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Bound states and solitons in the Gross-Neveu model*

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The results for the spectrum of bound states and of solitons first deduced by Dashen, Hasslacher, and Neveu for a model of interacting fermions by techniques of functional integration are obtained here by methods based on Heisenberg field mechanics analogous to those applied previously to models of self-interacting bosons. The method of solution is suggested by a simplified physical picture of the bound states: These are computed in a Hartree approximation in which the self-consistent potential is a sum of contributions from the fermions (and antifermions) occupying orbitals in the conventional many-body picture and from the vacuum fluctuations of single-closed-loop type. In the same approximation the self-consistent field generated by the heavy soliton is a result of the vacuum fluctuations alone. As the main new technical contribution, we deduce and solve directly equations determining the self-consistent fields as well as the amplitudes ("wave functions") from which these are constructed. We comment on the degeneracy of the heavy soliton state.

I. INTRODUCTION

This paper continues our program of studying the quantum properties of model nonlinear field theories based directly on the solution of matrix elements of the field equations.[1-4] Our previous work has been confined to the study of self-coupled boson models in $1+1$ dimension. Here we turn to a model of self-coupled fermions in $1+1$ dimension and, as in most of our previous work, we have been guided in the choice of a tractable and instructive model by another of the pioneering studies by Dashen, Hasslacher, and Neveu,[5] this time of the model of Gross and Neveu.[6]

The specifics of the model will be reviewed below, but let us first try to make clear what we think we have contributed to the subject. To start with, we have formulated our approximations in such a way as to render it transparent that what is involved for the bound states is a Hartree approximation, in the sense of atomic or nuclear physics, except that the self-consistent field is generated not only by the motion of the particles present, but also by the vacuum polarization field which they produce (at the one-loop level). Second, we have calculated the self-consistent field by direct construction and solution of an equation for it. We have then checked the self-consistency by solving for the "wave functions" which the potential generates and by which it is defined. The same method is applied to establish the properties of the soliton, first derived for this theory by Gross and Neveu.[6]

Moreover, it appears that the same methods can be applied, at least approximately, to other models which cannot be solved in closed form, even in the Hartree approximation. Thus just as we concluded previously[4] that we could in principle solve in a systematic way any self-coupled boson Hamiltonian of polynomial form in $1+1$ dimension, we shall attempt, in a later work, to show that, based on the methods of this paper, we can do the same for any self-coupled fermion field with quartic self-coupling in $1+1$ dimension, such as a generalized class of massive Thirring models.

Turning then to the specific model studied in this work, it is for our purpose defined by the Hamiltonian density

$$\mathcal{H}(x) = -i\psi^\dagger(x)\alpha\partial_x\psi(x) - \tfrac{1}{2}g_0{}^2[\bar\psi(x)\psi(x)]^2 , \qquad (1.1)$$

which we take seriously as a field operator. Here x is the spatial point and time is fixed, for example, at the origin. The operator $\psi(x)$ has N internal degrees of freedom so that, for instance,

$$\bar\psi(x)\psi(x) = \sum_{i=1}^{N} \psi_i^\dagger(x)\beta\psi_i(x) \qquad (1.2)$$

is a double scalar, under Lorentz transformations and under internal SU(N) transformations, and α, β are conventionally chosen Pauli matrices (see Appendix A). We shall henceforth write

$$g_0{}^2 = g^2 Z^{-1} \qquad (1.3)$$

to take care of one of the renormalization requirements of this theory; the other is the subtraction of the vacuum energy from (1.1) (see again Appendix A).

We base our study totally on the equations of motion

$$\begin{aligned} i\partial_t\psi_i(x) &= [\psi_i(x), H] \\ &= -i\alpha\partial_x\psi_i(x) - \tfrac{1}{2}g^2 Z^{-1}\{\beta\psi_i(x), \bar\psi(x)\psi(x)\} , \end{aligned} \qquad (1.4)$$

which follows from (1.1) and the anticommutators

$$\{\psi_i(x), \psi_j^\dagger(x')\} = \delta_{ij}\delta(x - x') \ . \tag{1.5}$$

In Sec. II, the equations of the Hartree approximation are derived and shown to be identical to the equations obtained by DHN using functional integration. In Sec. III these equations are shown to have a soliton solution; in Sec. IV they are solved for the bound states for which they were originally formulated. Finally in Sec. V, we discuss our results briefly with particular allusion to degeneracy of the soliton state.[7] Two appendixes complete this account: In Appendix A, we review the vacuum and one-fermion sector of the theory; in Appendix B some details of the solution of the Dirac equation in the self-consistent fields are given. We cannot pass to the body of this work without reference to several more papers containing some discussion of fermions related to the present context,[8-10] and which have provided some stimulus to the present work.[10a]

II. SELF-CONSISTENT-FIELD PICTURE OF THE GROSS-NEVEU MODEL

DHN[5] have shown that the bound-state spectrum of the model of GN (Ref. 6) possesses a large degeneracy, the energy depending in their semiclassical approximation only on the total number $n<N$ of "quanta" (fermions plus antifermions). For each n the solutions were then ordered according to irreducible representations of O($2N$), the symmetry group of the model. Starting from a physical picture, we shall derive directly from the equations of motion some[10a] of the results found by functional integration.

