\,, \tag{14}$$

where $\vec{\pi}$ is the momentum density of the fluid:

$$\pi^m=\frac{c^2}{G}\left[\mu v^m+\frac{1}{2c^2}\mu v^2 v^m+\int\frac{\mu\mu'}{|\vec{x}-\vec{x}'|}(3v^m-4v'^m)d^3x'-\frac{1}{2}\int\mu\mu'|\vec{x}-\vec{x}'|_{,mk'}v'^k d^3x'+\frac{1}{c^2}\mu v^m\int_0^p\frac{dp}{\rho}\right]. \tag{15}$$

To obtain the c.m. theorem we can use either of the two methods discussed in Sec. I. The second one is more appropriately discussed in connection with the derivation of the other conservation laws.[7] Here we shall present the first method. Thus we must determine $\vec{R}$, where now the expressions $\vec{P}$ and E in Eq. (3) must be written as integrals over the fluid body of some vector density functions. Here, we may be guided by the experience of the c.m. result for a system of particles in PN approximation, i.e., by the resultant expression (11). It shows that the PN correction of the Newtonian c.m. vector is equivalent to "correcting" the mass of each particle by taking into account the mass-energy equivalence relation, i.e., by replacing the rest mass of each particle by its total energy divided by c^2; to obtain this energy, the interaction energy between any two particles had to be distributed equally between them. The problem now is to obtain the correct hydrodynamic analog, guided by this heuristic idea. If $\rho^*\epsilon$ is the internal energy of compressibility per unit volume, it seems reasonable to guess that

$$\vec{K}=\int\rho^*\vec{x}\, d^3x+\frac{1}{c^2}\left(\frac{1}{2}\int\rho^* v^2\vec{x}\, d^3x-\frac{G}{2}\int\int\frac{\rho^*\rho^{*\prime}}{|\vec{x}-\vec{x}'|}\vec{x}\, d^3x\, d^3x'+\int\rho^*\epsilon\vec{x}\, d^3x\right)\,, \tag{16}$$

$$\epsilon=\int_0^p\frac{dp}{\rho}-\frac{p}{\rho}\,,$$

will be the correct volume integral whose total time derivative will be $\vec{P}$. For the first term in the integral is the moment of the matter density, the second term is the moment of the kinetic energy density, the third term is the moment of the gravitational potential energy density at point $\vec{x}$, halved in accordance with the sharing rule found to work for particles, and the last term is the moment of the elastic compressibility energy density, the last three energy terms being divided by c^2 to get their contribution to the effective PN mass in the calculation of the c.m. density. At any rate, no matter how plausible we have made this guess, the task now is to verify that Eq. (9) holds to the order indicated.

To do this we shall have to use the equation of mass conservation for ρ^*

$$\frac{\partial\rho^*}{\partial t}+(\rho^* v^s)_{,s}=0\,, \tag{17}$$

and the Newtonian equations of motion for the fluid

$$\rho\frac{d\vec{v}}{dt}=-\nabla p+\rho\nabla\Phi\,, \tag{18}$$

where Φ is the Newtonian gravitational potential, for which

$$\nabla^2\Phi=-4\pi G\rho\,. \tag{19}$$

Note that in all terms in which a c^{-2} occurs, i.e., all but the first term in Eq. (16), the PN matter density ρ^* may be replaced by the Newtonian matter density ρ, since the calculation is only correct to order $(v/c)^2$ in any case. For a locally adiabatic flow we also have

$$\frac{d\epsilon}{dt}=-\frac{p}{\rho}\nabla\cdot\vec{v}\,. \tag{20}$$

The following mathematical lemma, familiar from fluid dynamics,[4] is also useful in computing the needed total derivatives:

$$\frac{d}{dt}\int_{\text{fluid}}\rho^* F(\vec{x},t)d^3x=\int_{\text{fluid}}\rho^*\frac{dF}{dt}d^3x\,. \tag{21}$$

