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Reggeon field theory in zero transverse dimensions*

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We study Reggeon field theory in zero transverse dimensions. We calculate the renormalization-group functions in closed form and show that there *is* an infrared-stable fixed point. We also calculate the critical indices exactly. We find that perturbation theory cannot be used directly to calculate the renormalization-group functions in the neighborhood of the fixed point, but that Padé and Borel-Padé approximants can. We prove that the only singularities of the partial-wave amplitude in the angular momentum plane are isolated poles. The intercept of the leading pole approaches unity only in the limit that the bare intercept goes to infinity.

I. INTRODUCTION

In the study of high-energy diffraction scattering it is essential to take into account Regge cuts as well as Regge poles. Gribov's Reggeon field theory (RFT) allows one to do this in a systematic way.[1] Reggeons are treated as quasiparticles in a space with one "time" dimension, $t=iy$, where y is the rapidity, and two "space" dimensions, $\vec{x}$, the impact parameter. The conjugate variables are $E=1-J$ (J = angular momentum) and $\vec{k}$, the transverse momentum of the Reggeon. When the intercept of the Regge pole, $\alpha(0)$, approaches the critical value, $\alpha(0)=1$, RFT exhibits the critical behavior associated with a second-order phase transition.[2-5] In particular the partial-wave amplitude satisfies a scaling law with scaling indices which are independent of the underlying parameters of the theory.

Much of the recent work on the scaling laws of RFT has made use of the ϵ expansion. One calculates in a space with $D=4-\epsilon$ transverse dimensions, rather than the physical number, $D=2$. For small values of ϵ the theory has an infrared-stable fixed point at which the dimensionless renormalized coupling constant, g, is of order $\epsilon^{1/2}$. One can therefore use perturbation theory to calculate the scaling indices as a power series in ϵ. To date the calculation has only been carried out to second order in ϵ, and for $\epsilon=D=2$ the second-order terms are comparable to the first-order terms.[6] It is not even clear whether the infrared-stable fixed point persists for $\epsilon=2$.

In this paper we calculate the renormalization-group functions exactly for $D=0$ ($\epsilon=4$). We find that there *is* an infrared-stable fixed point, which occurs at $g=2$ with our normalization of the coupling constant. We therefore expect that the fixed point found in the ϵ expansion does persist through $D=2$. We calculate the critical indices exactly at $D=0$.

The $D=0$ problem is a good laboratory for testing approximation techniques which one would like to use in two dimensions, where the fixed-point value of g is also expected to be moderately large. For example, we find that perturbation theory approximations to the Gell-Mann–Low function, $\beta(g)$, collapse long before g reaches its critical value, while Padé and Borel-Padé approximants do considerably better. The failure of perturbation theory is understandable; we find that $\beta(g)$ is singular both at $g=0$ and $g=2$. The bare perturbation series for the Green's functions diverges, but it is Borel summable when the bare intercept, α_0, is less than unity. The Green's functions for $\alpha_0>1$ must be calculated by analytic continuation from $\alpha_0<1$.

In Sec. II we study the exact Pomeron propagator for $D=0$. We find that the only singularities in the angular momentum plane are isolated poles. (The discontinuities across Regge cuts vanish as $D\to 0$.) The renormalized intercept of the Pomeron reaches unity only as the bare intercept goes to infinity, so the theory we study does not have a bona fide second-order phase transition.

For $D=0$ we evaluate in closed form the propagator, its energy derivative, and the three-point function when the energies of all incoming Pomerons are zero. In Sec. III we show that for arbitrary D this is sufficient information to determine the scaling behavior of the theory. We carry out the calculation of the renormalization-group functions and scaling indices for $D=0$ in Sec. IV. We also show that $\beta(g)$ is singular at $g=0$ and $g=2$ (the infrared-stable fixed point). Section V gives numerical results for the function $\beta(g)$. We compare the exact β with perturbation theory results and with Padé and Borel-Padé approximants, which use perturbation theory to extract information on $\beta(g)$. We find that at $D=0$ the Padé and Borel-Padé approximants converge rather slowly to the exact answer, but at $D=2$ the convergence of the Borel-Padé approximants appears to be quite rapid. In Sec. VI we summarize our results and discuss their relevance to the $D=2$ problem.

II. GREEN'S FUNCTIONS FOR $D=0$

In D dimensions the Hamiltonian for noninteracting Pomerons is

$$H_0=\int d^Dx[\alpha_0'\vec{\nabla}\psi^\dagger(\vec{x})\cdot\vec{\nabla}\psi(\vec{x})+\Delta_0\psi^\dagger(\vec{x})\psi(\vec{x})]$$
$$=\int d^Dk(\alpha_0'\vec{k}^2+\Delta_0)a^\dagger(\vec{k})a(\vec{k}). \tag{1}$$

α_0' is the bare slope parameter, $\Delta_0=1-\alpha_0$ is the bare intercept gap, and $a(\vec{k})$ is the destruction operator for a bare Pomeron with transverse momentum $\vec{k}$ and "energy" $E=1-J=\alpha_0'\vec{k}^2+\Delta_0$. $\psi(\vec{x})$ is the Fourier transform of $a(\vec{k})$. In order to study the infrared ($J\approx 1$, $t=-\vec{k}^2\approx 0$) behavior of the theory it is sufficient to include only the triple-Pomeron coupling.[1,7] We therefore take the interaction Hamiltonian to be

$$H_I=\tfrac{1}{2}ir_0\int d^Dx[\psi^\dagger(\vec{x})\psi(\vec{x})^2+\psi^\dagger(\vec{x})^2\psi(\vec{x})]. \tag{2}$$

r_0 is the bare triple-Pomeron coupling constant.

For $D=0$ the transverse coordinates disappear from the problem and the Hamiltonian becomes

$$H=H_0+H_I$$
$$=\Delta_0a^\dagger a+\tfrac{1}{2}ir_0(a^{\dagger 2}a+a^\dagger a^2). \tag{3}$$

a and $a^\dagger$ satisfy the usual harmonic-oscillator commutation relations. The Heisenberg field operators $\psi(t)$ and $\bar{\psi}(t)$ satisfy the boundary conditions $\psi(0)=a$, $\bar{\psi}(0)=a^\dagger$. The bare vacuum $|0\rangle$, which satisfies $a|0\rangle=0$, is also the vacuum of the full Hamiltonian, $H|0\rangle=0$. The propagator is

$$G^{1,1}(E)=\int_{-\infty}^{\infty}dt\,e^{iEt}\langle 0|T[\psi(t)\bar{\psi}(0)]|0\rangle$$
$$=\int_{-\infty}^{\infty}dt\langle 0|a\,e^{-i(H-E)t}a^\dagger|0\rangle$$
$$=i\langle 0|a(E-H)^{-1}a^\dagger|0\rangle. \tag{4}$$

Introducing the usual harmonic-oscillator operators

$$N=a^\dagger a,$$
$$x=2^{-1/2}(a+a^\dagger), \tag{5}$$

Eq. (4) can be simplified through the identity

$$a(E-H)=(E-H')a, \tag{6}$$

where

$$H'=(N+1)(\Delta_0+ir_0x/\sqrt{2}). \tag{7}$$

Then

$$G^{1,1}(E)=i\langle 0|(E-H')^{-1}aa^\dagger|0\rangle$$
$$=i\langle 0|[E/(N+1)-\Delta_0-ir_0x/\sqrt{2}]^{-1}|0\rangle. \tag{8}$$

By parity we see that the propagator is an even function of r_0, which we henceforth take to be positive.

The propagator is particularly simple at $E=0$, where we may use the coordinate representation to find

$$iG^{1,1}(0)=\pi^{-1/2}\int_{-\infty}^{\infty}dx(\Delta_0+ir_0x/\sqrt{2})^{-1}e^{-x^2}$$
$$=\int_0^{\infty}d\sigma\exp(-\Delta_0\sigma-r_0^2\sigma^2/8)$$
$$=(2\pi)^{1/2}r_0^{-1}\exp(+h^{-2})\mathrm{erfc}(h^{-1}), \tag{9}$$

with

$$h=r_0/\sqrt{2}\,\Delta_0. \tag{10}$$

In deriving Eq. (9) we assumed that h was positive. We take the view that the theory is initially defined for $\Delta_0>0$ ($\alpha_0<1$), and then determined for $\Delta_0<0$ by analytic continuation. Our assertion is based on the observation that the path integral defining $G^{1,1}(E)$ is convergent without distortion of the contours of integration only for $\Delta_0>0$. In addition, only for $\Delta_0>0$ are the individual terms in the perturbation expansion of $G^{1,1}(E)$ free of unphysical singularities.