We understand the O($2N$) symmetry to signify that the force between two fermions (antifermions) is the same as that between fermion and antifermion. We characterize the states $|n\rangle$ as independent-particle states (subject to the Pauli principle). In more detail $n = n_f + \bar{n}_f$, where n_f is the number of fermions, $\bar{n}_f$ the number of antifermions. Each particle occupies the same space-spin orbital, but the fermions are characterized by a set of distinct indices from the set $1, \ldots, N$ as are the antifermions (independently). We are then asserting that the approximate eigenstates are, in configuration space, Slater determinants, or rather a product of such determinants, one for the fermions, one for the antifermions.

To calculate the energy and other properties of this system, we define a set of Dirac "wave functions," both for bound states and continuum states. For this purpose, we require, in addition to the bound states $|n\rangle$, the scattering states (with suitable boundary conditions) $|n-1,p\rangle$ for the scattering of a fermion (momentum p, mass m) by the previously defined bound states of $n-1$ particles. We shall then work with the amplitudes

$$\langle n-1|\psi_i(x)|n\rangle \equiv \psi_n(x) \ \text{ or } \ 0 \ , \tag{2.1}$$

$$\langle n+1|\psi_i(x)|n\rangle \equiv \chi_n(x) \ \text{ or } \ 0 \ , \tag{2.2}$$

$$\langle n-1|\psi_i(x)|n-1,p\rangle \equiv \psi_{np}(x) \ \text{ or } \ 0, \tag{2.3}$$

$$\langle n-1,p|\psi_i(x)|n-1\rangle \equiv \chi_{np}(x) \ \text{ or } \ 0. \tag{2.4}$$

The matrix elements are thus independent of the internal index i and are nonvanishing if, in (2.1), the state i is an occupied fermion state in n and unoccupied in $n-1$, in (2.2) if i is an unoccupied antifermion state in $|n\rangle$ but occupied in $|n+1\rangle$, in (2.3) and (2.4) if i is the index characterizing the particle of momentum p.

In the Hartree approximation valid for $N \gg 1$, these amplitudes will be solutions of a one-particle Dirac equation in a self-consistent external field. The form of this equation, and of its adjoint, to be derived, is

$$\epsilon_A\psi_A(x) = -i\alpha\partial_x\psi_A(x) + m(x)\beta\psi_A(x) \ , \tag{2.5}$$

$$\bar{\psi}_A(x)\epsilon_A = -i\partial_x\bar{\psi}_A(x)\,\alpha + \bar{\psi}_A(x)\,\beta m(x) \ . \tag{2.6}$$

Here $A = n$ or (n,p). For $A = n$, $\epsilon_A \equiv \omega_n = E_n - E_{n-1}$, where E_n is the energy of the state n; for $A = (n,p)$, $\epsilon_A = E(p) = (p^2+m^2)^{1/2}$, the energy of the physical fermion. The self-consistent field $m(x)$ is

$$m(x) = Z^{-1}g\sigma(x) \ , \tag{2.7}$$

where

$$\sigma(x) = -g\langle n|\bar{\psi}(x)\,\psi(x)|n\rangle \tag{2.8}$$

will turn out to be the self-consistent field defined by DHN. Furthermore, the amplitudes $\chi_A(x)$ defined in (2.2) and (2.4) satisfy the same equations as $\chi_A(x)$ with $\epsilon_A \to -\epsilon_A$. It follows that we may choose

$$\chi_A = i\beta\alpha\psi_A \ , \tag{2.9}$$

and

$$\bar{\psi}_A\psi_A = -\bar{\chi}_A\chi_A \ .$$

We illustrate the derivation of these equations with (2.5). Forming a nonvanishing matrix element (2.1), we derive from the equation of motion (1.4) the equation

$$\omega_n\psi_n(x) = -i\alpha\partial_x\psi_n(x) - \tfrac{1}{2}g^2Z^{-1}\sum_I\left[\ \langle n-1|\psi_i(x)|I\rangle\langle I|\bar{\psi}(x)\,\psi(x)|n\rangle + \langle n-1|\bar{\psi}(x)\,\psi(x)|I\rangle\langle I|\beta\psi_i(x)|n\rangle\right] , \tag{2.10}$$

