Melosh symmetries of vector-meson decays and a criterion for rapid convergence of semilocal averages

Donald E. Neville

Department of Physics, Temple University, Philadelphia, Pennsylvania 19122

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The rapid $t = 0$ convergence of the classic superconvergence relation $\rho\pi \to \rho\pi$ and its SU(6) generalizations $VM \to VM$ (V = vector nonet, M = pseudoscalar or vector nonet) is reexamined. It is suggested that the rapid convergence is due to a zero of the amplitude in t (crossed-channel energy) rather than the rapid falloff of the amplitude with s ("superconvergence" in s), as is usually assumed. In a separate investigation involving only symmetry arguments, we find that the Melosh-transform parameterization for decays to VV states involves only two undetermined scalar couplings, rather than the four (six) required in related processes involving vector particles, $B^* \to B\gamma$ ($B^* \to BV$) decays, where B is a decuplet or octuplet baryon. A critical discussion is given of the Gilman-Kugler-Meshkov and related prescriptions used to correct SU(6)-symmetric couplings for the effects of mass splitting.

I. INTRODUCTION

One puzzling aspect of the very earliest work on superconvergence relations is the special role played by the point $t=0$.[1] The original $\rho\pi \to \rho\pi$ relations gave quite reasonable results at this point if saturated with only a few resonances. Usually this rapid convergence was attributed to a rapid asymptotic falloff of the form

$$s^{\alpha(t)-2} \tag{1.1}$$

[for a $\rho\pi$ amplitude with $\lambda_3 - \lambda_1 = 2$ and leading trajectory $\alpha(t)$ in the crossed channel].[2] If convergence were due to Eq. (1.1), then one might anticipate almost as good convergence in the neighborhood of $t=0$ as at $t=0$ itself and perhaps even better convergence for negative t, where $\alpha(t) < \alpha(0)$; but no such pattern of convergence has emerged from later phenomenological work. Nor have subsequent theoretical developments (duality, Veneziano models) shed much light on the rapid convergence at $t=0$.[3,4] Semilocal duality, for example, predicts that direct-channel resonances should average to a (small, but) finite quantity of the same order as expression (1.1); but duality does not single out the point $t=0$ especially. Similarly, the Veneziano amplitude will obey sum rules analogous to superconvergence relations. [The spinless Veneziano amplitude goes only as $s^{\alpha(t)}$ but can be made to "superconverge" by taking $\alpha(t)$ sufficiently negative.] However, the only point singled out by these sum rules is the point where they diverge. Clearly, that point will not be a rapid superconvergence point, and in any case that point depends on $\alpha(t)$, not t.

We believe the rapid $t=0$ superconvergence found historically does not have to be dismissed as just a fluke, but rather can be understood as follows: Contributions from direct-channel resonances to (say) $VM \to VM$ scattering will average especially quickly to zero at $t=0$ if the crossed channel is dominated by an MM or VV Reggeon coupling which has a zero at $t=0$. We can sum up concisely the difference between this explanation and the usual one as follows. The usual explanation says that the asymptotic behavior of (for example) the $\lambda_1 - \lambda_3 = 2$ $VM \to VM$ amplitude is given by expression (1.1), and the rapid convergence at $t=0$ is due to the factor s^{-2}. The present explanation says that the asymptotic behavior of the $\lambda_3 - \lambda_1 = 2$ amplitude is given by

$$ts^{\alpha(t)-2} \tag{1.2}$$

and the rapid convergence at $t=0$ is due to the factor t. This explanation preserves rapid convergence at $t=0$ without requiring rapid (or more rapid) convergence at points near $t=0$. If the factor of t occurs in $MV \to MV$ scattering at vertices $VV \to$ (natural-parity Reggeons), as seems to be the case, then by factorization of Regge residues the same factor of t should occur in $BV \to BV$ scattering and should produce rapid convergence only for those amplitudes dominated by natural-parity trajectories.[5]

Since we are attributing rapid convergence more to a factor of t than to a factor of s^{-2}, conceivably a semilocal average over resonances could converge rapidly to zero even when the amplitude itself does not superconverge, i.e., even when the s^{-2} factor is missing from expression (1.2). However, in all the cases we shall consider here, the amplitude contains both a t and an s^{-2} factor.

Evidently if our hypothesis is correct and we wish to locate amplitudes which are good candidates for semilocal averaging, we can no longer simply look for those amplitudes which have rapid falloff in s, i.e., those which have large helicity flip in a crossed channel. We must locate Regge residues which have zeros in t. Regge residues are known to develop zeros when they pass through "nonsense" values of angular momentum, but we shall not exploit this source of zeros in the present paper. Here we shall try to understand the origin of the classic "superconvergence" zeros at $t=0$. We show in Sec. II that these zeros have nothing to do with nonsense values of J but are rather consequences of symmetry. At $t=0$, resonance contributions to $V\Pi \to V\Pi$ scattering average rapidly to zero if the resonance couplings obey the version of SU(6) symmetry suggested by recent work on the Melosh transformation.[6-10] If our hypothesis is correct, this rapid averaging means that the crossed $VV\to$ (Reggeon) couplings have a zero. We then assume Regge residues factorize, so that the zero propagates to $VV\to VV$ and $VB \to VB$ reactions, hence these become good candidates for semilocal averaging. Thus we use symmetries to locate the zeros initially, then use factorization to find additional amplitudes to be semilocally averaged.

In $V\Pi$ scattering, convergence of resonance averages is extremely rapid. A zero is obtained if one averages only resonances having a single value of $\mathcal{L}$ (= quark angular momentum). For VV and VB scattering we shall demand that the convergence is equally rapid. In the present paper, the term "rapid" will always imply an average over resonances having a single $\mathcal{L}$ value.