The formal perturbation expansion for $G^{1,1}(0)$ gives rise to the asymptotic series

$$iG^{1,1}(0)=\Delta_0^{-1}\sum_{n=0}(2n)!(n!)^{-1}(-h^2/4)^n. \tag{11}$$

This divergent series has a Borel sum which agrees with Eq. (9) only for $\Delta_0>0$. In analytically continuing $G^{1,1}(0)$ to $\Delta_0<0$ we must distort the contour of the x integration in Eq. (9) to avoid the pole at $x=ih^{-1}$. One never sees such a distortion in perturbation theory. We can derive an asymptotic expansion for $\Delta_0<0$ by using the fact $\mathrm{erfc}(-y)=2-\mathrm{erfc}(y)$. We find

$$iG^{1,1}(0)=(8\pi)^{1/2}r_0^{-1}\exp(h^{-2})$$
$$+\Delta_0^{-1}\sum_{n=0}(2n)!(n!)^{-1}(-h^2/4)^n. \tag{12}$$

So in continuing to $\Delta_0<0$ we pick up an extra term which has an essential singularity at $r_0=0$.

From Eq. (9) we see that $G^{1,1}(0)$ is finite for all finite values of $\Delta_0, r_0\neq 0$. The Pomeron pole can approach $E=0$ only in the limit $\Delta_0\to-\infty$, i.e., $\alpha_0\to+\infty$. To investigate the spectrum of $G^{1,1}(E)$

we make use of the linear integral equation

$$G(E) \equiv [\Delta_0 + ir_0 x/\sqrt{2} - E(N+1)^{-1}]^{-1} = G(0) + G(0)E(N+1)^{-1}G(E). \tag{13}$$

Equation (13) is originally defined for $\Delta_0 > 0$ and then analytically continued to $\Delta_0 < 0$. In the Appendix we show that $G(E)$ has a Fredholm solution and that $G^{1,1}(E)$ is given by the ratio of two power series in E, each of which is an entire function. Thus the only singularities of $G^{1,1}(E)$ for finite E are isolated poles. For $\Delta_0 > 0$ and $r_0 \to 0$ these poles are located at the points $E_n = n\Delta_0$, $n = 1, 2, \ldots$. As r_0 and Δ_0 vary, they never move off the positive real E axis.

It is tempting to associate the $n \geq 2$ poles with the $t=0$ intercept of the branch points of the Regge cuts, but this is incorrect. For $r_0 \neq 0$, the poles are not spaced by integers. One can check in second-order perturbation theory that for $D > 0$, where the cuts are present, the poles remain, but we expect that most of them will not be on the physical sheet of the E plane.

For Re $E < 0$, $\Delta_0 > 0$ the perturbation series is again Borel-summable and uniquely determines $G^{1,1}(E)$. In continuing to $\Delta_0 < 0$ it is essential that $r_0 \neq 0$, for we must not miss the contribution from the pole in $G(0)$ at $x = i/h$ in the coordinate representation of Eq. (9). Indeed in the limit $\Delta_0 \to -\infty$, this pole gives the dominant contribution. As a leading approximation we drop the integrals along the real axis in Eq. (13) and find

$$G(0) \xrightarrow[\Delta_0 \to -\infty]{} 8^{1/2}\pi\, r_0^{-1} \,|x = ih^{-1}\rangle\langle x = ih^{-1}| \tag{14}$$

and

$$G(E) \xrightarrow[\Delta_0 \to -\infty]{} [8^{-1/2}\pi^{-1} r_0 - E\langle x = ih^{-1}|(N+1)^{-1}|x = ih^{-1}\rangle]^{-1} |x = ih^{-1}\rangle\langle x = ih^{-1}| . \tag{15}$$

Here the state $|x = ih^{-1}\rangle$ means that we must evaluate for real coordinate eigenstates $|x\rangle$ and then analytically continue. Using the identity[8]

$$\langle x|(N+1)^{-1}|x'\rangle = \int_0^1 dz \langle x|z^N|x'\rangle = \pi^{-1/2}\int_0^1 dz\,(1-z^2)^{-1/2}\exp\left[-\frac{1}{4}\frac{1-z}{1+z}(x+x')^2 - \frac{1}{4}\frac{1+z}{1-z}(x-x')^2\right], \tag{16}$$

we find

$$iG^{1,1}(E) = \langle 0|G(E)|0\rangle \xrightarrow[\Delta_0 \to -\infty]{} -2h^{-2}[E - (2\pi)^{-1/2} r_0 h^{-2}\exp(-h^{-2})]^{-1}. \tag{17}$$

We see that the pole position, or renormalized Pomeron intercept gap, is

$$\Delta = (2\pi)^{-1/2} r_0 h^{-2}\exp(-h^{-2}), \tag{18}$$

so the renormalized intercept approaches unity very rapidly as $h \to 0-$, the infrared-stable fixed point.

In our renormalization-group work we shall need the Green's function $G^{(1,2)}(E_1, E_2)$, in which one Pomeron of energy $E_1 + E_2$ is dissociated into two Pomerons of energies E_1 and E_2:

$$\begin{aligned} G^{1,2}(E_1,E_2) &= \int_{-\infty}^{\infty}\int dt_1 dt_2 e^{i(E_1 t_1 + E_2 t_2)}\langle 0|T[\psi(t_1)\psi(t_2)\bar\psi(0)]|0\rangle \\ &= \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \langle 0|a\, e^{-iH(t_2-t_1)} a\, e^{-iHt_1} a^\dagger|0\rangle e^{i(E_1 t_1 + E_2 t_2)} + (E_1 \leftrightarrow E_2) \\ &= (-i)^2\langle 0|a(H - E_1 - i\epsilon)^{-1} a(H - E_1 - E_2 - i\epsilon)^{-1} a^\dagger|0\rangle + (E_1 \leftrightarrow E_2). \end{aligned} \tag{19}$$

We use Eq. (6) and its adjoint with $r_0 \to -r_0$ to obtain

$$G^{1,2}(E_1,E_2) = (-i)^2\langle 0|(H' - E_1 - i\epsilon)^{-1} a a a^\dagger (H' - E_1 - E_2 - i\epsilon)^{-1}|0\rangle + (E_1 \leftrightarrow E_2). \tag{20}$$

At zero energy and positive Δ_0, we have

$$\begin{aligned} G^{1,2}(0,0) &= -2\langle 0|(\Delta_0 + ir_0 x/\sqrt{2})^{-1}(N+1)^{-1} a (\Delta_0 + ir_0 x/\sqrt{2})^{-1}|0\rangle \\ &= -2\int_0^\infty d\sigma_1 d\sigma_2\, e^{-\Delta_0(\sigma_1+\sigma_2)}\int_0^1 dz \langle 0|e^{-ir_0 x\sigma_1/\sqrt{2}} z^N a\, e^{-ir_0 x\sigma_2/\sqrt{2}}|0\rangle. \end{aligned} \tag{21}$$

Using the identities

$$\exp(\alpha a^\dagger + \beta a) = \exp(\alpha a^\dagger)\exp(\beta a)\exp(\alpha\beta/2)$$
$$= \exp(\beta a)\exp(\alpha a^\dagger)\exp(-\alpha\beta/2) \tag{22}$$

and

$$\exp(\alpha a)z^N = z^N\exp(\alpha z a) \tag{23}$$

Eq. (21) reduces to

$$G^{1,2}(0,0) = ir_0\int_0^\infty d\sigma_1 d\sigma_2\sigma_2\, e^{-(\sigma_1^2+\sigma_2^2)r_0^2/8}\, e^{-(\sigma_1+\sigma_2)\Delta_0}\int_0^1 dz\, e^{-r_0^2 z\sigma_1\sigma_2/4}. \tag{24}$$

This integral can be simplified by the substitutions $\sigma_1 = W\sqrt{8}/r_0$, $\sigma_2 = (x+y)\sqrt{8}/r_0$, $z = x/(x+y)$. The integrals over y and W can be evaluated leaving

$$G^{1,2}(0,0) = \frac{i\pi 2^{3/2}}{\Delta_0^2}h^{-2}e^{h^{-2}}\int_{h^{-1}}^\infty du\, e^{u^2}[\operatorname{erfc}(u)]^2. \tag{25}$$

The last Green's function we shall need is the derivative

$$i\frac{\partial G^{(1,1)}}{\partial E}(0) = \langle 0|(H')^{-2}|0\rangle$$
$$= \left\langle 0\left|\frac{1}{\Delta_0 + ir_0 x/\sqrt{2}}\,\frac{1}{N+1}\,\frac{1}{\Delta_0 + ir_0 x/\sqrt{2}}\right|0\right\rangle. \tag{26}$$

Using the techniques applied to the vertex, this is

$$i\frac{\partial G^{1,1}}{\partial E}(0) = \int_0^\infty\int d\sigma_1 d\sigma_2 e^{-(\sigma_1^2+\sigma_2^2)r_0^2/8-(\sigma_1+\sigma_2)\Delta_0}\int_0^1 dz\, e^{-r_0^2 z\sigma_1\sigma_2/4}. \tag{27}$$

Comparing with Eq. (24), we see there is a simple relation between this derivative and the vertex

$$G^{1,2}(0,0) = \frac{r_0}{2}\frac{\partial}{\partial\Delta_0}\left[\frac{\partial G^{(1,1)}}{\partial E}(0)\right]_{r_0}. \tag{28}$$

We collect these results together by defining functions A, B, and C through

$$iG^{1,1}(0) = \frac{A}{\Delta_0},\quad i\frac{\partial G^{1,1}}{\partial E}(0) = \frac{B}{\Delta_0^2},\quad G^{1,2}(0,0) = \frac{ir_0}{\Delta_0^3}C. \tag{29}$$

Convenient expressions for A, B, and C are

$$A(h) = 2h^{-1}\exp(h^{-2})\int_{h^{-1}}^\infty du\, e^{-u^2},$$
$$C(h) = 2h^{-3}\exp(h^{-2})\int_{h^{-1}}^\infty du\, u^{-2}e^{-u^2}[A(u^{-1})]^2, \tag{30}$$
$$B(h) = 2h^{-2}\int_{h^{-1}}^\infty du\, u^{-3}C(u^{-1}).$$

The equation for B follows from integrating Eq. (28). These formulas are used in Sec. V, where A, B, and C are evaluated on a computer by integrating down from $h^{-1} = +\infty$. Notice that in this form the analytic continuation to $\Delta_0 < 0$ is automatic—there is no singularity of any of the integrals at $h^{-1} = 0$.