where $|I\rangle$ is any intermediate state. To simplify (2.10), the relevant observation is that $\langle n|\bar{\psi}(x)\psi(x)|n\rangle$ is of order N (see below), whereas any other choice of intermediate state $I \neq n$ yields contributions down by N^{-1}. (The detailed considerations needed to substantiate this assertion will be omitted and no such small contributions will be included.) To the same order, we can replace $(n-1)$ in $\langle n-1|\bar{\psi}(x)\psi(x)|n-1\rangle$ by n. The result is Eq. (2.5) for $A=n$. A similar consideration in which we assume

$$\langle n-1,\ p|\bar{\psi}(x)\psi(x)|n-1,\ p\rangle \simeq \langle n|\bar{\psi}(x)\psi(x)|n\rangle$$

yields (2.4) with $A=n,p$. Furthermore, it appears that a less cavalier treatment of the sum over states would allow consideration of corrections to the present treatment.

The physical picture underlying our treatment and its implementation by simple but approximate completeness arguments lead directly to expressions for the self-consistent field and for the energy of the state $|n\rangle$. Thus utilizing (2.1)–(2.4) and subsequently (2.9)

$$\begin{aligned}\langle n|\bar{\psi}(x)\psi(x)|n\rangle &= \sum_I \langle n|\bar{\psi}(x)|I\rangle\langle I|\psi(x)|n\rangle \\ &= n_f\bar{\psi}_n(x)\psi_n(x) + (N-\bar{n}_f)\bar{\chi}_n(x)\chi_n(x) \\ &\quad + N\sum_p \bar{\chi}_{np}(x)\chi_{np}(x) \\ &= (n-N)\bar{\psi}_n(x)\psi_n(x) - N\sum_p \bar{\psi}_{np}(x)\psi_{np}(x)\,. \end{aligned} \tag{2.11}$$

In this expression the part proportional to n is the self-consistent field generated by the "real" particles present in the state n, whereas the remainder is the contribution of the vacuum polarization. For purposes of actual computation, the decomposition in (2.11) into bound-state and continuum contributions is the convenient one, as we shall see.

From the Hamiltonian (1.1), we compute the energy E_n,

$$E_n = \langle n|H|n\rangle - \mathcal{E}_{\rm vac}L\ , \tag{2.12}$$

where $\mathcal{E}_{\rm vac}$ is the additive constant computed in Appendix A and L is the size of the system. Utilizing the Hartree approximation explained above, we find straightforwardly and with the help of Eqs. (2.5) and (2.6)

$$\begin{aligned} E_n &= (n-N)\,\omega_n - N\sum_p (E_p - E_p^{(0)}) \\ &\quad + \tfrac{1}{2}Z^{-1}\int dx\,[\sigma^2(x) - \sigma_0{}^2]\,. \end{aligned} \tag{2.13}$$

Here

$$\begin{aligned}\sigma_0 &= -g\langle \mathrm{vac}|\bar{\psi}(0)\psi(0)|\mathrm{vac}\rangle \\ &= gN\sum_p \frac{m}{E_p^{(0)}}\ , \end{aligned} \tag{2.14}$$

and the superscript on $E_p^{(0)}$ is used to distinguish the discrete values of p assumed by a free particle confined to the line of length L as opposed to the values for an interacting particle. The approximate connection[5]

$$p \cong p^{(0)} - [2\delta_n(p)/L]\ , \tag{2.15}$$

where $\delta_n(p)$ is the phase shift which will be determined from a study of the amplitudes $\psi_{np}(x)$, permits (2.14) to be rewritten in the form

$$\begin{aligned} E_n &= (n-N)\omega_n + \frac{N}{\pi}\int_{-\infty}^{\infty}\frac{p\,dp}{E_p}\,\delta_n(p) \\ &\quad + \tfrac{1}{2}Z^{-1}\int dx[\sigma^2(x) - \sigma_0{}^2]\ . \end{aligned} \tag{2.16}$$

It is finally noteworthy that (2.13) is stationary with respect to variations of $\sigma(x)$,

$$\delta E_n/\delta\sigma(x) = 0\ . \tag{2.17}$$

This is verified by noting first that (2.17) applied to (2.13) yields the condition

$$Z^{-1}\sigma(x) = -(n-N)\frac{\delta\omega_n}{\delta\sigma(x)} - N\sum_p \frac{\delta E_p}{\delta\sigma(x)}\,. \tag{2.18}$$

Furthermore, from the integrated form of (2.5) we conclude that

$$\delta\epsilon_A/\delta\sigma(x) = Z^{-1}g\bar{\psi}_A(x)\psi_A(x)\ , \tag{2.19}$$

which together with (2.18) yields (2.11).