In Sec. III we investigate the $VV\to VV$ amplitudes. In a preliminary investigation of symmetry constraints on VV couplings, we obtain a result which is of interest in its own right: An SU(6) parameterization of VV couplings to resonances of a given $\mathcal{L}$ requires only two unknown scalars, rather than the four (six) required in parameterizations of the closely related couplings $B^*\to B\gamma$ ($B^*\to BV$). Having determined the symmetry constraints on the VV couplings, we then demand that VV resonances contributing to $\lambda_3-\lambda_1=2$ amplitudes average rapidly to zero. Conceivably, imposing this requirement on the couplings could lead to relationships between scalar coupling parameters (the two scalars per $\mathcal{L}$ value mentioned above). This does not occur; the symmetry constraints are so strong that by themselves they force the resonance contributions to average rapidly to zero. This happens also in $VB\to VB$ reactions (which can be investigated using the same techniques as those described in Secs. II and III). The symmetry constraints on VB couplings are considerably weaker, since there are six undetermined scalars per $\mathcal{L}$ value rather than two; nevertheless, symmetry by itself is enough to produce rapid convergence.

Since the masses observed in nature do not obey an exact SU(6) symmetry, all calculations done in the SU(6) limit involve assumptions as to how mass splittings change coupling strengths. Getting these assumptions correct is of course crucial if contact with the real world is to be made successfully, and the present calculation is an excellent "laboratory" in which to test out the effects of various mass-splitting prescriptions. Usually exact-SU(6)-limit calculations are corrected for the effects of mass splitting by factoring out a function f depending on the masses from each scalar coupling C in the theory

$$C=f\overline{C}. \tag{1.3}$$

Then it is assumed that only f, but not $\overline{C}$, changes when the masses are split. Choosing f equal to a threshold factor of $\mathcal{S}_{12}$ [see Eq. (2.4)] gives the $\mathcal{S}_{12}$ prescription which we have investigated in a previous paper.[11] In the limit that one of the external masses m_2 is a pion, much lighter than any other mass in the problem, the $\mathcal{S}_{12}$ prescription reduces to the ΔM^2 prescription proposed by Gilman, Kugler, and Meshkov (GKM) for pion decays.[7]

In Ref. 11 (which considered only the $\mathcal{L}=0$ case) the $\mathcal{S}_{12}$ prescription seemed favored because it allowed residue zeros discovered in the SU(6) limit to persist to the broken mass limit. However, in Sec. III of the present paper we locate zeros which do not persist, even when $\mathcal{S}_{12}$ is used. In Sec. IV we present additional evidence that $\mathcal{S}_{12}$ cannot be the correct mass-splitting presecription for VV reactions. We also present less compelling but still quite strong evidence that $\mathcal{S}_{12}$ cannot be correct for $B\Pi$ reactions.

II. DUALITY CONSTRAINTS ON ΠV COUPLINGS

In constructing invariant helicity amplitudes suitable for superconvergence relations or semilocal averaging, we must cross direct- (s-) channel amplitudes to the t channel.[12] We must therefore investigate how the standard Melosh couplings behave under such crossing. We shall find it convenient to write the Melosh couplings in a way which makes their crossing behavior more explicit.

In this section and the next we shall construct only the "st" part of the direct-channel couplings. As explained in Ref. 11, we assume the direct-channel resonance amplitude can be split up into an "st" and an "su" part, with only the st part converging rapidly at $t=0$. (This splitup is motivated by the Veneziano model and other dual models: The st part, for example, contains s-channel resonances dual to t-channel Reggeons.) We shall not need the su part here. In Ref. 11 we explain how to construct the *su* part when $\mathcal{L}=0$ and the procedure for $\mathcal{L}>0$ is identical.

Since explicit detail probably would be more confusing than enlightening, we shall rely on a diagrammatic notation to explain the helicity structure of our amplitudes and shall write out explicit Clebsch-Gordan coefficients (CGC's) only when absolutely necessary. Figure 1 diagrams the SU(2) (helicity and W-spin) structure of the ΠV vertices allowed by the Melosh analysis. We postpone introduction of SU(3) for the moment. Each vertex in the diagram represents an SU(2) CGC coupling, the three angular momentum lines meeting at that vertex. The lines forming the equilateral triangle are spin-$\frac{1}{2}$ quarks. The wiggly lines represent orbital angular momenta. The diagram represents a special way of coupling the two initial spins S_1, S_2 plus three orbital angular momenta L_i to a resultant of J. This amplitude can be expressed in terms of the more familiar LSJ amplitudes using standard recoupling theory. In particular, if we recoupled L_1, L_2, and L_{12} in all possible ways, we would find all the values of L (the usual orbital angular momentum of the LSJ coupling) allowed by the vertex. We want orbital angular momentum lines $L_1=L_2=1$ attached to the quark loop, in order that the vertex exhibit W-spin symmetry.[11] To see what L_1 and L_2 have to do with W spin, consider the two vertices involving L_1 and S_1 in Fig. 1. These vertices constitute one possible way of coupling the four angular momenta L_1, S_1, Q_H, and Q_V. (Q_H = the quark exiting horizontally from the S_1 vertex; Q_V = the quark exiting vertically from the L_1 vertex.) Suppose we recouple in the order $L_1\times S_1=W_1, W_1=Q_H\times Q_V$:

$$\sum_m \langle \tfrac{1}{2}m\,\tfrac{1}{2}m_H|S_1\lambda_1'\rangle\langle L_1=1\,0\,\tfrac{1}{2}m|\tfrac{1}{2}m_V\rangle$$
$$=\sum_{W_1} C(W_1)\langle \tfrac{1}{2}m_H\,\tfrac{1}{2}m_V|W_1W_{1z}\rangle\langle L_1 0 S_1\lambda_1'|W_1W_{1z}\rangle . \qquad (2.1)$$