For $\Delta_0 > 0$ it is also desirable to have the perturbation series for A, B, and C. $A(h)$ can be read off from Eq. (11). For the other functions we have from Eq. (30)

$$C(h) = \frac{2\exp(h^{-2})}{h^3}\int_0^h dh'\exp(-h'^{-2})[A(h')]^2,$$
$$B(h) = \frac{2}{h^2}\int_0^h dh'\, h'\, C(h'). \tag{31}$$

$C(h)$ is developed as a power series by integration by parts after Eq. (11) is substituted for $A(h)$. A, B, and C are all unity at $h = 0$.

It is clear that by these methods closed expressions can be derived for any derivative of any Green's function at zero energy. These three suffice for the renormalization-group studies which are our central concern.

III. RENORMALIZATION-GROUP FUNCTIONS AND SCALING EXPONENTS

In Ref. 4 RFT is studied for general values of the Pomeron intercept. A complete set of scaling laws and scaling exponents is obtained for small values of E, k^2, and Δ. In that work the renormalization-group functions and scaling exponents are calculated in terms of the Pomeron propagator and three-point function evaluated with $\Delta=0$ and with the energies of the external Pomerons different from zero. We have just seen that for $D=0$ the Pomeron propagator and three-point function can be evaluated in closed form for $E=0$ and $\Delta\neq 0$. We now show that this is sufficient information to determine whether the theory has an infrared fixed point and to calculate the scaling exponents. Thus we can determine the behavior of the Green's functions for $\Delta \ll E \ll r^{4/\epsilon}\alpha'^{-D/\epsilon}$ in terms of critical indices which can be calculated for $E=0$ and $\Delta\neq 0$.

We wish to develop a formalism which will be applicable for all transverse dimensions in the range $4\geq D\geq 0$, since we would eventually like to be able to study the renormalization-group functions and scaling exponents as a function of D. As usual we introduce a renormalized field operator, $\psi_R(\vec{x})$, coupling constant, r, and slope parameter, α', through the relations

$$\psi_R(\vec{x}) = Z_3^{-1/2}\psi(\vec{x}), \tag{32}$$

$$r = Z_3^{3/2}Z_1^{-1}r_0, \tag{33}$$

$$\alpha' = Z_3 Z_2^{-1}\alpha_0'. \tag{34}$$

We shall work in terms of the single-particle-irreducible proper vertex functions. The renormalized vertex function for n incoming and m outgoing Pomerons is denoted by $\Gamma_R^{n,m}$ and the unrenormalized one by $\Gamma^{n,m}$. They are related by

$$\Gamma_R^{n,m} = Z_3^{(n+m)/2}\Gamma^{n,m}. \tag{35}$$

We impose the normalization conditions

$$i\Gamma_R^{1,1}(E,k^2)\big|_{E=k^2=0} = -\mu = -Z_3\mu_0, \tag{36}$$

$$\frac{\partial}{\partial E} i\Gamma_R^{1,1}(E,k^2)\bigg|_{\substack{E=-E_N\\ k^2=0}} = 1, \tag{37}$$

$$\frac{\partial}{\partial k^2} i\Gamma_R^{1,1}(E,k^2)\bigg|_{\substack{E=-E_N\\ k^2=0}} = -\alpha', \tag{38}$$

$$\Gamma_R^{1,2}(E_1,E_2,E_3,\vec{k}_1,\vec{k}_2,\vec{k}_3)\big|_{\substack{E_1=2E_2=2E_3=-E_N\\ \vec{k}_i=0}} = r. \tag{39}$$

These normalization conditions are a generalization of the one used in Ref. 4 in that we do not require that $\mu=0$, i.e., $\Delta=0$. It is convenient to use μ or μ_0 as an independent variable rather than $\eta_0 \equiv \alpha_{0c}-\alpha_0$, as was done in Ref. 4, because at $D=0$, $\alpha_{0c}=\infty$. Note that μ is not the physical Pomeron intercept gap.

Making use of the fact that[4]

$$\mu_0 \underset{\alpha_0\to\alpha_{0c}}{\sim} K(\alpha_{0c}-\alpha_0)^{(1-\gamma)/(1-\kappa)} \tag{40}$$

the scaling law for the unrenormalized Pomeron propagator can be written in the form[4]

$$i\Gamma^{1,1}(E,k^2) \underset{E,k^2,\mu_0\to 0}{\sim} K_0\mu_0 F\left(K_1\frac{E}{\mu_0^{1/(1-\gamma)}}, K_2\frac{\alpha'k^2}{\mu_0^{(1-\tau)/(1-\gamma)}}\right). \tag{41}$$

The K's are constants and γ, τ, and κ are the scaling indices. It follows from the results of Ref. 9 that

$$\gamma = \lim_{E_N\to 0}\left(E_N\frac{\partial}{\partial E_N}\ln Z_3\bigg|_{\mu=0}\right), \tag{42}$$

$$\tau = \lim_{E_N\to 0}\left[E_N\frac{\partial}{\partial E_N}\ln(Z_3Z_2^{-1})\bigg|_{\mu=0}\right]. \tag{43}$$

Thus in order to calculate the scaling indices γ and τ it is sufficient to calculate the renormalization constants Z_2 and Z_3.

To study the renormalization constants we introduce a set of renormalization-group functions

$$\gamma_E = E_N\frac{\partial}{\partial E_N}\ln Z_3\bigg|_{r_0,\alpha_0',\mu_0\,\text{fixed}},\quad \gamma_\mu = \mu_0\frac{\partial}{\partial\mu_0}\ln Z_3\bigg|_{r_0,\alpha_0',E_N\,\text{fixed}}, \tag{44}$$

$$\tau_E = E_N\frac{\partial}{\partial E_N}\ln(Z_3Z_2^{-1})\bigg|_{r_0,\alpha_0',\mu_0\,\text{fixed}},\quad \tau_\mu = \mu_0\frac{\partial}{\partial\mu_0}\ln(Z_3Z_2^{-1})\bigg|_{r_0,\alpha_0',E_N\,\text{fixed}}, \tag{45}$$

$$\beta_E = E_N\frac{\partial}{\partial E_N}g\bigg|_{r_0,\alpha_0',\mu_0\,\text{fixed}},\quad \beta_\mu = \mu_0\frac{\partial}{\partial\mu_0}g\bigg|_{r_0,\alpha_0',E_N\,\text{fixed}}. \tag{46}$$

Here g is the dimensionless, renormalized coupling constant

$$g = \frac{r}{\alpha'^{D/4}\mu^{\epsilon/4}} = z\frac{r_0}{\alpha_0'^{D/4}\mu_0^{\epsilon/4}} \equiv zg_0, \tag{47}$$

where $z = Z_3^{1/2}Z_1^{-1}Z_2^{D/4}$. Notice that our definition of g differs from the conventional one[4] by a factor of $(E_N/\mu)^{\epsilon/4}$.