Equations (2.11)–(2.19), derived as Hartree approximations in this section, agree in all particulars with corresponding equations in the work of DHN. These equations were solved in the latter work by invoking the techniques of inverse scattering theory. In the succeeding sections we shall obtain all required results by straightforward calculations.

III. THE HEAVY-SOLITON SECTOR

Further study of the problem defined in the preceding section will be deferred until the next section. It is a remarkable fact that the equations for the self-consistent field possess a solution corresponding to $n=0$ and $\omega_n=0$ in (2.11) and (2.13). We are dealing here with a sector of Hilbert space completely orthogonal to the space built upon the vacuum with the addition of any finite number of fermions and antifermions. Tentatively we find this portion of Hilbert space to consist of (i) A heavy particle—the soliton, (ii) other states of the same energy (2^N such states in all; see Sec. V) but differing from the soliton by the value of fermion number—loosely speaking, fermions of zero-energy bound to the soliton, and (iii) scattering states of normal fermions and the solitons. In this section we shall develop the theory of this sector.

Let us define the quantities

$$\sigma_A \equiv \bar{\psi}_A(x)\psi_A(x), \tag{3.1}$$

$$\rho_A(x) \equiv \psi_A^\dagger(x)\psi_A(x). \tag{3.2}$$

With the help of Eqs. (2.5) and (2.6), we derive

$$\frac{d}{dx}\sigma_A(x) = 2i\epsilon_A\bar{\psi}_A(x)\alpha\psi_A(x). \tag{3.3}$$

$$\frac{d^2}{dx^2}\sigma_A(x) = -4\epsilon_A{}^2\sigma_A(x) + 4\epsilon_A m(x)\rho_A(x), \tag{3.4}$$

$$\epsilon_A\frac{d}{dx}\rho_A(x) = m(x)\frac{d}{dx}\sigma_A(x), \tag{3.5}$$

where

$$m(x) = -\sum_A g^2 Z^{-1}\sigma_A(x), \tag{3.6}$$

and we also define the weighted sum utilized below,

$$s(x) = -\sum_A g^2 Z^{-1}\frac{\sigma_A(x)}{\epsilon_A{}^2}. \tag{3.7}$$

As we shall see below, $\sigma_A = 0$ for A = bound state, and, by definition, since $\epsilon_A = 0$ for this case, we exclude this term from the sum in the definition (3.7).

From (3.3) to (3.7), it is straightforward to derive the equation

$$\frac{d^3 s(x)}{dx^3} - \left[\frac{d}{dx}\ln m(x)\right]\frac{d^2 s(x)}{dx^2} - 4m^2(x)\frac{ds(x)}{dx} = 0. \tag{3.8}$$

This equation also holds for the bound-state problem of the preceding section, where the sum (3.7) now *includes* the bound state, though for reasons to be seen in the next section we have found it to be useful only for the soliton problem.

We shall now prove that (3.8) admits a solution of the form

$$\frac{ds(x)}{dx} \propto \frac{dm(x)}{dx}. \tag{3.9}$$

With this ansatz, (3.8) becomes

$$\frac{d}{dx}\left[m(x)\frac{d^2m(x)}{dx^2} - \left(\frac{dm}{dx}\right)^2 - m^4\right] = 0, \tag{3.10}$$

or

$$m(x)\frac{d^2m(x)}{dx^2} - \left(\frac{dm}{dx}\right)^2 - m^4(x) = -m_0{}^4, \tag{3.11}$$

where $m_0 = m(+\infty)$ is the mass of a fermion. This equation has the solutions

$$m(x) = \pm m_0 \tanh m_0(x - x_0). \tag{3.12}$$

For definiteness in the following discussion, we choose the plus sign and $x_0 = 0$.

We shall now check that Eq. (3.12) does indeed provide a solution to the problem posed. We shall do this by finding the solutions of (2.5) in the potential (3.12) and checking that they regenerate $m(x)$ by using (2.11). Toward this end we utilize the representation $\alpha = \sigma_1$, $\beta = \sigma_3$. We thus look for solutions of

$$\sigma_1\frac{d}{dx}\psi_A - m(x)\psi_A = -\sigma_3\epsilon_A\psi_A. \tag{3.13}$$

For $\epsilon_A = 0$, the bound state, we find

$$\psi_A \sim \begin{pmatrix}(\cosh m_0x)^{-1}\\(\cosh m_0x)^{-1}\end{pmatrix}, \tag{3.14}$$

$$\bar{\psi}_A\psi_A = 0. \tag{3.15}$$

For the scattering solutions, we write

$$z = m_0x, \quad p = m_0k, \quad E_p = \epsilon m_0, \tag{3.16}$$

$$\psi_p(x) = e^{ikz}\begin{pmatrix}f(z)\\g(z)\end{pmatrix}. \tag{3.17}$$

Then f, g satisfy the coupled equations

$$\begin{aligned}&-ikg - \epsilon f + (\tanh z)f - \frac{dg}{dz} = 0,\\&-ikf + \epsilon g + (\tanh z)g - \frac{df}{dz} = 0.\end{aligned} \tag{3.18}$$

