Comments and Addenda

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Mass spectra in some two-dimensional models*

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We show that a simple "closure" assumption of the bound-state poles leads in relativistic two-dimensional models to spectra of the form $m_n = 2M\cos(n\theta_0)$.

The application of a new field-theoretic WKB approximation[1,2] to the "doublet states" of the sine-Gordon system[2,3] yielded[4] the spectrum

$$m_n = 2M\sin(n\theta_0)\,, \quad n = 1, 2, \ldots, \frac{\pi}{2\theta_0} \tag{1}$$

where M is the soliton mass and θ_0 is determined by the mass and coupling in the sine-Gordon equation. In the equivalent field theory[5,6] of the massive Thirring model M is the "elementary fermion" mass and (1) describes then the various fermion-antifermion bound states. Equation (1) was verified to nontrivial orders both in the weak- and strong-coupling limits and most recently proved by a direct lattice approach to the Thirring model.[7] An application of the WKB method to another one-dimensional field theory, the Gross-Neveu model, has also led to a fermion-antifermion spectrum

$$m_n = 2M\sin\frac{n(\pi/2)}{N}, \quad n = 1, 2, \ldots, N-1\,. \tag{2}$$

The simplicity and similarity of Eqs. (1) and (2) are clearly striking. In particular they become identical if the parameters in the sine-Gordon equation are chosen so that

$$\theta_0 = \frac{\pi/2}{N}, \tag{3}$$

a choice which is particularly natural from some points of view.[3]

We would like to show that Eq. (2) can be derived from the following assumptions:

(i) The two-dimensional field theory considered has a spectrum consisting of a fermion (antifermion) of mass M and a discrete set of bound states $m_n \le 2M$ characterized by *one* quantum number n. We note that this assumption implies in particular that the $m_i m_j$ continuum (unlike the fermion-antifermion continuum) is suppressed and does not play an important role in the decay of a higher bound state $(m_n \ge m_i + m_j)$.

(ii) An S matrix describing the scattering of the fermion states and the bound state exists. In particular we might attempt to describe it in terms of diagrams with internal fermionic lines. The particular case shown in Fig. 1 is a scattering of two bound states via the simplest nontrivial rearrangement diagram.

(iii) We assume that if we have a process involving N external "bosonic" legs ($N-1$) which lie on the mass shells of the bound-state poles (m_n^2) and the internal fermion lines are on the mass shell so that the Nth line has an invariant mass squared $S_N \le 4M^2$, then S_N ought to equal one of the bound-state values m_i^2.

Essentially the motivation for the last assump-

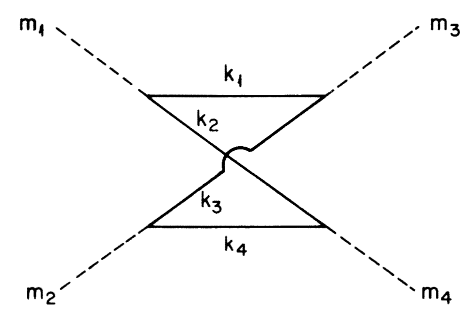

FIG. 1. A simple scattering diagram for mesons.

tion is that one can write diagrams for such a process as that of Fig. 1 with the fermion lines on the mass shell which have *a priori* no reason to be canceled. Consistency would then demand that they be interpretable as a scattering of the known states in the theory. The unkown missing system in Fig. 2 could *a priori* be some $m_i m_j$ combination, but in the spirit of (i) we neglect this possibility and since $S_4 \leq 4M^2$ the only remaining option is some of the bound states m_i. Stated differently in terms of the $m-m$ rearrangement diagram of Fig. 3, assumption (iii) amounts to the requirement that if we chose three fermion-antifermion pairs to be on some bound state's mass shell so will be the fourth. We will retrun to assumption (iii) later.