Since the Z's and the renormalization-group function are dimensionless, they can be expressed as functions of g and the dimensionless parameter

$$x=\frac{E_N}{\mu}=Z_3^{-1}\frac{E_N}{\mu_0}\equiv Z_3^{-1}x_0. \tag{48}$$

Equation (44) can therefore be written in the form

$$\gamma_E(g,x)=\beta_E\frac{\partial}{\partial g}\ln Z_3(g,x)+(1-\gamma_E)\frac{\partial}{\partial \ln x}\ln Z_3(g,x), \tag{49}$$

$$\gamma_\mu(g,x)=\beta_\mu\frac{\partial}{\partial g}\ln Z_3(g,x)-(1+\gamma_\mu)\frac{\partial}{\partial \ln x}\ln Z_3(g,x). \tag{50}$$

The partial derivative with respect to g is to be taken with x fixed and vice versa. Equations (49) and (50) can be inverted to give

$$\frac{\partial \ln Z_3}{\partial g}=\frac{\gamma_E+\gamma_\mu}{\bar\beta}, \tag{51}$$

$$\frac{\partial \ln Z_3}{\partial \ln x}=\frac{\gamma_E\beta_\mu-\gamma_\mu\beta_E}{\bar\beta}, \tag{52}$$

with

$$\bar\beta(g,x)=\beta_E(1+\gamma_\mu)+\beta_\mu(1-\gamma_E). \tag{53}$$

Similarly from Eqs. (45) and (46) we find

$$\frac{\partial \ln(z_3z_2^{-1})}{\partial g}=\frac{\tau_E(1+\gamma_\mu)+\tau_\mu(1-\gamma_E)}{\bar\beta}, \tag{54}$$

$$\frac{\partial \ln(Z_3Z_2^{-1})}{\partial \ln x}=\frac{\tau_E\beta_\mu-\tau_\mu\beta_E}{\bar\beta}, \tag{55}$$

and

$$\frac{\partial \ln z}{\partial g}=g^{-1}+\frac{\epsilon}{4}\,\frac{1-\gamma_E}{\bar\beta}, \tag{56}$$

$$\frac{\partial \ln z}{\partial \ln x}=\frac{\epsilon}{4}\,\frac{\beta_E}{\bar\beta}. \tag{57}$$

Let us begin by integrating Eq. (56) with respect to g for fixed x. Using the fact that $Z_i(g,x)|_{g=0}=1$, we see from Eqs. (47) and (56) that

$$g_0=\frac{r_0}{\alpha_0'^{D/4}\mu_0^{\epsilon/4}}=g\exp\left\{-\int_0^g dg'\left[g'^{-1}+\frac{\epsilon}{4}\,\frac{1-\gamma_E(g',x)}{\bar\beta(g',x)}\right]\right\}. \tag{58}$$

We are interested in the limit $\mu_0\to 0$ for fixed x, as we shall verify in a moment, so we must determine how the integral in Eq. (58) can diverge. For small values of ϵ this happens because $\bar\beta(g,x)$ has a simple zero as a function of g. We shall show shortly that this is also the case for $D=0$. Therefore, we shall assume that for a general value of D there exists a $g_1(x)$ such that

$$\bar\beta(g,x)\underset{g\to g_1}{\simeq}\bar\beta'(x)[g-g_1(x)]. \tag{59}$$

Then

$$\ln z(g,x)\underset{g\to g_1}{\simeq}C\ln[g_1(x)-g]+\ln\bar z(x)+O(g-g_1), \tag{60}$$

where

$$C=\frac{\epsilon}{4}\,\frac{1-\gamma_E(g,x)}{\bar\beta'(x)}\bigg|_{g=g_1(x)}. \tag{61}$$

Since $g\to g_1(x)$ as $\mu_0\to 0$, we learn from Eqs. (47) and (60) that

$$g_1(x)-g\underset{\mu_0\to 0}{\sim}[g_1/\bar z(x)g_0]^{1/C}=\mu_0^{\epsilon/4C}[g_1\alpha_0'^{D/4}/\bar z(x)r_0]^{1/C}. \tag{62}$$

It is crucial that C is independent of x. To see this we substitute Eq. (60) into Eq. (57). Unless $\partial C/\partial x=0$, the left-hand side of Eq. (57) will have a logarithmic singularity at $g=g_1(x)$ which is not present on the right-hand side of the equation.

We can now integrate Eqs. (51) and (54), and find

$$Z_3(g,x)\underset{\mu_0\to 0}{\sim}\bar Z_3(x)[g_1(x)-g]^{C_3}=\mu_0^{\epsilon C_3/4C}\bar Z_3(x)[g_1\alpha_0'^{D/4}/\bar z(x)r_0]^{C_3/C} \tag{63}$$

and

$$Z_3Z_2^{-1}\underset{\mu_0\to 0}{\sim}\bar Z_{23}(x)[g_1(x)-g]^{C_2}=\mu_0^{\epsilon C_2/4C}\bar Z_{23}(x)[g_1\alpha_0'^{D/4}/\bar z(x)r_0]^{C_2/C}, \tag{64}$$

where

$$C_3=[\gamma_E(g,x)+\gamma_\mu(g,x)]/\bar\beta'(x)\big|_{g=g_1(x)} \tag{65}$$

and

$$C_2=[\tau_E(1+\gamma_\mu)+\tau_\mu(1-\gamma_E)]/\bar\beta'(x)\big|_{g=g_1(x)}. \tag{66}$$

The absence of logarithmic singularities at $g=g_1$ in Eqs. (52) and (55) requires that C_2 and C_3 be independent of x.

The limit $\mu_0\to 0$ in Eqs. (62)–(64) is to be taken for fixed x. From Eqs. (48) and (63) we see that

$$x_{\mu_0\to 0}\sim Z_3^{-1}(g,x)\frac{E_N}{\mu_0}=\bar{Z}_3^{-1}(x)\frac{E_N}{\mu_0^{1+\epsilon C_3/4C}}, \tag{67}$$

so fixed x corresponds to a fixed value of $E_N/\mu_0^{1+\epsilon C_3/4C}$. We can therefore rewrite Eq. (63) in the form

$$Z_3 \underset{\mu_0, E_N\to 0}{\sim} \mu_0^{\epsilon C_3/4C} f_3(E_N/\mu_0^{1+\epsilon C_3/4C}). \tag{68}$$

If we now set $\mu_0=0$ for small but finite E_N, Z_3 neither vanishes nor blows up, so

$$Z_3\big|_{\mu_0=0} \underset{E_N\to 0}{\sim} k_3 E_N^{(\epsilon C_3/4C)/(1+\epsilon C_3/4C)}. \tag{69}$$

Similarly

$$Z_3 Z_2^{-1}\big|_{\mu_0=0} \underset{E_N\to 0}{\sim} k_{23} E_N^{(\epsilon C_2/4C)/(1+\epsilon C_3/4C)}. \tag{70}$$

Here k_3 and k_{23} are constants. Comparing these results with Eqs. (42) and (43) we see that

$$\gamma=\frac{\epsilon C_3/4C}{1+\epsilon C_3/4C}=\frac{\gamma_E+\gamma_\mu}{1+\gamma_\mu}\bigg|_{g=g_1}, \tag{71}$$

$$\tau=\frac{\epsilon C_2/4C}{1+\epsilon C_3/4C}=\left[\tau_E+\tau_\mu\left(\frac{1-\gamma_E}{1+\gamma_\mu}\right)\right]\bigg|_{g=g_1}. \tag{72}$$

Since γ and τ are independent of x the right-hand side of Eqs. (71) and (72) can be evaluated for any convenient value of x. To all orders of perturbation theory the Z_i are finite if either E_N or μ (but not both) are set equal to zero. If one takes $\mu=0$ as in Ref. 4 then $\gamma_\mu=\tau_\mu=0$ and Eqs. (71) and (72) become

$$\gamma=\gamma_E\big|_{g=g_1;\ \mu=x^{-1}=0}, \tag{73}$$

$$\tau=\tau_E\big|_{g=g_1;\ \mu=x^{-1}=0}. \tag{74}$$

On the other hand, if we set $E_N=0$ as is convenient for $D=0$, then $\gamma_E=\tau_E=0$ and we see that

$$\gamma=\frac{\gamma_\mu}{1+\gamma_\mu}\bigg|_{g=g_1;\ E_N=x=0}, \tag{75}$$

$$\tau=\frac{\tau_\mu}{1+\gamma_\mu}\bigg|_{g=g_1;\ E_N=x=0}. \tag{76}$$

We must now investigate whether $\bar\beta$ has a zero as a function of g. When $E_N=0$, $\beta_E=0$, so

$$\bar\beta\big|_{E_N=x=0}=\beta_\mu\big|_{E_N=x=0}. \tag{77}$$

With this normalization we must study $\beta_\mu(g,x=0)$. In order to study $\bar\beta$ in the alternative limit, $\mu\to 0$, it is convenient to introduce the more conventional renormalized coupling constant

$$\tilde g=gx^{-\epsilon/4}. \tag{78}$$

(For general values of x either g or $\tilde g$ can be used as the dimensionless renormalized coupling constant.) The corresponding renormalization-group functions are

$$\tilde\beta_E=E_N\frac{\partial}{\partial E_N}\tilde g\bigg|_{r_0,\alpha_0',\mu_0 \text{ fixed}}=\beta_E x^{-\epsilon/4}-\frac{\epsilon}{4}(1-\gamma_E)\tilde g, \tag{79}$$

$$\tilde\beta_\mu=\mu_0\frac{\partial}{\partial \mu_0}\tilde g\bigg|_{r_0,\alpha_0',E_N \text{ fixed}}=\beta_\mu x^{-\epsilon/4}+\frac{\epsilon}{4}(1+\gamma_\mu)\tilde g. \tag{80}$$

From Eq. (53)

$$\bar\beta(g,x)=[\tilde\beta_E(1+\gamma_\mu)+\tilde\beta_\mu(1-\gamma_E)]x^{-\epsilon/4}\equiv\bar{\tilde\beta}(\tilde g,x)x^{-\epsilon/4}. \tag{81}$$

From simple power counting we see that $\bar{\tilde\beta}(\tilde g,x)$ is finite in the limit $x\to\infty$. If $\bar\beta(g,x)$ has a zero at $g=g_1(x)$, then $\bar{\tilde\beta}(\tilde g,x)$ must have one at $\tilde g=\tilde g_1(x)=g_1(x)x^{-\epsilon/4}$. It is $\tilde g_1(x)$ which approaches a finite limit as $x\to\infty$, whereas $g_1(x)$ approaches a finite limit as $x\to 0$. For $\mu=0$, $\tilde\beta_\mu=0$, so

$$\bar{\tilde\beta}(\tilde g,x)\big|_{\mu=x^{-1}=0}=\tilde\beta_E\big|_{\mu=x^{-1}=0}. \tag{82}$$

Thus we can determine whether the theory has an infrared-stable fixed point by asking either whether $\tilde\beta_E|_{x^{-1}=0}$ or $\beta_\mu|_{x=0}$ has a zero with a positive slope.