Now, it is just a matter of calculating the total time derivative of Eq. (16), which we proceed to do, term by term, omitting the details of the essentially trivial intermediate steps. We get successively

$$\frac{d}{dt}\int\rho^*\vec{x}\,d^3x=\int\rho^*\vec{v}\,d^3x \tag{22}$$

using (17) and (21);

$$\frac{d}{dt}\int\rho v^2\vec{x}d^3x=\int\rho v^2\vec{v}\,d^3x-2\int\vec{v}\cdot\nabla p\vec{x}\,d^3x-2G\int\int\rho\rho'\frac{\vec{v}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\vec{x}\,d^3x\,d^3x' \tag{23}$$

using (17), (18), and (21);

$$\frac{d}{dt}\int\int\frac{\rho\rho'\vec{x}}{|\vec{x}-\vec{x}'|}d^3x\,d^3x'=\int\int\frac{\rho\rho'\vec{v}}{|\vec{x}-\vec{x}'|}d^3x\,d^3x'-\int\int\frac{\rho\rho'(\vec{x}+\vec{x}')\,(\vec{x}+\vec{x}')\cdot\vec{v}}{|\vec{x}-\vec{x}'|^3}d^3x\,d^3x'\,; \tag{24}$$

$$\frac{d}{dt}\int\rho\epsilon\vec{x}\,d^3x=-\int p(\nabla\cdot\vec{v})\vec{x}\,d^3x+\int\rho\left(\int_0^p\frac{dp}{\rho}-\frac{p}{\rho}\right)\vec{v}\,d^3x \tag{25}$$

using (20) and the integral expression for ϵ given in Eq. (16).

Combining Eqs. (22)–(25), we get

$$\frac{d\vec{K}}{dt}=\int\rho^*\vec{v}\,d^3x+\frac{1}{c^2}\left[\frac{1}{2}\int\rho v^2\vec{v}\,d^3x-\frac{G}{2}\int\int\frac{\rho\rho'(\vec{x}-\vec{x}')(\vec{x}-\vec{x}')\cdot\vec{v}}{|\vec{x}-\vec{x}'|^3}d^3x\,d^3x-\frac{G}{2}\int\frac{\rho\rho'\vec{v}}{|\vec{x}-\vec{x}'|}d^3x\,d^3x'\right.$$
$$\left.-\int\vec{v}\cdot\nabla p\vec{x}\,d^3x-\int p(\nabla\cdot\vec{v})\vec{x}\,d^3x+\int\rho\vec{v}\int_0^p\frac{dp}{\rho}d^3x-\int p\vec{v}\,d^3x\right]\,. \tag{26}$$

The four last, pressure-dependent, terms combine to give

$$\int\rho\vec{v}\int_0^p\frac{dp}{\rho}d^3x-\oint p\vec{x}\vec{v}\cdot d\vec{S}\,, \tag{27}$$

and the surface integral vanishes (either taking the surface outside the body, or noting that the pressure vanishes on the surface of the body). The third term of (26) can be rewritten by noting that

$$|\vec{x}-\vec{x}'|_{,ab'}v'^b=-\frac{v'^a}{|\vec{x}-\vec{x}'|}+\frac{(x^a-x'^a)(\vec{x}-\vec{x}')\cdot\vec{v}'}{|\vec{x}-\vec{x}'|^3}\,. \tag{28}$$

When this is done, we finally get the desired result: Eq. (9), with the $\vec{K}$ and $\vec{P}$ of Eqs. (14)–(16).

Note added in proof. In a recently published paper by G. Contopoulos and N. Spyrou, Astrophys. J. 205, 592 (1976), it is shown that the momentum of the PN fluid, in the form given by Chandrasekhar,[4] may be written as the total time derivative of a vector equivalent to our $\vec{K}$ by methods similar to those of our Sec. II. This result is then used to define a c.m. point. As shown in Ref. 6, Chandrasekhar's definition of the momentum is equivalent to that given here. Contopoulos and Spyrou do not seem to be aware that the c.m. expression for a fluid was given by Fock some time ago, as discussed in Sec. I of this paper.

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*The results presented in Sec. II constitute part of a thesis submitted to Boston University in partial fulfillment of the requirements for the Ph.D. degree.