$C(W)$ is a recoupling coefficient, and we use a bra-ket notation for the SU(2) CGC's. If the right-hand side of Eq. (2.1) is computed explicitly, it becomes (for S_1 a spin-1 meson)

$$\langle \tfrac{1}{2}m_H\tfrac{1}{2}m_V|W_1W_{1z}\rangle[(-1)\delta(W_1,0) +\lambda_1\delta(W_1,1)]\delta(W_{1z},\lambda_1)(-3^{-1/2}), \qquad (2.2)$$

where $\delta(a,b)$ is a Kronecker delta. Equations (2.1) and (2.2) mean that the two CGC's involving L_1 and S_1 can be replaced by a single CGC coupling an external spin W_1 to the two internal quarks. W_1 may be identified with the usual W spin of S_1, since the Kronecker deltas impose the usual "W-S flip" and $W_z=S_z$ selection rules. In the square brackets the factors of (-1) and $\lambda_1=\pm1$ are just the standard

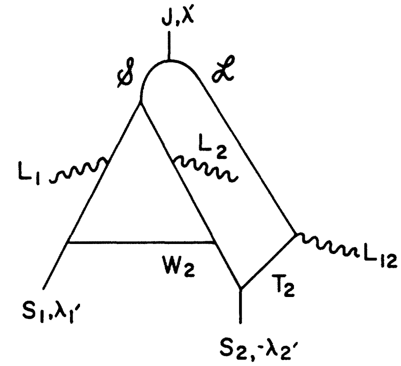

FIG. 1. Diagram showing the SU(2) Clebsch-Gordan coefficients making up the coupling $(S_1, \lambda_1)+(S_2,-\lambda_2')\to(J,\lambda')$. All unlabeled internal lines are spin-$\frac{1}{2}$ quarks.

phase differences between S-spin and W-spin states,[13] and the last factor in Eq. (2.2) may be absorbed into a scalar coupling constant multiplying the vertex. Thus the L_1 line in Fig. 1 converts S_1 to W_1; similarly the L_2 line converts S to W; the SU(2) quark triangle then couples $W_1\times W_2=W$. Now imagine that this $\mathrm{SU}(2)_W$ quark triangle is multiplied by an SU(3) quark triangle with external legs μ_1, μ_2, and μ ($\mu=\underline{1}$ or $\underline{8}$). The loop in Fig. 1 would then become an $\mathrm{SU}(6)_W$ quark loop coupled to three external $\mathrm{SU}(6)_W$ states $\mu_1I_1Y_1W_1$, $\mu_2I_2Y_2W_2$, and μIYW. By recoupling theory this loop is just an $\mathrm{SU}(6)_W$ CGC times a constant independent of the μ's I's, Y's, and W's. Thus the quark loop in Fig. 1, if multiplied by an SU(3) loop, gives just the $\mathrm{SU}(6)_W$ CGC required by the Melosh analysis. Furthermore, the SU(2) CGC at the top of Fig. 1 is just $\langle\mathcal{L}\mathcal{L}_z\mathcal{S}\mathcal{S}_z|J\lambda'\rangle$, also familiar from the Melosh work. Finally, on the lower right-hand side of the figure, if we take $S_2=0$, $W_2=1$ (values appropriate to the pion), therefore $T_2=1$, we find that L_{12} must be $\mathcal{L}+1$ or $\mathcal{L}-1$. ($L_{12}=\mathcal{L}$ is forbidden by parity conservation.) One linear combination of the $L_{12}=\mathcal{L}+1$ and $L_{12}=\mathcal{L}-1$ amplitudes will vanish unless $\mathcal{L}_z$ equals zero; this linear combination is the classic $\mathrm{SU}(6)_W$-symmetric amplitude. A second, orthogonal linear combination will vanish unless $\mathcal{L}_z$ equals ±1; this linear combination is the new amplitude suggested by Melosh. This completes the verification that the coupling of Fig. 1 satisfies the Melosh constraints.

In the exact-$\mathrm{SU}(6)_W$ limit (no mass splittings) the two tensors of Fig. 1 ($L_{12}=\mathcal{L}+1$ and $L_{12}=\mathcal{L}-1$) would be multiplied by scalar coupling constants $C(L_{12})$ not determined by the theory but independent of J. To take care of mass-splitting effects we shall replace $C(L_{12})$ by scalar couplings $C(L_{12},J)$ depending on J as well as L_{12}.

We now construct the imaginary part of the direct-channel amplitude by squaring the coupling of Fig. 1, multiplying by

$$\pi\delta(s-s_J)d^J_{\lambda'\mu'}(\theta_s)=\pi\delta(s-s_J)\delta(\lambda',\mu')$$

(sharp resonance approximation at $t=\theta_s=0$) and summing over the L_{12}, L_{34} (from the final state), and J quantum numbers. We cross to the t channel using a helicity crossing matrix $\prod d^{s_i}_{\lambda_i'\lambda_i}(-\pi/2)$. (All crossing angles χ_i reduce to $-\pi/2$ for elastic scattering at $t=0$.) We set $\lambda_3=-\lambda_1=1$ and divide by a factor $\sin^2\theta_t$ to remove kinematic s and t singularities.[14] At $t=0$,

$$\sin^2\theta_t=(\text{const})\mathcal{S}^2_{12}. \tag{2.3}$$

We use standard abbreviations for the threshold and pseudothreshold kinematic factors $\mathcal{S}_+$, $\mathcal{S}_-$, and $\mathcal{S}^2_{12}$:

$$\mathcal{S}_\pm=s-(m_1\pm m_2)^2,$$
$$\mathcal{S}^2_{12}=\mathcal{S}_+\mathcal{S}_-=4sp, \tag{2.4}$$

where p is the c.m. momentum. The s in $\mathcal{S}^2_{12}$ gets replaced by s_J when we semilocal average the t-channel amplitude by integrating $\int\delta(s-s_J)ds$. We can remove this $\mathcal{S}^2_{12}$ factor altogether by defining reduced couplings $\overline{C}$,

$$C_{12}(L_{12},J)=\mathcal{S}_{12}\overline{C}(L_{12},J), \tag{2.5}$$

and similarly for the $C(L_{34},J)$. Since the C's occur squared in the amplitude, the numerator will contain two factors of $\mathcal{S}_{12}$ which just cancel the $\mathcal{S}^2_{12}$ in the denominator coming from Eq. (2.3). There is now no more explicit mass dependence in the invariant amplitude. We now assume that the $\overline{C}$'s are the constants which obey the $SU(6)_W$ symmetry after the masses are broken:

$$\overline{C}(L_{34},J)=\overline{C}(L_{12},J)=\overline{C}(L_{12}). \tag{2.6}$$