The approach-to-scaling index of Frazer and Moshe is defined to be[10]

$$\lambda=\frac{\partial\bar{\tilde\beta}}{\partial\tilde g}\bigg|_{\tilde g=\tilde g_1;\ x^{-1}=0}=\frac{\partial\bar\beta}{\partial g}\bigg|_{g=g_1;\ x^{-1}=0}=\frac{\epsilon/4C}{1+\epsilon C_3/4C}. \tag{83}$$

For $E_N=x=0$

$$\lambda=\frac{\partial\beta_\mu}{\partial g}\Big/(1+\gamma_\mu)\bigg|_{g=g_1;\ x=0}. \tag{84}$$

γ, τ, and λ form a complete set of exponents which can be constructed from C, C_2, and C_3.

Finally we turn to the scaling exponent κ defined in Eq. (40). Following Ref. 4 we write

$$Z_4^{-1}=-\frac{\partial}{\partial\Delta_0}\,i\Gamma^{1,1}(E,k^2,\mu_0)\bigg|_{E=-E_N;\ k^2=0} \tag{85}$$

and

$$\kappa_E=E_N\frac{\partial}{\partial E_N}\ln(Z_3Z_4^{-1})\bigg|_{r_0,\alpha_0',\mu_0' \text{ fixed}},$$
$$\kappa_\mu=\mu_0\frac{\partial}{\partial\mu_0}\ln(Z_3Z_4^{-1})\bigg|_{r_0,\alpha_0',E_N \text{ fixed}}. \tag{86}$$

Proceeding in the usual way we find that

$$Z_3 Z_4^{-1} \underset{\mu_0\to 0}{\sim} \mu_0^{\epsilon C_4/4C} \bar{Z}_{34}(x)[g_1 \alpha_0'^{D/4}/\bar{z}(x) r_0]^{C_4/C}, \tag{87}$$

where

$$C_4 = [\kappa_E(1+\gamma_\mu) + \kappa_\mu(1-\gamma_E)]/\bar{\beta}'(x)|_{g=g_1(x)}. \tag{88}$$

Again C_4 is independent of x. Now for $E_N = 0$

$$\frac{\partial \Delta_0}{\partial \mu_0} = Z_4|_{x=0}, \tag{89}$$

so

$$\Delta_0 - \Delta_{0C} \underset{\mu_0\to 0}{\sim} \mu_0^{1-\epsilon(C_3-C_4)/4C} k_{34}. \tag{90}$$

Comparing this result with Eq. (40) we see that

$$\begin{aligned}\kappa &= \frac{\epsilon C_4/4C}{1+\epsilon C_3/4C}\\ &= \kappa_E|_{g=g_1;\,\mu=x^{-1}=0}\\ &= \frac{\kappa_\mu}{1+\gamma_\mu}\bigg|_{g=g_1;\,E_N=x=0}.\end{aligned} \tag{91}$$

IV. INFRARED SCALING IN THE $D=0$ THEORY

The scaling exponents introduced in Sec. III can be inferred by what we know about the propagator in the $D=0$ theory. We can adjust the renormalized Pomeron gap to be zero by taking $\Delta_0 \to -\infty$ for fixed r_0. This pole is isolated, so $i\Gamma_R^{1,1}(\mu=0) = aE + bE^2 = aE^{1-\gamma}[1+(b/a)E^\lambda]$. From this we read off the indices $\gamma=0$, $\lambda=1$. The total cross section is constant at high energy which is the maximum allowed by unitarity. The index κ is unity. This is seen by rewriting Eq. (90)

$$\mu_0 \underset{\Delta_0\to\Delta_{0c}}{\sim} \left(\frac{1}{\Delta_{0c}} - \frac{1}{\Delta_0}\right)^{(1-\gamma)/(1-\kappa)}. \tag{92}$$

At $D=0$, $1/\Delta_{0c}=0$. Since μ_0 goes to zero faster than any power of $1/\Delta_0$ as $\Delta_0 \to -\infty$, κ must be unity. The fourth exponent τ, which controls the shrinkage of the diffraction peak, is not defined at $D=0$, so we cannot calculate its limit as $D\to 0$.

These results can be calculated from our solution of the theory at $E=0$. Equations (36), (37), (39), and (47) give

$$\begin{aligned}g &= \frac{r_0}{\mu_0} Z_3^{1/2} Z_1^{-1}\\ &= r_0[iG^{1,1}(0)]\left[\frac{\partial}{\partial E}\frac{i}{G^{(1,1)}(E)}\bigg|_{E=0}\right]^{1/2} \frac{1}{r_0}\frac{G^{(1,2)}(0,0)}{[G^{(1,1)}(0)]^3}\\ &= \frac{\sqrt{2}\,hC}{AB^{1/2}},\end{aligned} \tag{93}$$

in the notation of Eq. (30). This gives g as a function of h and can be inverted to give the mapping $h(g)$. The exponent γ is calculated from Eq. (75). For general couplings the function γ_μ is

$$\begin{aligned}\gamma_\mu &= -\frac{\mu_0}{Z_3^{-1}}\frac{\partial Z_3^{-1}}{\partial \mu_0}\bigg|_{r_0}\\ &= -\frac{\mu_0 \partial Z_3^{-1}/\partial h|_{r_0}}{Z_3^{-1}\partial\mu_0/\partial h|_{r_0}}\\ &= \frac{hA^3(d/dh)(B/A^2)}{B(d/dh)(hA)}.\end{aligned} \tag{94}$$

Similarly, κ is given by Eq. (91), where

$$\kappa_\mu = -\frac{hAB}{[(d/dh)(hA)]^2}\frac{d}{dh}\left[\frac{1}{B}\frac{d}{dh}(hA)\right]. \tag{95}$$

To evaluate these functions at $g=g_1$, we note that this limit corresponds to h going to zero from below ($r_0>0$, $\Delta_0\to-\infty$). For this limit Eqs. (12) and (30) give

$$\begin{aligned}A(h) &\sim \frac{2\pi^{1/2}}{h} e^{1/h^2} + a(h) \quad (|h|\to 0,\ \mathrm{Re}h<0),\\ B(h) &\sim 2\pi e^{2/h^2} b_2(h) + \frac{2}{h} e^{1/h^2} b_1(h) + b_0(h),\\ C(h) &\sim -\frac{4\pi}{h^2} e^{2/h^2} C_2(h) - \frac{2}{h^3} e^{1/h^2} C_1(h) + C_0(h),\end{aligned} \tag{96}$$

where

$$\begin{aligned}a(h) &= 1 - \frac{h^2}{2} + \frac{3h^4}{4} + \cdots,\\ b_2(h) &= [C_2(h)]^2,\\ b_1(h) &= 4\pi^{1/2} C_2(h)[\xi + \ln(-1/h)] - \pi^{1/2} h^2\left(1 + \frac{13h^2}{4} + \cdots\right),\\ b_0(h) &= \frac{\eta}{h^2} + \left(1 - \frac{5h^2}{2} + \frac{8h^4}{3} + \cdots\right),\\ C_2(h) &= 1 + \frac{h^2}{2} + \frac{3h^4}{4} + \cdots,\\ C_1(h) &= 4\pi^{1/2}\left[\xi + \ln(-1/h) + \frac{h^2}{4} - \frac{3h^4}{16} + \cdots\right],\\ C_0(h) &= \left(1 - \frac{5h^2}{2} + 8h^4 + \cdots\right).\end{aligned} \tag{97}$$

The expression for A in Eq. (96) is valid when $\mathrm{Re}h<0$. The remaining expressions are obtained by treating the rest of Eqs. (30) as differential equations in h^{-1}. ξ and η are integration constants, and η is actually zero. This can be shown by using Eqs. (11) and (31) to develop the analog of Eq. (96) for $\mathrm{Re}h>0$. No integration constant appear, and matching asymptotic formulas along the imaginary axis yields $\eta=0$. The imaginary axis is a Stokes line across which the asymptotic behaviors of A, B, and C change.