The reader will then verify that the (unnormalized) solution to this equation with a definite phase shift is

$$\begin{aligned}&f = 1 + (\epsilon - ik)^{-1}\tanh z,\\&g = (\epsilon - ik)^{-1}(\epsilon + ik - \tanh z).\end{aligned} \tag{3.19}$$

Since $m(x) \propto \sum_p \bar{\psi}_p(x)\psi_p(\psi)$, the elementary verification that

$$\bar{\psi}_p(x)\psi_p(x) \propto (|f|^2 - |g|^2) \propto \tanh z, \tag{3.20}$$

independently of p, completes the proof that (3.12) is indeed our self-consistent potential. Furthermore, we have

$$\begin{aligned}&\frac{f(\infty)}{f(-\infty)} = \frac{2k}{(\epsilon-1)^2 + k^2}(k+i),\\&\frac{g(\infty)}{g(-\infty)} = \frac{2k}{(\epsilon+1)^2 + k^2}(k+i),\end{aligned} \tag{3.21}$$

demonstrating that the phase shift $\delta(k)$ is given by

$$\delta(k) = \tfrac{1}{2}\tan^{-1}(1/k). \tag{3.22}$$

We shall finally use our results to calculate the energy from (2.16) with the first term of the latter absent. Since

$$\sigma(x)=\sigma_0\tanh Z^{-1}g\sigma_0 x \tag{3.23}$$

$(m_0=Zg\sigma_0)$, we calculate

$$\tfrac{1}{2}Z^{-1}\int dx\,[\sigma^2(x)-\sigma_0^2]=-\sigma_0/g\,, \tag{3.24}$$

$$\sigma_0=\frac{Ng}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{m\,dp}{E_p}\,, \tag{3.25}$$

where Λ is a cutoff. Adding this to the integral over the phase shift, we obtain for the energy, i.e., the mass of the soliton after integrating by parts in a way familiar from previous work

$$M=(Nm/\pi)\,, \tag{3.26}$$

the result quoted by DHN. Further discussion of this state and of the states degenerate with it is delayed until Sec. V.

IV. BOUND STATES IN THE GN MODEL

We now return to the problem posed in Sec. II. The method found to establish the existence of the soliton sector is not directly useful here because Eqs. (3.8)–(3.11) contain only the scale associated with the fundamental fermion, its Compton wavelength. In the present problem we need a formulation which brings in the scales associated with the bound states proper.

Toward this end, we utilize Eqs. (3.4) and (3.5). Let us write

$$\begin{aligned} m(x)&\equiv m_0+m_B+m_C\\ &\equiv m_0+\mu(x)\,, \end{aligned} \tag{4.1}$$

where the subscripts distinguish bound-state and continuum contributions. We first consider (3.4) and (3.5) for $A=B$ and eliminate ρ_B between these two equations by integrating the second of them. We thus find an equation to characterize $m_B(x)$, namely,

$$\begin{aligned} 0=-\frac{d^2m_B(x)}{dx^2}&+4\beta^2m_B(x)+8m_0\mu(x)\,m_B(x)+4\mu^2(x)\,m_B(x)\\ &-4[m_0+\mu(x)]\int_{-\infty}^{x}m_B(x')\frac{d\mu(x')}{dx'}\,, \end{aligned} \tag{4.2}$$

where $\beta^2=m^2-\omega^2$ determines the size of the bound state.

For the scattering states, in order to integrate (3.5), we define the amplitudes

$$r_p\equiv\rho_p(x)-\rho_p(-\infty)=\rho_p-L^{-1}\,, \tag{4.3}$$

$$\tau_p\equiv\sigma_p(x)-\sigma_p(-\infty)=\sigma_p-(m/E_pL)\,. \tag{4.4}$$

Eliminating r_p between (3.4) and (3.5), we find an equation for $\tau_p(x)$,

$$\begin{aligned} 0=-\frac{d^2\tau_p}{dx^2}&-4p^2\tau_p+(4E_p/L)\,\mu(x)\\ &+8m_0\mu(x)\,\tau_p+4\mu^2(x)\,\tau_p-4[m_0+\mu(x)]\int_{-\infty}^{x}\tau_p(x')\frac{d\mu}{dx'}\,. \end{aligned} \tag{4.5}$$