The entire J, $\mathcal{L}$, and $\mathcal{S}$ dependence of the couplings is therefore given explicitly by the CGC's in Fig. 1, and we can use CG completeness to carry out sums over J and $\mathcal{S}$. The amplitude then opens up into the box form shown in Fig. 2. [The figure shows only the helicity structure of the direct-channel amplitude; additional factors, the crossing matrices $d_{\lambda_i'\lambda_i}(-\pi/2)$ and the reduced couplings $C(L_{12})C(L_{34})$, are not indicated in the diagram.]

We can now delete the line L_1 thru L_4 from the diagram because two adjacent orbital-angular-momentum lines on the same quark line reduce to a Kronecker delta. E.g., the two CGC's at the adjacent L_1 and L_3 vertices,

$$\sum_{m_V}\langle L_1 0 \tfrac12 m|\tfrac12 m_V\rangle\langle L_3 0\tfrac12 m'|\tfrac12 m_V\rangle,$$

reduce to $\delta(m,m')/3$, by inspection of a standard table of SU(2) CGC's. The left-hand edge of Fig. 2 now involves only two CGC's (those coupling the internal quarks to S_1 and S_3) plus two crossing matrix factors

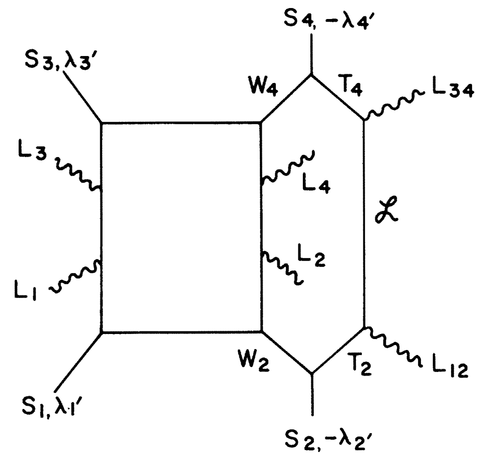

FIG. 2. Helicity structure of the direct-channel amplitude after the sums over $\mathcal{S}$ and J have been carried out.

$$\sum_{m,\lambda_i'}\langle\tfrac12 m\tfrac12 m_H|S_1\lambda_1'\rangle\langle\tfrac12 m\tfrac12 m_H'|S_3\lambda_3'\rangle \times d^{S_1}_{\lambda_1'\lambda_1}(-\pi/2)d^{S_3}_{\lambda_3'\lambda_3}(-\pi/2). \tag{2.7}$$

Using the rotation properties of CGC coefficients, we move the d's in Eq. (2.7) through to the quark indices, then use some standard identities to prove that the two d's on the vertical quark line equal $d(\chi_3-\chi_1)$, a Kronecker delta, and therefore cancel each other. The left-hand edge of Fig. 2 then becomes

$$d^{1/2}_{m_H n_H}(-\pi/2)d^{1/2}_{m_H' n_H'}(-\pi/2)\langle\tfrac12 m\tfrac12 n_H|S_1\lambda_1\rangle\langle\tfrac12 m\tfrac12 n_H'|S_3\lambda_3\rangle. \tag{2.8}$$

For $\lambda_1=-1$, $\lambda_3=+1$, this expression must vanish because m cannot equal $-\frac12$ in the first CGC and $+\frac12$ in the second. Hence the couplings of Fig. 1 produce a zero at $t=0$ in the crossed t-channel amplitude having $\lambda_3-\lambda_1=2$, Q.E.D. Notice that no restrictions were placed on L_{12} or L_{34}, so that both types of Melosh couplings (those with $\mathcal{L}_z=0$ and those with $\mathcal{L}_z=\pm1$) average rapidly to zero and do so independently.[10]

III. DUALITY CONSTRAINTS ON VV COUPLINGS

Again, Fig. 1 gives the most general structure for the $VV\to(\underline{35},\mathcal{L})$ vertex allowed by the Melosh analysis, provided we now take $S_2=1$ rather than 0. We must allow W_2, T_2, and L_{12} to range over all possible values allowed by conservation of angular momentum and parity ($W_2\leq 1$, $1\leq T_2\leq 2$, $L_{12}=\mathcal{L}\pm1$), giving six possible amplitudes in all. However, these six amplitudes are not linearly independent. Just as the matrix elements for pion decays are obtained by studying matrix elements of the axial-vector current and then assuming partial conservation of axial-vector current (PCAC), so also the matrix elements for vector-meson decays are obtained by studying matrix elements of the electromagnetic current and then assuming the vector-dominance model (VDM). Since both external particles are vector mesons, we can analyze either $\langle J\lambda'|J_\mu^{\rm em}|S_1\lambda_1'\rangle$ (obtaining the structure shown in Fig. 1 or $\langle J\lambda'|J_\mu^{\rm em}|S_2-\lambda_2'\rangle$ (obtaining a structure identical to Fig. 1 but with T_2, W_2 relabeled T_1, W_1 and the T_1 line attached to S_1). Of course both analyses must yield the same helicity couplings. This will not be automatically true unless we impose constraints.