Let us denote by $k_i \phi_i$ the momenta or boost angles of the fermions (external in Fig. 3 or internal in Fig. 1) with momenta going on-shell. We take odd i (even) to correspond to fermion (antifermion). Typically the invariant masses are $(k_1+k_2)^2 = 2M^2 + 2M^2\cosh(\phi_2-\phi_1) \geq 4M^2$. However, we will be interested in the regime where we have a bound state $(k_1+k_2)^2 \leq 4M^2$. This corresponds to continuation to purely imaginary boost angles $\phi_i \to i\phi_i$ so that (choosing $\phi_1 = 0$) our assumptions are

$$(k_1+k_2)^2 = m_{i_1}{}^2 = 2M^2(1+\cos\phi_2)\,, \tag{4a}$$

$$(k_3+k_4)^2 = m_{i_2}{}^2 = 2M^2[1+\cos(\phi_4-\phi_3)]\,, \tag{4b}$$

$$(k_2+k_3)^2 = m_{i_3}{}^2 = 2M^2[1+\cos(\phi_3-\phi_2)]\,, \tag{4c}$$

and our "consistency" or "closure" assumption (iii) amounts then to the demand that also (see Fig. 3)

$$(k_1+k_4)^2 = 2M^2(1+\cos\phi_4) = m_{i_4}{}^2\,, \tag{4d}$$

where $m_{i_1}{}^2, \ldots, m_{i_4}{}^2$ belong in the set of bound states.

Since we have four masses appearing in (4a)–(4d) but only three independent relative boost angles, a constraint on the spectrum ($m_i{}^2$) is implied. In principle we can envision more complicated scattering processes involving more external bosons.

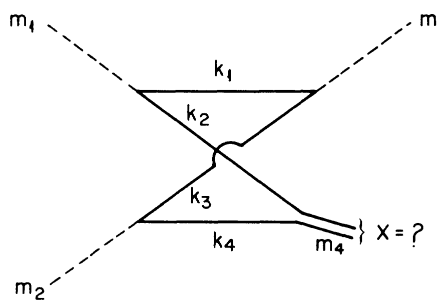

FIG. 2. Illustrating the closure hypothesis.

In all cases all invariant pair masses involve various differences of "boost" angles.

The simplest choice of boost angles which will satisfy the closure assumption is

$$\theta_n = n\theta_0\,, \quad n = 1, 2, \ldots \tag{5}$$

since then all the differences $\theta_n - \theta_m = (n-m)\theta_0$ also belong in the same family of integer times θ_0. (Negative integers are equivalent to positive integers because $\cos\theta$ is even.) In particular if we want to have only a finite number of bound states then we have to adopt the choice $\theta_0 = \pi/N$, as in Eq. (3). We note that the "closure" property could be ensured by more complicated choices for the boost angle, e.g.,

$$\theta_{n_1 n_2} = n_1\theta_1 + n_2\theta_2\,, \quad \theta_1, \theta_2 \text{ noncomeasurable}\,, \tag{6}$$

etc. The choice of (5) is dictated by our assumption that the bound states are to be characterized by *one* quantum number, and (5) with $\theta_0 = \pi/N$ follows from demanding a finite number of bound states. Thus we conclude that

$$\begin{aligned} m_n{}^2 &= 2M^2(1+\cos\theta_n) \\ &= 4M^2\cos^2(\tfrac{1}{2}\theta_0 n) \\ &= 4M^2\cos^2(n\theta_0)\,, \end{aligned} \tag{7}$$

where we rescaled $\theta_0 \to \theta_0/2$. Clearly for $\theta_0 = (\pi/2)/N$ the spectrum deduced from (6) and the spectrum (2) are identical. This is not the case for Eq. (1). The set $4M^2\cos^2(n\theta_0)$ can be made to overlap with $4M^2\sin^2(n\theta_0')$ by choosing $\theta_0 = \pi/2 - \theta_0'$ only for odd n. We find the fact that the simple ansatz on the "closure" of bound states in multiple-particle scattering does reproduce results for the sine-Gordon and Gross-Neveu models very intriguing. A common feature of these models is that they can be (exactly or in a certain approximation) reformulated in terms of bosonic fields only, and it may well be that it is this feature which is being implicitly made use of in assumptions (iii) and (i). Clearly it is a very unlikely possibility that in all two-dimensional field theories[8] bound-state spectra have always the form of Eq. (2).

If the fact that Eq. (2) is reproduced is not a mere coincidence then one conclusion may be that

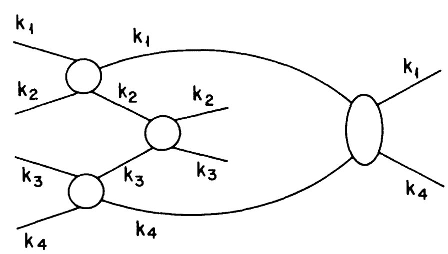

FIG. 3. A contribution to 4-4 scattering of fermions.