Using Eq. (96) we obtain

$$g_1=2\ ,\quad \gamma_\mu=0\ ,\quad \kappa_\mu=1\ ,\quad \gamma=0\ ,\quad \kappa=1\ , \tag{98}$$

where we have taken $B^{1/2}$ to change sign at $h^{-1}=0$ where it is linear. The function $\overline{\beta}$ is

$$\overline{\beta}=\beta_\mu=-\frac{hA}{(d/dh)(hA)}\frac{d}{dh}\left(\frac{\sqrt{2}\,hC}{AB^{1/2}}\right). \tag{99}$$

From Eq. (96) we have

$$\overline{\beta}\underset{h\to 0-}{\sim}\pi^{1/2}he^{-1/h^2}(1+2h^2+\cdots)\ , \tag{100}$$

$$g\underset{h\to 0-}{\sim}2+\pi^{-1/2}he^{-1/h^2}\left(1+\frac{3h^2}{2}+\cdots\right). \tag{101}$$

We verify that $\overline{\beta}(g=g_1)=0$, which justifies our association $g\to g_1$ with $h\to 0-$. Equations (100) and (101) can be inverted

$$h\underset{g\to 2}{\sim}-[-\ln(2-g)]^{-1/2}\ , \tag{102}$$

$$\overline{\beta}(g)\underset{g\to 2}{\sim}(g-2)-\frac{(g-2)}{2[-\ln(2-g)]^{1/2}} + \text{smaller terms}\ . \tag{103}$$

We see immediately from Eq. (83) that $\lambda=1$, as predicted. We also see that $g=2$ is a branch point of $\overline{\beta}(g)$. The second derivative of $\overline{\beta}$ is infinite at $g=2$, and $\overline{\beta}$ is complex for $g>2$. This establishes that $g=2$ is a true upper bound on the renormalized coupling constant. If we try to force $g>2$, the parameters in the unrenormalized Hamiltonian become complex.

We can also show that singularities of $\overline{\beta}(g)$, $\gamma(g)$, etc. accumulate at $g=0$ along lines which are asymptotic to the rays $\arg g=\pm 3\pi/4$. We begin with the observation that the mapping $h(g)$ will be singular for those g's such that

$$\begin{aligned}0&=\frac{dg}{dh}\\ &=\frac{d\ln g}{dh}\\ &=\frac{1}{hABC}\left(ABC+hABC'-\tfrac{1}{2}hAB'C-hA'BC\right),\end{aligned} \tag{104}$$

where primes mean differentiation with respect to h. We look for solution of this equation where $\arg h\approx\pm 3\pi/4$. We must therefore retain all the exponentials in Eq. (96) because $e^{1/h^2}=O(1)$ when $\arg h=\pm 3\pi/4$. On the other hand, we take $|h|$ small, so in Eq. (104) we may use

$$\begin{aligned}A&=2h^2x+1\ ,\\ B&=2h^6x^2+8h^2x\ln(-1/h)+1\ ,\\ C&=-4h^4x^2-8x\ln(-1/h)+1\ ,\end{aligned} \tag{105}$$

where

$$x=\pi^{1/2}e^{1/h^2}h^{-3}\ . \tag{106}$$

We now further assume x is small when $|h|$ is small, so that $|\arg h|$ must be a little less than $3\pi/4$. In this regime, A, B, and C are all well approximated by unity. The derivatives in Eq. (104) are due chiefly to the variation supplied by the exponential $dx/dh\sim -2x/h^3$. Accordingly, A' and B' are small compared to C', and Eq. (104) becomes simply

$$\begin{aligned}0&\sim 1+hC'=1+16xh^{-3}\ln(-1/h)\ ,\\ x&=-\frac{h^3}{16\ln(-1/h)}\ .\end{aligned} \tag{107}$$

We see that x is small when h is, thereby justifying the simplifications made in A, B, and C. Substituting for x and taking logarithms, we find a sequence of solutions h_n:

$$\frac{1}{h_n^{\,2}}=(2n+1)\pi i+6\ln(h_n)-\ln(16\pi^{1/2}) -\ln\ln(-1/h_n)\ . \tag{108}$$

This equation can be solved iteratively by first ignoring all terms on the right except $(2n+1)\pi i$. For large n, we find

$$h_n=\frac{e^{\pm 3i\pi/4}}{[(2n+1)\pi+O(\ln n)]^{1/2}}\ . \tag{109}$$

The corresponding value of g_n is

$$g_n=\frac{e^{\pm 3i\pi/4}}{[(n+\frac{1}{2})\pi+O(\ln n)]^{1/2}}\ . \tag{110}$$

At each of these points $\overline{\beta}(g)$, $\gamma(g)$, etc. have a square root branch point, and these branch points accumulate at zero. These functions cannot be expanded in a power series in g around $g=0$, so the perturbation series for them is divergent.

V. NUMERICAL RESULTS AND COMPARISONS WITH PERTURBATION THEORY

In this section we present the numerical results for $\beta(g)$. Having an exact solution and also a simple perturbation expansion makes the model an ideal proving ground for techniques of extracting meaningful information about the fixed point from the limited information one can get from perturbation theory in a realistic model.

We first turn to the method of extracting numerical results. We have found that $G^{1,1}$, $G^{1,2}$, and $\partial G^{1,1}/\partial E$ are given by successive integrations on h^{-1} using Eq. (30). Using the expansions (11) and (31) for large positive h^{-1} as a starting point, we may integrate from $h^{-1}=+\infty$ through the entire range of h, evaluating $g(h)$ and $\beta(h)$. In Fig.1 we show the result for $\beta(g)$. The curve appears unexceptional, a smooth curve just as one would have sketched as a guess of what β might look like

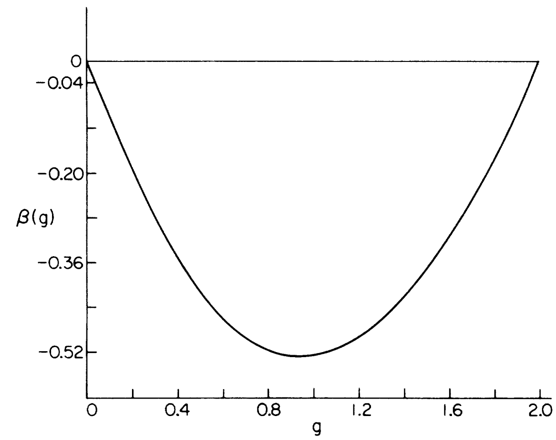

FIG. 1. The β function for Reggeon field theory in zero transverse dimensions.

if the theory had an infrared-stable fixed point. The fact that β does cross the axis at $g=2$ to very good accuracy provides a check on the integration procedure.

Actually the graph hides a peculiar effect which takes place in an exceedingly small region near $g=2$. From Eq. (103) $d\beta(g)/dg=1$ at $g=2$, but a straight-edge fit to Fig. 1 gives 1.08. This is due to an abrupt change in the derivative within $\Delta g \approx 0.001$ of the fixed point, a sign that the second derivative blows up at $g=2$.

The fixed point $g_1=2$ is not small. Away from $D=0$, we do not have exact solutions and we usually rely on results of perturbation theory truncated

TABLE I. The coefficients in the power-series expansion for $\beta(g)$ through 21 loops. $\beta(g)=\sum_{n=0} C_n g^{2n+1}$.

C_0	$=-1$
C_1	$=+0.875$
C_2	$=-1.041$
C_3	$=+2.120$
C_4	$=-5.849$
C_5	$=+19.875$
C_6	$=-79.037$
C_7	$=+375.020$
C_8	$=-1796.945$
C_9	$=+9944.826$
C_{10}	$=-59\,935$
C_{11}	$=+390\,475$
C_{12}	$=-2\,733\,767$
C_{13}	$=+20\,468\,237$
C_{14}	$=-163\,213\,379$
C_{15}	$=+1\,381\,095\,000$
C_{16}	$=-12\,362\,000\,000$
C_{17}	$=+116\,724\,000\,000$
C_{18}	$=-1\,159\,499\,000\,000$
C_{19}	$=+12\,089\,607\,000\,000$
C_{20}	$=-132\,023\,649\,000\,000$
C_{21}	$=+1\,507\,057\,000\,000\,000$

at some fixed order. We may easily reproduce these results, as it is not at all difficult to calculate asymptotic expansions for $g(h)$ and $\beta(h)$ to very high order. Using Eqs. (11) and (31) we derive a formal power series for A, B, and C. Then Eqs. (93) and (99) give $g(h)$ and $\beta(h)$ as power series. The series $g=g(h)$ can be inverted and substituted into β to get $\beta(g)$, as a power series in g. Truncating after g^{2n+1} gives the n-loop approximation to β. In Table I we present the coefficients involved up to the 21-loop approximation. One can readily see that the expansion is asymptotic rather than convergent. In Fig. 2(a), we have plotted the n-loop approximation for β, together with the exact result. We see that no approximation gives a good fit anywhere near the critical point, and that, as we would expect of an asymptotic expansion, higher approximations disintegrate sooner. That they are in fact an improvement for small g is shown in Fig. 3, where we present the error in the functions evaluated with the 2-, 3-, 4-, 5-, 7-, and 21-loop approximations.