These constraints may be computed as follows. We first write down the CGC's making up the bottom half of Fig. 1, i.e., those involving S_1, W_2, T_2, and L_{12}:

$$\langle S_1\lambda_1|\tfrac{1}{2}m_{V1}\,\tfrac{1}{2}m_H\rangle(-1)^{1/2-m_H}\langle W_2W_{2z}|\tfrac{1}{2}m_{V2}\tfrac{1}{2}-m_H\rangle \times[C_2(1,0)+(-\lambda_2')C_2(0,1)+C_2(0,0)+(-\lambda_2')C_2(111)-\mathcal{L}_zC_2(110)+(-\lambda_2')C_2(2,1)]\delta(\mathcal{L}_z+W_{2z},-\lambda_2'). \quad (3.1)$$

The first two CGC's couple S_1 and W_2 to the two vertical quark lines and one horizontal quark line (azimuthal quantum numbers m_{V1}, m_{V2}, and m_H, respectively). The square brackets contains six terms, each a scalar coupling constant $C_2(|\mathcal{L}_z|, W_2)$ or $C_2(|\mathcal{L}_z|, W_2, |\lambda_2'|)$ multiplied by linear combinations of the CGC's involving T_2 and L_{12}. The linear combinations allow each term only the $|\mathcal{L}_z|$, W_2, and $|\lambda_2'|$ values occurring in the argument of the C_2 belonging to that term. Wherever there is no danger of ambiguity, we have dropped the $|\lambda_2'|$ argument of the C_2's. Certain constants occurring in the T_2 and L_{12} CGC's ($\sqrt{3}$, $\sqrt{10}$, etc.) have been absorbed into the C_2's. In short we have chosen our basis set of amplitudes to have definite $|\lambda_2'|$ and $|\mathcal{L}_z|$ rather than definite T_2 and L_{12}. This choice facilities comparison with the work of Cashmore *et al.* on $B^*\to BV$ decays.[8] If S_1 and J were baryons, then the six scalar couplings in Eq. (3.1) would be (in order of their appearance) just the A, B, a_0, C, a_1, and D parameters of Ref. 8. Equation (3.1) is the Melosh expansion when the T line occurs attached to S_2. We can relabel Eq. (3.1) to obtain the Melosh expansion when the T line is attached to S_1: Replace S_1, λ_1', W_2, W_{2z} by W_1, W_{1z}, $S_2-\lambda_2'$ in the first two CGC's; elsewhere in Eq. (3.1), replace each subscript 2 by subscript 1 and $(-\lambda_2')$ by λ_1'. We now demand that the T_1 and T_2 expansions shall be identically equal, for all values of m_{Vi}, λ_i', and $\mathcal{L}_z$. Setting the lower halves of the T_1 and T_2 coupling diagrams equal in this manner is enough to guarantee that the T_1 and T_2 couplings as a whole will be equal, because the CGC's in the upper half are common to both coupling diagrams and will cancel out when we set the two coupling expressions equal. More precisely, CGC's involving L_1, L_2, and SU(3) variables will cancel out. The $\langle\mathcal{L}\mathcal{L}_z\mathcal{S}\mathcal{S}_z|J\lambda\rangle$ CGC constitutes a unitary transformation from $\mathcal{L}\mathcal{S}$ to $J\lambda$ coupling and may be removed by using CGC completeness, and similarly for the CGC linking $\mathcal{S}$ to the two vertical quarks. In carrying out these unitary transformations, we assume no J or $\mathcal{S}$ dependence in the C_i's. I.e., we derive the constaints in the limit of exact $SU(6)_W$ symmetry, then assume that the constraints continue to hold after the masses are broken (at least, for C_i's with appropriate kinematic factors removed).

We now choose various values of m_{Vi}, λ_i' and $\mathcal{L}_z$, evaluate Eq. (3.1) involving C_2's as well as the relabeled Eq. (3.1) involving C_1's, and set the two expressions equal. In this way we obtain relations between the C_1's and C_2's. We can then eliminate (say) the C_1's and obtain the following equations involving the C_2's alone:

$$\begin{aligned} &C(2,1)=0,\\ &C(1,0)=C(111)=-C(110), \qquad (3.2)\\ &C(0,1)=-C(00). \end{aligned}$$

We have omitted subscripts 1 or 2; the equations are understood to hold for either choice of subscript. The first line implies there is no $|\mathcal{L}_z|=2$ coupling. The second and third lines allow both $SU(6)_W$-breaking ($|\mathcal{L}_z|=1$) and $SU(6)_W$-conserving ($\mathcal{L}_z=0$) terms to be present. The various $|\mathcal{L}_z|=1$ and 0 contributions occur in just the proportions demanded by the $^3P_0$ model.[15] In fact the constraints (3.2) allow Fig. 2 to be recoupled to the structure shown in Fig. 3. Figure 3 brings out the $^3P_0$ structure and makes it explicit that the coupling now treats S_1 and S_2 symmetrically. (Note: The $^3P_0$ duality diagrams usually drawn in the literature are for baryon-meson rather than meson-meson decays. The former diagrams look a bit different because they are lacking L_1 and L_2 lines. These lines would "W-S flip" the lines S_1 and S, as explained in Sec. II, and are not needed in the baryon case.)

We now square these couplings, cross to the t channel, divide out kinematical singularities, and check for rapid convergence at $t=0$, exactly as

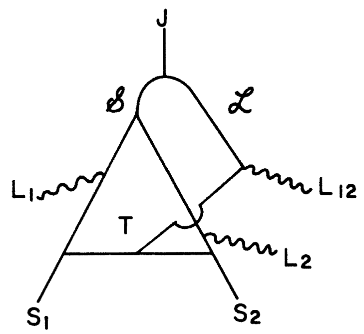

FIG. 3. Helicity structure for meson-meson decay couplings in the ${}^3P_0$ model. All unlabeled internal lines are spin-$\frac{1}{2}$ quarks.

for the $S_4 = S_2 = 0$ case discussed in Sec. II. We must verify convergence for six different amplitudes, those having

$$(\lambda_4, \lambda_2) = (+1,+1), (0,0), (\pm 1, \mp 1), (\pm 1, 0). \tag{3.3}$$

All other amplitudes having $|\lambda_3 - \lambda_1| = 2$ are related to these six by P or CP symmetries.