Equations (4.2) and (4.5) are to be solved in conjunction with the definition

$$\mu(x)=m_B(x)+Z^{-1}g^2N\sum_p\tau_p(x)\,. \tag{4.6}$$

From (4.5) we see that for large p

$$\tau_p=(LE_p)^{-1}\mu(x)+O(p^{-3})\,. \tag{4.7}$$

Since m_B is $O(Z^{-1})$, Eq. (4.7) verifies (4.6) to order unity in Z^{-1}, but still leaves us without a method of solution for $\mu(x)$. The limiting form (4.7) suggests, however, that quite generally τ_p might be proportional to $\mu(x)$, and consequently we must also have

$$m_B(x)\propto\mu(x)\,. \tag{4.8}$$

The assumption (4.8) would, in any event, present itself as the natural starting point for an iteration procedure from the mere examination of (4.2). In the present instance we are grateful to find it to be exact.

With the help of (4.8), (4.2) is now transformed into the simple equation

$$-\frac{d^2\mu(x)}{dx^2}+4\beta^2\mu(x)+6m_0^2\,\mu^2(x)+2\mu^3(x)=0\,. \tag{4.9}$$

With $z=\beta x$, (4.9) has the solution

$$\begin{aligned} \mu(z)&=\frac{-2\beta^2}{\omega\cosh 2z+m}\\ &=\frac{-2\beta^2}{2\omega\cosh^2 z+(m-\omega)}\\ &=\beta[\tanh(z-b)-\tanh(z+b)]\,, \end{aligned} \tag{4.10}$$

where

$$b=\tfrac{1}{4}\ln\frac{m+\beta}{m-\beta}\,. \tag{4.11}$$

The last form is exhibited because it is the one given by DHN.

To calculate the energy, we need, according to Eq. (2.16), the potential derived above and the phase shift. From (2.3) and the representation

$$\psi_k=e^{ikz}\begin{pmatrix}\xi(z)\\ \dfrac{ik}{E+m}\eta(z)\end{pmatrix}, \tag{4.12}$$

where all momenta and energies are measured in units of β, we derive the coupled equations

$$\frac{1}{ik}\frac{d\eta}{dz}+(\eta-\xi)-\frac{1}{(E-m)}\left(\frac{1}{\omega}\,\frac{1}{\cosh^2 z+b}\right)\xi=0\,, \tag{4.13}$$

$$\frac{1}{ik}\frac{d\xi}{dz}+(\xi-\eta)+\frac{1}{(E+m)}\left(\frac{1}{\omega}\,\frac{1}{\cosh^2 z+b}\right)\eta=0\,, \tag{4.14}$$

where $b=(m-\omega)/2\omega$. With the definitions (4.12) ξ and η become equal for large $|z|$. We find that Eqs. (4.13) and (4.14) possess the exact solution

$$\xi=1+\frac{ia\sinh z\cosh z}{\cosh^2 z+b}+\frac{a_2}{\cosh^2 z+b}\,, \tag{4.15}$$

$$\eta=1+\frac{ia\sinh z\cosh z}{\cosh^2 z+b}+\frac{b_2}{\cosh^2 z+b}\,, \tag{4.16}$$

where

$$a=k^{-1}\,, \tag{4.17}$$

$$a_2=[2\omega(E+m)]^{-1} \tag{4.18}$$

$$b_2=[2\omega(E-m)]^{-1}\,. \tag{4.19}$$

In verification of (4.7), we find exactly

$$L\tau_p=(E_p)^{-1}\mu(x)[1+\omega^2/(p^2+\beta^2)]\,. \tag{4.20}$$

Equations (4.15)–(4.17) imply the phase shift

$$\delta(k)=\tan^{-1}(1/k)=\tan^{-1}(\beta/p)\,. \tag{4.21}$$

The bound-state wave function, not required for the energy calculation, is given in Appendix B.

The calculation of the energy now parallels precisely that given by DHN: We first calculate

$$\tfrac{1}{2}Z^{-1}\int[\sigma^2(x)-\sigma_0{}^2]=-\frac{2\sigma_0\beta}{gm}\equiv-\frac{2\sigma_0\sin\theta}{g}\,. \tag{4.22}$$

On the other hand, for the integral over the phase shift, we obtain after integrating by parts

$$\begin{aligned}\frac{N}{\pi}\int_{-\infty}^{\infty}dE_p\delta(k)&=\frac{2N}{\pi}\beta+\frac{N}{\pi}\int_{-\infty}^{\infty}\frac{dpE_p}{k^2+1}\frac{1}{\beta}\\&=\frac{2\sigma_0\sin\theta}{g}+\frac{2N\beta}{\pi}+\frac{N}{\pi}\beta^{-1}(m^2-\beta^2)\int_{-\infty}^{\infty}\frac{d\nu}{(\nu^2+1)(\nu^2+\csc^2\theta)^{1/2}}\\&=\frac{2\sigma_0}{g}\sin\theta+\frac{2N\beta}{\pi}+\frac{2Nm}{\pi}\cos\theta(\tfrac{1}{2}\pi-\theta)\,.\end{aligned} \tag{4.23}$$