A popular method of extrapolating a power-series expansion is to use Padé approximants. That is, one finds polynomials (in g^2) of order m and n such that the ratio is a power series with the first $m+n+1$ terms agreeing with the known coefficients for $\beta(g)/g$. We have found that the best results are near $m=n$ provided there is no zero pole pair on

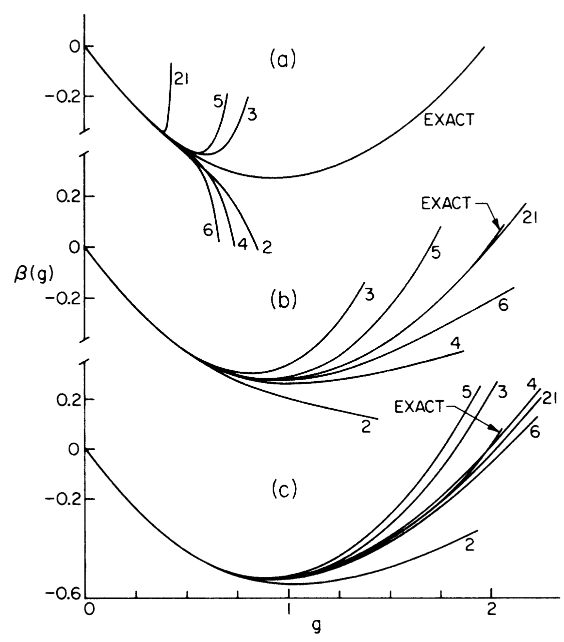

FIG. 2. The β function for $D=0$ calculated exactly and in perturbation theory. (a) The n-loop (g^{2n+1}) approximation. (b) The $(n-[n/2],\ [n/2])$ Padé approximant, where $[n/2]$ is the greatest integer $\leq n/2$. (c) The $(n-[n/2],\ [n/2])$ Borel-Padé approximant.

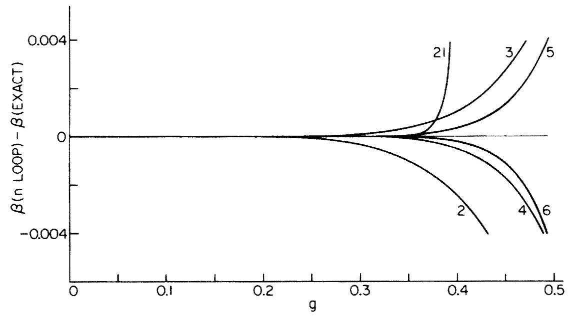

FIG. 3. The error in the n-loop (g^{2n+1}) approximation for $\beta(g)$. For sufficiently small g each additional term is an improvement.

the real axis. In Fig. 2(b) we show the middle approximant for 2, 3, 4, 5, 6, and 21 loops. We see that using Padé approximants certainly helps and that higher-order calculations give better results in general, but only converge slowly. One must go to five loops to get 10% accuracy for g_1. The slowness of convergence is due to the rapidity with which the coefficients blow up. Under such circumstances it may be better[11] to write β as a Borel expansion

$$\beta(g)=g\int_0^\infty e^{-t}\,\varphi(g^2t)\,. \tag{111}$$

The function $\varphi(x)$ is a power series with the coefficients divided by $n!$, and appears to have a finite radius of convergence in our case. If we do a Padé approximation to φ, and then evaluate β, we generally get much better results, as shown in Fig. 2(c) and Table II. Nonetheless, we do not have reliable good results until we get to 6 loops. The residual errors for $n \geq 6$ may well be due to round-off errors in our computer calculations.

TABLE II. Values for g_1 as evaluated by Padé and Borel-Padé approximants using the information available from the n-loop approximation. In each case the $(n-[n/2],[n/2])$ approximant is used, where $[n/2]$ is the greatest integer $\leq n/2$.

Number of loops	g_1-Padé	g_1-Borel-Padé
2		2.714
3	1.47925	1.836
4	3.971	2.009
5	1.709	1.774
6	2.480	2.095
7	1.844	2.024
8	2.235	2.053
10	2.143	2.065
13	1.995	2.040
17	2.015	2.035
21	2.013	2.044

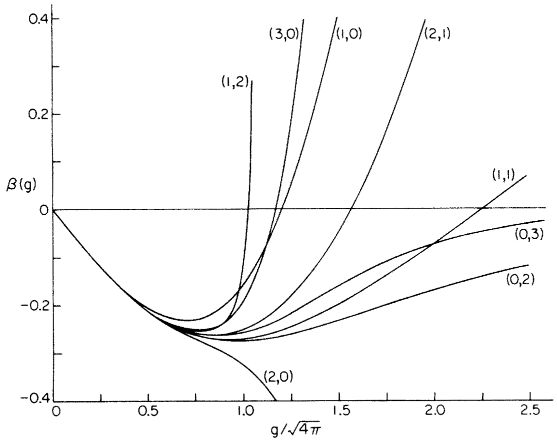

FIG. 4. The (n,m) Padé approximants to $\beta(g)$ in two transverse dimensions. The polynomials $(n,0)$ resemble those for $D=0$ [Fig. 2(a)]. g is normalized differently than in the $D=0$ work.

The conclusion is that it is possible, by means of Borel-Padé approximants, to extract fixed points at large g from perturbation theory, but that results from a few loops are not very reliable, at least at $D=0$.

The expansion of the renormalization-group functions in powers of g has been done in the physical case, $D=2$, through order g^7 by Harrington.[12] He finds that the zero of β in the 3-loop approximation is not far from that of the 1-loop approximation. In Fig. 4 we show the 1-, 2-, and 3-loop polynomial expansions as well as Padé approximants for the 2- and 3-loop results. The polynomials, (1,0), (2,0), (3,0), look very similar to Fig. 2(a), and we expect the correct β lies between the even- and odd-loop expansions. Unfortuantely the Padé approximants do not agree very well with each other. When we turn to the Borel-Padé approximants shown in Fig. 5, we find that all the reason-

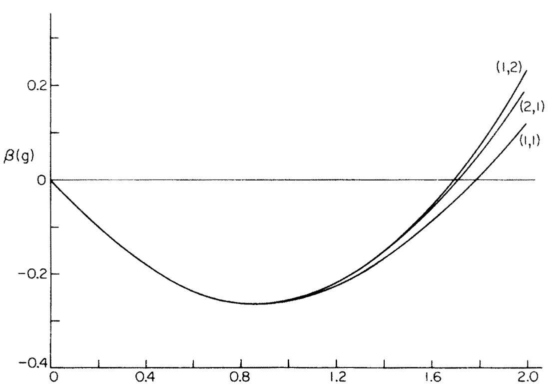

FIG. 5. The (n,m) Borel-Padé approximants to $\beta(g)$ at $D=2$ for two and three loops.

able choices [excluding $(n,0)$ and $(0,n)$] agree well, with $g_1 \simeq 1.7(4\pi)^{1/2}$, which is much larger than the value $1.17(4\pi)^{1/2}$ obtained from the polynomial expression.[12] Using the same Borel approximants for $z(g)$ and $\gamma(g)$ gives $z(g_1) \simeq 1.13$ and $\gamma(g_1) = -0.20$ to -0.25 for the 3-loop results. These results are considerably larger in magnitude than Harrington's values of $z(g_1) = 1.07$, $\gamma(g_1) = -0.13$ from the power series.

VI. SUMMARY

Our principal result is that $\beta(g)$ has an infrared-stable zero for $D=0$ just as it does for D near 4. It therefore seems quite likely that such a zero exists in the physical number of dimensions, $D=2$.[13] $\beta(g)$ has singularities both at $g=0$ and at $g=g_1=2$. The loop expansion gives rise to an asymptotic series, and we have seen in Fig. 2 that it cannot be used directly to determine g_1. Padé and Borel-Padé approximants can be used to calculate g_1 and the critical indices, but the rate of convergence is disappointingly slow at $D=0$. For $D=2$, $g_1/(4\pi)^{1/2} \approx 1-2$, and Padé and Borel-Padé approximants again appear to be useful. The agreement among the Borel-Padé approximants for the 2- and 3-loop calculations is very encouraging. It would of course be desirable to have the 4-loop results in order to be certain that the agreement is not fortuitous, but that would require a formidable calculation.