In this section we want to investigate critically our choice of the mass-splitting prescription. Accordingly, we shall not specialize at once to the choice $f = \mathcal{S}_{12}$ [f defined at Eq. (1.1)], as we did in Sec. II; also, we shall work initially in the SU(6)-symmetric limit (masses dependent on $\mathcal{L}$ but independent of $\mathcal{S}$ and J). In this limit Wang kinematic-singularity-removing (KSR) factors will be independent of $\mathcal{S}$ and J, so that we may carry out sums over these variables (required in the proof of rapid convergence; see Sec. II); SU(6) symmetry guarantees that f is also independent of $\mathcal{S}$ and J and dependent only on L and $\mathcal{L}$. Therefore, any f will allow rapid convergence in the SU(6)-symmetric limit. Once we have established what the results are in this limit, then we can introduce the $f = \mathcal{S}_{12}$ prescription and see if it preserves those results to the mass-split limit, as it did for the $\mathcal{L} = 0$ case studied in Ref. 11.

The KSR factor occurring in the denominator of the first four amplitudes (3.3) is

$$(\sin\theta_t/2)^{|\lambda-\mu|}(\cos\theta_t/2)^{|\lambda+\mu|}, \tag{3.4}$$

where $\lambda = \lambda_3 - \lambda_1$ and $\mu = \lambda_4 - \lambda_2$. The first two amplitudes in the list (3.3) have $\mu = 0$, and for these the factor (3.4) reduces to $\mathcal{S}_{12}^2$ at $t = 0$. Therefore, the contribution from each resonance will have a mass dependence $f^2/\mathcal{S}_{12}^2$ or $f^2/(\mathcal{S}_\pm)^2$. Provided only f is independent of $\mathcal{S}$, J, and λ_i', this dependence can be factored out and the proof of convergence for the $V\Pi$ case, Eqs. (2.3) thru (2.8), goes through unchanged for the first four VV cases.

Next we consider the last two amplitudes on list (3.3), which have $\mu = \pm 1$. These amplitudes must be divided by not only the KSR factor (3.4) but also an additional KSR factor of $t^{1/2}$. That is, these amplitudes have a kinematic zero at $t = 0$. Since we are only interested in dynamical zeros, we should expand the $\mu = \pm 1$ amplitudes near $t = 0$, and look for a factor $t^{1/2}$ times a power series in t with no constant term. If there is a dynamical zero, the $\mu = \pm 1$ amplitudes should start off as $t^{3/2}$ rather than $t^{1/2}$.

However, when we actually perform the expansions we find the leading $t^{1/2}$ terms do not vanish and there is no dynamical zero. The proof of this statement involves straightforward algebra and we sketch only the highlights. In the sharp-resonance approximation each resonance contributes

$$\langle \lambda_3'\lambda_4'|H|J\mu'\rangle d^J_{\lambda'\mu'}(\theta_s)\langle \lambda_1'\lambda_2'|H|J\lambda'\rangle \pi\delta(s - s_J) \tag{3.5}$$

to the imaginary part of the direct-channel amplitude. After crossing Eq. (3.6) to the s channel and averaging over s, we obtain the $\mu = \pm 1$ amplitudes

$$\langle \lambda_3'\lambda_4'|H|J\mu'\rangle d^J_{\lambda'\mu'}\langle \lambda_1'\lambda_2'|H|J\lambda'\rangle \prod d_{\lambda_i'\lambda_i}(\chi_i)(\pm \mathcal{S}_{12}\mathcal{S}_\pm)^{-1}. \tag{3.6}$$

The last factor is the KSR term, Eq. (3.4), evaluated at $t = 0$; also $\lambda_3 = +1$, $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_4 = \pm 1$. Since we are interested in only the leading $t^{1/2}$ behavior, we expand each d function about $t = 0$:

$$d_{\lambda'\mu'}(\theta_s) \cong \delta(\lambda'\mu') - i(J_y)_{\lambda'\mu'}\sin\theta_s, \tag{3.7}$$

$$\begin{aligned} d_{\lambda'\lambda}(\chi) &= \sum_m d_{\lambda' m}(-\pi/2) d_{m\lambda}(\chi + \pi/2) \\ &\cong \sum_m d_{\lambda' m}(-\pi/2)[\delta(m,\lambda) - i(J_y)_{m\lambda}\cos\chi]. \end{aligned} \tag{3.8}$$

J_y is the usual rotation-group generator, and we have used the fact that both $\sin\theta_s$ and $\cos\chi$ are of order $t^{1/2}$ at $t = 0$. If we insert expansions (3.7) and (3.8) into Eq. (3.6), the terms involving only Kronecker deltas vanish, as expected, while the terms linear in J_y give us the leading term we wish to examine. We can carry out the sum over J and $\mathcal{S}$ and open the amplitude up into the box form, Fig. 3, just as we did in the $V\Pi$ case, Sec. II, provided we take the precaution of rotating the $d^J(\theta_s)$ function off the J and $\mathcal{S}$ lines before inserting expansion (3.7). (That is, by repeated use of the rotation property of CGC's, we can remove the d^J from the J line and replace it by three $d(\theta_s)$'s located on the $\mathcal{L}$ line and the two vertical quark lines of Fig. 3; these d's will not interfere with the sum over J and $\mathcal{S}$.) All the terms linear in J_y then vanish except the three where the J_y occurs on the external lines λ_1' or λ_3' or the vertical quark line connecting those two lines. These three J_y terms sum to a quantity which is highly f-dependent, but it we consider the simplest case for purposes of illustration and choose f independent of L and helicities, we obtain

$$|\overline{C}_1|^2[2s^{-1/2}+(2m)^{-1}](f/\mathcal{S}_{12})^2 \tag{3.9}$$

and

$$[|\overline{C}_1|^2-|\overline{C}_0|^2][2s^{1/2}/\mathcal{S}_+ + s/2m\mathcal{S}_+](f/\mathcal{S}_{12})^2. \tag{3.10}$$

Equations (3.9) and (3.10) are the surviving J_y terms contributing to the $\mu=-1$ and $\mu=+1$ amplitudes, respectively. KSR factors (3.4) and $t^{1/2}$ have been divided out, and $\overline{C}_n$ is a scalar coupling for decay of resonances with $|\mathcal{L}_z|=n$. Evidently if we demand that these terms vanish, we obtain only the trivial solution $\overline{C}_1=\overline{C}_0=0$.