The energy, Eq. (2.16), becomes

$$E_n=nm\cos\theta+\frac{2Nm}{\pi}\sin\theta-\frac{2N}{\pi}\theta\cos\theta\,. \tag{4.24}$$

This expression is still variational with respect to choice of θ.
We find

$$\frac{dE}{d\theta}=0\rightarrow\theta=(n\pi/2N) \tag{4.25}$$

and

$$E_n=\frac{2Nm}{\pi}\sin\frac{n\pi}{2N}\,. \tag{4.26}$$

This formula suggests a connection with the sine-Gordon model, but we shall make no effor to pursue this matter, which is under study elsewhere.[11]

As has been discussed,[5] when $n=N$, we obtain a limiting bound state which can decay into a soliton-antisoliton pair. Thus such objects must exist with fermion number as large in absolute value as N. This point is explained in the next section.

V. DEGENERACY IN THE SOLITON SECTOR

Let $\psi_B(x)$ be the normalized version of the zero-energy bound-state wave function (3.14). Following Jackiw and Rebbi[7] we define the mode operators $a_i^\dagger$, $i=1,\ldots,N$,

$$a_i^\dagger=\int dx\,\psi_B(x)\hat{\psi}_i^\dagger(x)\,, \tag{5.1}$$

where the caret indicates the field operator, where ambiguity exists. If $|s\rangle$ is the soliton state, then the ensemble of 2^N states

$$|s\rangle, a_i^\dagger|s\rangle, a_{i_1}^\dagger a_{i_2}^\dagger|s\rangle,\ \ldots\ , a_1^\dagger\ldots a_N^\dagger|s\rangle \tag{5.2}$$

is degenerate in energy with total degeneracy 2^N. Jackiw and Rebbi have argued that to maintain symmetry under fermion number conjugation these states should be assigned fermion numbers $-\frac{1}{2}N$, $-\frac{1}{2}N+1,\ \ldots,\frac{1}{2}N$. We see no objection to this assignment, but on the other hand, there is not, within the present model, any "experimental" way of distinguishing between this assignment and that in which one chooses, for instance, the values $0,1,\ldots,N$, *as long as for the antisoliton space* [the minus sign in Eq. (3.12)] *we choose the reversed signs of these quantum numbers*. This last assignment does, however, give up fermion number conjugation symmetry, except in the vacuum sector of Hilbert space.

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APPENDIX A

We review briefly the properties of the GN model for the vacuum and one-particle states needed in the text to carry out renormalization procedures. We first study matrix elements connecting the vacuum to particle and antiparticle states assumed to be of mass m

$$\langle \mathrm{vac}|\psi_i(x)|p,j(m)\rangle = \delta_{ij}e^{ipx}\psi_+(p,m)\,,$$
$$\langle \bar{p},j(m)|\psi_i(x)|\mathrm{vac}\rangle = \delta_{ij}e^{-ipx}\psi_-(-p,m)\,. \tag{A1}$$

The spinors $\psi_\pm(p,m)$ are solutions of the equations

$$\pm E(p)\psi_\pm(p,m) = (\alpha p + \beta m)\psi_\pm(p,m)\,, \tag{A2}$$

$$E(p) = (p^2+m^2)^{1/2}\,. \tag{A3}$$

If we choose the representation

$$\alpha = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A4}$$

then the solutions normalized to $\psi^\dagger\psi = 1$ are

$$\psi_+ = [(m+E)/2E]^{1/2}\begin{pmatrix} 1 \\ ip/(m+E) \end{pmatrix},$$
$$\psi_- = \sigma_x\psi_+\,. \tag{A5}$$

To derive an expression for m, we form the particle matrix element (A1) in (A3). In a Hartree approximation valid when $n \gg 1$, we find

$$E(p)\psi_+(p,m) = \alpha p\psi_+(p,m) - g^2Z^{-1}\langle \mathrm{vac}|\bar{\psi}(0)\psi(0)|\mathrm{vac}\rangle\psi_+(p,m)\,. \tag{A6}$$

Comparison with (A2) and a consequent sum over intermediate states yields

$$\begin{aligned} m &= -Z^{-1}g^2\langle \mathrm{vac}|\bar{\psi}(0)\psi(0)|\mathrm{vac}\rangle \\ &= -Z^{-1}g^2N\sum_p \psi_-^\dagger(p,m)\beta\psi_-(p,m) \\ &= mZ^{-1}g^2N\sum_p \frac{1}{E(p)}\,. \end{aligned} \tag{A7}$$

This yields a formula for Z,

$$Z = g^2N\sum_p \frac{1}{E(p)}\,. \tag{A8}$$

Furthermore, the "γ_5" invariance of H reflects itself in (A7), in that the sign of m is not determined.