We have shown that for $D=0$ the only singularities of the partial-wave amplitude in the E plane are isolated poles which lie on the positive real axis. The fact that the only singularity for which $\mathrm{Re}E=0$ is at $E=0$ justifies the renormalization-group treatment of the Pomeron as an infrared problem. Because the Pomeron is a simple pole for $D=0$, the critical indices take on the trivial values $\gamma=0$, $\kappa=\nu=1$. The intercept of the renormalized pole reaches $E=0$ $(J=1)$ only in the limit that the bare intercept goes to infinity, so there is not a true phase transition.

While this work was being completed we received three reports in which the $D=0$ problem is treated from a different point of view.[14-16] Where our calculations overlap the results are in qualitative agreement with those of Refs. 15 and 16. In these papers it is also found that the only J-plane singularities of the partial-wave amplitude are simple poles and that the leading pole does not reach unity for finite values of Δ_0/r_0.

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APPENDIX

We start by showing that $iG^{1,1}(E)$ has a Fredholm construction. For this purpose it is convenient to write

$$iG^{1,1}(E) = iG^{1,1}(0) + E\frac{\partial}{\partial E}iG^{1,1}(0) + \langle\psi_0|(1-K)^{-1}K|\psi_1, \tag{A1}$$

where

$$K = Ee^{-\rho p}U^{1/2}G(0)U^{1/2}e^{\rho p}, \tag{A2}$$

$$\langle\psi_0| = E^{1/2}\langle 0|G(0)U^{1/2}e^{\rho p}, \tag{A3}$$

$$|\psi_1\rangle = E^{1/2}e^{-\rho p}U^{1/2}G(0)|0\rangle. \tag{A4}$$

Here we have defined $U=(N+1)^{-1}$, and $p = 2^{-1/2}i(a^\dagger - a)$ is the "momentum operator." $\rho \equiv h^{-1} = \sqrt{2}\,\Delta_0/r_0$ is the inverse dimensionless coupling constant.

Our first task is to show that K is an L^2 operator. We shall work in the momentum representation where

$$\langle q|U^{1/2}|q'\rangle = 2\pi^{-1/2}\int_0^\infty d\sigma\langle q|e^{-(N+1)\sigma^2}|q'\rangle = \pi^{-1}\int_0^1 dz[(-\ln z)(1-z^2)]^{-1/2}\exp\left[-\frac{1}{4}\frac{1-z}{1+z}(q+q')^2 - \frac{1}{4}\frac{1+z}{1-z}(q-q')^2\right] \tag{A5}$$

$(z=e^{-\sigma^2})$, and

$$\langle q|e^{-\rho p}G_0e^{\rho p}|q'\rangle = \frac{\sqrt{2}}{r_0}\theta(q'-q) \le \frac{\sqrt{2}}{r_0}. \tag{A6}$$

As a result

$$\mathrm{tr}KK^\dagger = |E|^2\mathrm{tr}[e^{-2\rho p}U^{1/2}G_0U^{1/2}e^{2\rho p}U^{1/2}G_0^\dagger U^{1/2}] \le \frac{2|E|^2}{r_0^2}\left[\int_{-\infty}^\infty dq\,dq'\langle q|e^{-\rho p}U^{1/2}e^{2\rho p}U^{1/2}e^{-\rho p}|q'\rangle\right]^2 \equiv \frac{2|E|^2}{r_0^2}I^2. \tag{A7}$$

But

$$I = 2\pi^{-1/2} \int_0^1 dz_1 dz_2 [\ln z_1)(\ln z_2)(1 - z_1{}^2 z_2{}^2)]^{-1/2} \exp\left\{2\rho^2\left[\frac{1 - z_1 z_2}{1 + z_1 z_2} - \frac{(z_1 + z_2)(1 - z_1)(1 - z_2) + \frac{1}{2}(z_1 - z_2)^2}{1 - z_1{}^2 z_2{}^2}\right]\right\}. \tag{A8}$$

I can be bounded by noting that the exponential is less than $e^{2\rho^2}$. The remaining integral can be evaluated giving

$$I \le \pi^{3/2} e^{2\rho^2}. \tag{A9}$$

So

$$\text{tr}KK^\dagger \le 2\pi^3 \frac{|E|^2}{r_0^2} e^{4\rho^2}. \tag{A10}$$

One can obtain considerably better bounds for $\Delta_0 > 0$, but this one is sufficient for our purposes. Equation (A10) tells us that the resolvent of K, $R = (1 - K)^{-1}K$, is an L^2 operator for all finite values of E except isolated points at which the Fredholm determinant, $D(E) = \det(1 - K)$, vanishes.

Next we must consider the norms of $|\psi_0\rangle$ and $|\psi_1\rangle$

$$\begin{aligned}\langle\psi_0|\psi_0\rangle &= |E|\langle 0|G_0 U^{1/2} e^{2\rho p} U^{1/2} G_0^\dagger|0\rangle \\ &\le \frac{2|E|}{r_0^2}\int_{-\infty}^{\infty} dq \langle 0|q\rangle e^{\rho q}\int_{-\infty}^{\infty} dq' e^{\rho q'}\langle q'|0\rangle I \\ &\le 4\pi^2 \frac{|E|}{r_0^2} e^{3\rho^2}.\end{aligned} \tag{A11}$$

The same bound applies to $\langle\psi_1|\psi_1\rangle$. As a result, we know from the standard Fredholm theorems that $G^{1,1}(E)$ exists for all finite values of E. Its only singularities are isolated poles arising from the zeros of the Fredholm determinant.

What can be said about the position of the poles? For $\Delta_0 \ge 0$ it is trivial to show from Eq. (13) that there are no poles in $G^{1,1}(E)$ in the left half E plane. The proof is more difficult for negative values of Δ_0. First we note that a pole can exist only if K has a normalizable eigenvector with eigenvalue 1. That is,

$$K|\psi\rangle = |\psi\rangle, \tag{A12}$$

with

$$\langle\psi|\psi\rangle < \infty. \tag{A13}$$

Defining $|\varphi\rangle = U^{1/2} e^{\rho p}|\psi\rangle$ Eqs. (A12) and (A13) are equivalent to

$$(\Delta_0 + i r_0 x/\sqrt{2})(N+1)|\varphi\rangle = E|\varphi\rangle \tag{A14}$$

and

$$\langle\varphi|(N+1)^{1/2} e^{-2\rho}(N+1)^{1/2}|\varphi\rangle < \infty. \tag{A15}$$

In the coordinate representation Eq. (A14) becomes a second-order differential equation

$$\left[\frac{d^2}{dx^2} - x^2 - 1 + 2^{3/2}(E/r_0)(ix + \rho)^{-1}\right]\varphi(x) = 0. \tag{A16}$$

We expect one solution to this equation to behave like $e^{-x^2/2}$ for large x, so we write $\varphi(x) = e^{-x^2/2}v(x)$ and find

$$\left[\frac{d^2}{dx^2} - 2x\frac{d}{dx} - 2 + 2^{3/2}(E/r_0)(ix + \rho)^{-1}\right]v(x) = 0. \tag{A17}$$

Equation (A17) has an irregular singular point at $x = \infty$. One solution has a power-series expansion of the form

$$v(x) = \sum_{n=1}^{\infty} C_n x^{-n}, \tag{A18}$$

which corresponds to $\varphi(x)$ having the asymptotic behavior

$$\varphi(x) \underset{x\to\infty}{\sim} x^{-1}e^{-x^2/2}.$$

Since the Wronskian of Eq. (A16) is a constant, the second solution will have the asymptotic behavior

$$\varphi(x) \underset{x\to\infty}{\sim} e^{x^2/2}.$$

As $x \to -\infty$ there will also be two independent solutions with these behaviors. Of course only for certain discrete values of E will there be a solution that goes to zero at both plus and minus infinity. Such a solution will satisfy Eq. (A15) because

$$(N+1)^{1/2}x^{-1}e^{-x^2/2} \underset{|x|\to\infty}{\sim} 2^{-1/2}x^{-1}e^{-x^2/2}. \tag{A19}$$

It is clear that for any value of E there can be at most one solution to Eqs. (A14) and (A15). In other words, it is not possible for two poles of $G^{1,1}(E)$ to coincide. On the other hand, if $\varphi(x)$ is a solution to Eq. (A16) with energy E, then $\varphi^*(-x)$ is a solution with energy E^*. So, if there were any complex poles they would have to come in complex-conjugate pairs. For $\Delta_0 > 0$ and $r_0 = 0$ the poles are on the real E axis at the points $n\Delta_0$, $n = 1, 2, 3, \ldots$. As we vary r_0 and Δ_0 they cannot leave the real axis because that would require two of them to coincide, which we have just proved is impossible. Since the poles are all on the positive real axis for $\Delta_0 > 0$ and no pole passes through the point $E = 0$ for finite Δ_0 and $r_0 \neq 0$, we conclude that $G^{1,1}(E)$ is analytic in the left half E plane. We again recall that one can obtain a sensible theory for $\Delta_0 < 0$ only if one continues from positive to negative values of Δ_0 with r_0 different from zero.

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