Before we examine more complex choices for the f's, let us ask whether the $\mu=\pm1$ amplitudes *should* vanish. Conceivably, there could exist a class of trajectories having nonzero residues at $t=0$ and contributing only to these two amplitudes because of selection rules. If so, the $\mu=\pm1$ amplitudes need not vanish. In fact the $J=\mathcal{L}$, $\mathcal{S}=1$ unnatural-parity trajectories (A_1, etc.) contribute only to $\mu=\pm1$, provided that these trajectories have Toller quantum numbers $M\neq1$. It is well known that unnatural-parity trajectories cannot contribute to the $\mu=0$ amplitude with $\lambda_2=\lambda_4=0$,[16] and additionally unnatural-parity trajectories will not contribute to any $\lambda=2$, $\mu=0$ or ±2 amplitudes because of a zero required by O(4) symmetry if $M\neq1$.[17] The phenomenological evidence (especially, the absence of a parity-doublet partner for the A_1) is consistent with the assignment $M=0$ for the A_1 trajectory.[18] An assignment $M=1$ has been suggested for the unnatural-parity trajectories with $\mathcal{S}=0$ (π, B, etc.), but these have odd CP and cannot couple to the $\lambda=2$ VV channel we are now considering. It is therefore reasonable to assume $M\neq1$ for unnatural-parity trajectories, and when this is done there is no longer any reason to expect the $\mu=\pm1$ amplitudes to vanish. Since we have already examined all the other amplitudes in list (3.3), our examination of the SU(6) limit is now concluded.

We now break the masses but choose the external mesons so that for the moment the reaction remains elastic. Then the Toller symmetry constraints at $t=0$ should continue to hold, and in particular unnatural-parity contributions to $\mu=\pm2$ should continue to vanish. We should again get rapid convergence of all amplitudes except the $\mu=\pm1$ ones; but we do not, at least we do not if we use the $\mathcal{S}_{12}$ prescription. Each resonance contribution to the $\mu=\pm2$ amplitudes is proportional to a factor $(\mathcal{S}_{12}/\mathcal{S}_\pm)^2$ which does not factor out or cancel out, so that the proof of convergence no longer goes through for these two amplitudes. Thus the $\mathcal{S}_{12}$ prescription does not allow the zeros found in the SU(6) limit to survive mass splitting. We shall discuss the implications of this result in Sec. IV.

At the beginning of this section we found that the VV couplings have the structure of the ${}^3P_0$ model. Furthermore, it is well known that ΠV couplings obtained from a Melosh analysis also have the structure of the ${}^3P_0$ model.[19] Historically, the ${}^3P_0$ model was derived and motivated entirely in a symmetry context, and in such a context it is natural to go one step further and assume that VV and ΠV scalar couplings are identical. Lest there be confusion, we emphasize we have not derived nor do we assume identical scalar couplings for VV and ΠV. Of course our ${}^3P_0$ results for VV couplings are derived using only symmetry arguments, rather than the duality arguments employed throughout the rest of the paper. Nevertheless, duality considerations supply the context, and in this context it would be unnatural for us to assume identity of VV and ΠV scalar couplings. The two sets of couplings are dual to two quite different set sets of Regge residues having quite different spin and energy dependences.

IV. SUMMARY

With benefit of hindsight, we conclude that the pioneering workers on $\pi\rho\to\pi\rho$ superconvergence relations got the right answer using the wrong logic. They could have assumed that the $\lambda=2$ natural-parity Reggeon coupling had a zero at $t=0$, then used semilocal duality to deduce relations between s-channel couplings. Or they could have reversed this logic, starting out with symmetry constraints on s-channel couplings and deducing zeros in crossed Regge residues (as we did here). About the only logic that should not have been used was the logic actually used (the only logic available at the time): Assume that rapid falloff in s will bring about rapid convergence of sums over resonances.

We would like to believe that zeros in t will bring about rapid convergence where falloffs in s would not. Certainly all the zeros investigated here (zeros at $t=0$ and symmetry-related) are associated with rapid convergence; but this rapidity could be symmetry-related, not caused by the zeros as such. Further work on zeros not so closely related to symmetry should clarify the situation.

In Secs. II and III we used the following simple prescription for correcting exact-SU(6)-limit calculations for the effects of mass splitting: Factor out a threshold function $\mathcal{S}_{12}$ [Eq. (2.4)] from each scalar coupling constant, then assume that only the $\mathcal{S}_{12}$ factor changes when masses are split. The initial argument in favor of $\mathcal{S}_{12}$ [that it allowed SU(6)-limit residue predictions to persist unchanged to the broken mass limit] evaporated at the end of Sec. III, when we found that the zeros of the $\mu=\pm 2$ amplitudes did not persist.

As pointed out in Sec. III, a wide class of prescriptions will yield residue zeros in the SU(6) limit. We have not explored all these possibilities, so that conceivably an f other than $\mathcal{S}_{12}$ exists which would allow zeros to persist. For the moment, however, the rapid convergence must be viewed as an SU(6)-limit effect. Even if this is the case, the rapid convergence should still be useful phenomenologically. Had the zeros at $t=0$ yielded constraints on BV resonance couplings (for example), we would have written these constraints in terms of the SU(6)-limit couplings $\overline{C}$ defined in Eq. (1.3), then tested the constraints using $\overline{C}$'s extracted from a Melosh fit to the data.