We use the same results and arguments to calculate the vacuum energy

$$\begin{aligned} \mathcal{E}_{\mathrm{vac}} &= \langle \mathrm{vac}|\mathcal{H}(x)|\mathrm{vac}\rangle \\ &= \sum_p N\psi_-^\dagger(p,m)\alpha p\psi_-(p,m) \\ &\quad - \tfrac{1}{2}g^2Z^{-1}[\langle \mathrm{vac}|\bar{\psi}(0)\psi(0)|\mathrm{vac}\rangle]^2\,. \end{aligned} \tag{A9}$$

Utilizing Eq. (A9) with $\psi_+ \to \psi_-$, $E \to -E$, (A9) is simplified and evaluated:

$$\mathcal{E}_{\mathrm{vac}} = -\sum_p NE(p) + \tfrac{1}{2}g^2Z^{-1}N^2\left(\sum_p \frac{m}{E(p)}\right)^2. \tag{A10}$$

This provides a needed subtraction constant to render the calculations in the main text finite.

APPENDIX B

Since the properties of the bound-state amplitudes were not required in Sec. IV, we give here some details of the calculation of this quantity. From Eq. (4.8), we know that (B referring to bound state)

$$\sigma_B(x) = \bar{\psi}_B(x)\psi_B(x) = C\mu(x)\,, \tag{B1}$$

where C is a (cutoff-dependent) constant. By integration of Eq. (3.5), we then find, using (4.10),

$$\begin{aligned} \omega\rho_B(x) &= Cm_0\mu(x) + \tfrac{1}{2}C\mu^2(x) \\ &= C\left\{\frac{-2m_0\beta^2}{2\omega\cosh^2 z + (m-\omega)} + \frac{2\beta^4}{[2\omega\cosh^2 z + (m-\omega)]^2}\right\}. \end{aligned} \tag{B2}$$

Now write

$$\psi_B(x) = R(x)\begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix}, \quad R = \rho^{1/2}\,. \tag{B3}$$

The Dirac equation becomes

$$\begin{aligned} \omega R\cos\phi &= \left(-\frac{d}{dx}R\right)\sin\phi - R\cos\phi\frac{d\phi}{dx} \\ &\quad + m(x)R\cos\phi\,, \end{aligned} \tag{B4}$$

$$\omega R\sin\phi = \frac{dR}{dx}\cos\phi - R\sin\phi\frac{d\phi}{dx} - m(x)R\sin\phi\,,$$

from which we can eliminate R and obtain

$$\omega = -\frac{d\phi}{dx} + m(x)(\cos^2\phi - \sin^2\phi)\,. \tag{B5}$$

But

$$\begin{aligned}\cos^2\phi - \sin^2\phi &= \frac{\overline{\psi}_B\psi_B}{\psi_B^\dagger\psi_B} \\ &= \frac{\sigma_B}{\rho_B} \\ &= \frac{2\omega(m+\omega)\cosh^2 z + \beta^2}{2m(m+\omega)\cosh^2 z - \beta^2}\,.\end{aligned} \tag{B6}$$

Inserting the expressions (B1) and (4.10) for σ_B, the expression (B2) for ρ_B and the expression $m(x) = m_0 + \mu(x)$ into (B5), we find the equation

$$\frac{d\phi}{dx} = \frac{-\beta^2(m+\omega)}{2m(m+\omega)\cosh^2\beta x - \beta^2}\,, \tag{B7}$$

or

$$\begin{aligned}\phi(x) &= -\tan^{-1}\frac{2\beta\tanh\frac{1}{2}z}{(m+\omega)[1+\tanh^2\frac{1}{2}z]} \\ &= -\tan^{-1}\left(\frac{m-\omega}{m+\omega}\right)^{1/2}\tanh z\,.\end{aligned} \tag{B8}$$

As a check (B6) can be verified from (B8). The "soliton" behavior of this phase has been emphasized by previous authors.[8,9]

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[10a]*Note added in proof.* The physical considerations of Sec. II and the developments attendant thereupon are essentially correct as far as they go, but do not embrace the full generality of DHN. The treatment in Sec. IV, for instance, corresponds only to their time-independent case, and the quantum number n discussed below is the Casimir invariant n_0 of the previous work. A treatment with full generality has now been carried out and will be communicated as part of a subsequent paper. We are grateful to David K. Campbell for a discussion which stimulated this note.

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