S-matrix methods may be a useful approach to two-dimensional field theory. Thus for the sine-Gordon theory the infinite number of classical conservation laws resulting in the separation of degrees of freedom into the soliton and pair variables prompted conjecturing a particular simple form for the Thirring model S matrix.[3] The S matrix is assumed to be diagonal in soliton (fermion) number and in their momenta as well so that (apart from permutations) it is just an overall phase shift. A particular simple way in which this could arise is if the connected part of any $n \to n$ amplitude vanishes and the complete S matrix reduces to the products of $2-2$ S matrices as illustrated (Fig. 3) for the $4-4$ amplitudes with all the intermediate quark lines on the mass shell. For the particular case of $3-3$ scattering in the tree approximation this conjecture (i.e., the vanishing of the principal-value contribution) has been directly verified.[9] If this conjectured S matrix is indeed correct then one could proceed to directly compute the S matrix, by using an appropriate S_{2-2} (see Ref. 10) which has to be iterated in a multiple scattering series only a finite number[11] of times to yield S_{n-n}.

Using the last comments let us return to the crucial assumption (iii) in an effort to clarify it and its relation to the spectrum of Eq. (2).

Consider meson-meson scattering $M_k + M_l \to M_m + M_n$ in the sine-Gordon or the equivalent Thirring model.

Following the indications of Ref. 9 and Faddeev's general conjecture we assume that in the Thirring model Feynman diagrams for this process (e.g., the diagram of Fig. 4) all internal fermion lines *have* to be on their mass shell.

The fact that the "mesons" is a bound state of $q_i\bar{q}_j$ (soliton-antisoliton) suggests that the four $M \to q_i\bar{q}_j$ vertices appearing in this diagram are nonvanishing. However, we neglect higher order $M \to q_i\bar{q}_j q_k\bar{q}_l$, etc., so that the diagram of Fig. 1 will be the only relevant diagram. Now we have the following two possibilities:

(a) The kinematics is such so as to disallow rearrangement collisions (Fig. 1) with the four intermediate fermions on the mass shell. This would mean that the meson-meson scattering amplitudes vanish.

(b) The choice of meson masses m_n is such that rearrangement scattering will in general be possible. It is this last case which corresponds to our assumption (iii).

The result that we presented here is that such nontrivial meson-meson rearrangement processes will be nonvanishing [for "excitation" quantum number (n) conserving process $l+k=n+m$] if the spectrum is of the form of Eq. (2).

Note added in proof. J. F. Schonfeld has observed (private communication) that the sequence $\theta = n\theta_0 + \pi$ satisfies our closure conditions just as well as $\theta = n\theta_0$. This choice has the following advantage: The bound-state squared masses are now

$$\begin{aligned} 2M^2 + 2M^2\cos\theta &= 4M^2\cos^2(\tfrac{1}{2}\theta) \\ &= 4M^2\cos^2(n\tfrac{1}{2}\theta_0 + \tfrac{1}{2}\pi) \\ &= 4M^2\sin^2 n\tfrac{1}{2}\theta_0 . \end{aligned}$$

Thus (replacing $\frac{1}{2}\theta_0$ by θ_0) we reproduce the formula $M_n = 2M\sin n\theta_0$ *without* the even-n gaps that followed from our original ansatz. Of course, neither ansatz tells us what should be the largest admissible n, when θ_0 is *not* a rational multiple of π. We thank Dr. Schonfeld for these remarks.

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[7] A. Luther, Harvard report, 1975 (unpublished).

[8] The diversity of possible field theories could very well reflect mainly in off-shell Green's functions, and most of the features of S-matrix theory (including bound-state poles) would be determined largely by unitarity, and analyticity, and the internal symmetry of the model.

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[10] See e.g. J. T. Cushing, Phys. Rev. 148, 1558 (1966). The construction of a one-dimensional S matrix with no production is particularly simple.

[11] The number of such multiple on-shell scatterings is the maximal number of multiple collisions which could occur for a one-dimensional classical N-particle system which is $O(N^2)$.