We shall not suggest an alternative to $\mathcal{S}_{12}$. However, we can show why $\mathcal{S}_{12}$ is *not* the correct prescription, and in doing so we shall uncover some of the requirements that a correct prescription must satisfy.

Firstly, the couplings cannot vary as $\mathcal{S}_{12}$ near the zeros $s=(m_1\pm m_2)^2$ of $\mathcal{S}_{12}$. At these values of s and for $t=0$, each invariant helicity amplitude is of the form

$$\text{(sum over } s\text{-channel resonances)/(KSR factor).} \tag{4.1}$$

The KSR (kinematic-singularity-removing) factor reduces to a product of $\mathcal{S}_\pm$ and $\mathcal{S}_{12}$ factors at $t=0$. If the $\mathcal{S}_{12}$ prescription is taken seriously not just at $s=(\text{resonance mass})^2$, but also down to $s=(m_1\pm m_2)^2$, then the numerator in Eq. (4.1) contains only two factors of $\mathcal{S}_{12}$ times constants, and some of the invariant and supposedly singularity-free helicity amplitudes will diverge at $s=(m_1 \pm m_2)^2$, an impossibility. The sums over resonances in Eq. (4.1) must obey constraints so as to avoid these divergences (compare the evasion and conspiracy constraints of standard Regge theory).[20,21]

It might be argued that we need not take into account $s=(m_1\pm m_2)^2$ constraints in a duality approximation because these constraints are the manifestation of threshold cut behavior which appears only when we consider the presumably small unitarity corrections to our basic dual "Born term." On the contrary, improper threshold behavior destroys properties which even a Born term amplitude ought to possess, namely superconvergence. In Eqs. (3.8) and (3.9) we calculated the contribution of the $\mathcal{L}$th level of resonances to the amplitudes $\mu=\pm 1$, assuming a mass-splitting factor f independent of helicities. We then stressed that each of these contributions need not vanish individually. However, the *sum* of all such contributions certainly must vanish. From superconvergence theory, Eq. (3.8) is just the contribution of the $\mathcal{L}$th level to the usual superconvergence relation (SCR) for the $\mu=-1$ $\lambda=2$ amplitude. Hence this SCR is the sum over all such contributions (3.8), and ought to vanish. Similarly, a sum over all contributions (3.9) ought to vanish. But if $f=\mathcal{S}_{12}$ each contribution (3.8) is positive-definite, so that the only solution to these two SCR's would be the trivial one. Furthermore, note the peculiar kinematic singularities at $s=\mathcal{S}_-=0$ multiplying each supposedly invariant contribution. Presumably the lack of superconvergence and the bizarre kinematic behavior are two effects of the same cause, a too simple choice for the mass-splitting function f. Even a dual Born term will have threshold singularities, simply because it is a helicity amplitude, and these must be removed properly.

There are also indications that $\mathcal{S}_{12}$ is not the correct prescription for $B\Pi$ scattering. For $B=\text{spin } \frac{3}{2}$, $\lambda=3$, the KSR factor at $t=0$ reduces to $\mathcal{S}_{12}^3$. The numerator in Eq. (4.1) turns out to vanish, exactly at $t=0$, so that to check whether $f=\mathcal{S}_{12}$ would remove the possible pole at $\mathcal{S}_{12}=0$, strictly speaking, we should compute the $\lambda=3$ amplitude slightly away from $t=0$. We have not had the courage to attempt this algebraically very difficult calculation. However, we can make an educated guess as to its outcome from experience with similar calculations done in Ref. 11. The KSR factor reduces to $\mathcal{S}_{12}^3$ at $t=0$, while away from $t=0$ it is $\phi^{3/2}$ (aside from irrelevant t-dependent factors), where ϕ is the Kibble polynomial. The numerator in Eq. (4.1) will contain rotation matrices with a ϕ dependence just right to cancel the ϕ dependence coming from the KSR factor. However, since the rotation matrices are dimensionless and ϕ is not, ϕ always occurs in these matrices divided by factors such as $\mathcal{S}_{12}$. When the factor $\phi^{3/2}$ cancels out, it gets replaced by a factor of $\mathcal{S}_{12}^3$. Thus we expect the denominator of the $\lambda=3$ amplitude to remain $\mathcal{S}_{12}^3$ even away from $t=0$, and the choice $f=\mathcal{S}_{12}$ probably will not be enough to remove a pole at $\mathcal{S}_{12}=0$.

For $V\Pi$ scattering the power of $\mathcal{S}_{12}$ in the KSR factor never gets higher than quadratic, so that $\mathcal{S}_{12}$ could be correct for this case.

We conclude with a final remark on mass-splitting prescriptions for decays to final states involving pions. As mentioned in the Introduction, the ΔM^2 mass-splitting prescription for pion decays is the $m_2\to 0$ limit of the $\mathcal{S}_{12}$ prescription. It follows that factoring out *both* a ΔM^2 and a p^L angular momentum barrier term from the couplings would amount to double counting. And in fact, Cashmore *et al.* tried using both factors and got worse fits to the data.[8]

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[2]Our notation is as follows. λ_i denotes the crossed t-channel helicity of particle i; λ_i' denotes its direct s-channel helicity; m_i and S_i denote its mass and spin. M denotes any member of the ground state ($\mathcal{L}=0$) $\underline{35}\oplus\underline{1}$ representation of SU(6) × O(6); V and Π denote any member of the ground-state vector and pseudoscalar nonets, respectively; B (B^*) denotes any $\mathcal{L}=0$ ($\mathcal{L}>0$) baryon. Script letters $\mathcal{L}$, $\mathcal{S}$ are reserved for quark orbital and spin angular momentum; Latin letters L, S are reserved for MM or BM orbital and spin angular momentum. J denotes the total spin of a resonance; sometimes (as in summations and subscripts of energy variables) J is used as a convenient shorthand for the entire set of scalar variables [$\mathcal{L}$, $\mathcal{S}$ plus SU(3) variables] needed to label the resonance.

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