[1]H. A. Lorentz and J. Droste, Versl. Koninklijke Akad. Wetensch. Amsterdam 26, 392 (1917) [English translation in H. A. Lorentz, *Collected Papers*, edited by P. Zeeman and A. D. Fokker (Martinus Nijhoff, The Hague, 1939), Vol. 5].

[2]In the limit of point particles, the equations of motion are identical with those obtained by a different method by Einstein, Infeld, and Hoffmann (EIH) twenty years later; the Lorentz-Droste Lagrangian then becomes identical with the Lagrangian obtained directly from the EIH equations another decade later by Fichtenholz. These and other aspects of the history of the general-relativistic equations of motion will be discussed by one of us (P. H.) in detail elsewhere.

[3]J. Plebański and S. Bażański, Acta Phys. Polon. 18, 307 (1959).

[4]S. Chandrasekhar, Astrophys. J. 142, 1488 (1965).

[5]T. Pascoe and J. Stachel, Bull. Am. Phys. Soc. 14, 69 (1969).

[6]T. Pascoe, Ph.D. thesis, Boston University, 1972 (unpublished).

[7]T. Pascoe and J. Stachel (unpublished).

[8]K. K. Tam, Can. J. Phys. 44, 2403 (1966); K. K. Tam and J. O'Hanlon, Nuovo Cimento 62B, 351 (1969). This variational principle was apparently rediscovered independently by B. F. Schutz, Phys. Rev. D 2, 2762 (1971).

[9]See L. Infeld and J. Plebański, *Motion and Relativity* (MacMillan, New York, 1960).

[10]V. Fock, Zh. Eksp. Teor. Fiz. 9, 375 (1935) [J. Phys. Acad. Sci. USSR 1, 81 (1939)]; V. Fock, *Theory of Space Time and Gravitation* (Pergamon, Oxford, 1964), 2nd revised English edition.

[11]One of us (J. S.), with T. Morrill, is currently working on the problem of the PN equations of motion for elastic materials (internal equations, in Fock's sense) based upon recent progress in the development of a general-relativistic elastic theory.

[12]E. Bessel-Hagen, Math. Ann. 84, 258 (1921).

[13]P. Havas and J. Stachel, Phys. Rev. 185, 1636 (1969).

[14]It was shown by Stachel and Havas [Phys. Rev. D 13, 1598 (1976)] that in phase space the transformations (2) are the generators of a three-parameter Lie group; the c.m. theorem can be rederived in Hamiltonian form, and the ten conserved quantities are a realization of the Lie algebra of the Poincaré group to order $(v/c)^2$.

[15]In general, one has to be rather careful in associating names such as energy and momentum carrying physical connotation with invariance under particular infinitesimal transformations, as discussed by P. Havas, Acta Phys. Austriaca 38, 145 (1973). However, in the PN case under consideration, the association is based on the corresponding well-defined association of Newtonian theory, and thus is as well founded as any other aspect of a theory intended to give small corrections to Newtonian motions. In a fully relativistic calculation the entire question has to be reconsidered, however.

[16]H. W. Woodcock and P. Havas, Phys. Rev. D 6, 3422 (1972). The general form of the Lagrangian obtained there also follows from the considerations of Ref. 14, which, instead of approximating an exact variational principle, starts from the Newtonian canonical formalism and obtains the most general two-body corrections to order c^{-2}. While the general problem of three-or-more-body interactions was not attacked in Ref. 14; it was shown there that simple three-body corrections of the type encountered in the various PN equations of motion can readily be included.

[17]Reference 13, Eqs. (64) and (66). In Eq. (66) a factor $\frac{1}{2}$ in front of V_k was omitted erroneously. Similarly, a factor $\frac{1}{2}$ was omitted in front of $(v_k/c)^2$ in Eq. (49), and in front of V_{ik} in Eq. (67); however, the discussion is appropriate for the correct equations.

[18]J. Stachel and P. Havas (unpublished).

[19]Compare also H. Steudel, Z. Naturforsch. 17a, 129 (1962).