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Semiclassical bound states in a model with chiral symmetry

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We discuss semiclassical bound states in a manifestly chiral-symmetric model, the Nambu–Jona-Lasinio model in 2 space-time dimensions. The mass spectrum of the bound states in the theory is found using the semiclassical method of Dashen, Hasslacher, and Neveu. We also discuss the connection between the classical confined solution of the nonlinear Dirac equation and the quantized version of the theory.

I. INTRODUCTION

In quantum field theory we have to deal with systems with infinite degrees of freedom. For any realistic model there is no hope of finding exact solutions. Some approximation methods of solutions have to be used. The covariant perturbation theory of Schwinger, Feynman, and Dyson has yielded in quantum electrodynamics many quantitative results which have been verified experimentally to a very high degree of accuracy. Unfortunately, perturbation theory does not provide a quantitative framework for discussing strong interactions. The quanta obtained from quantizing the free fields are too far away from hadrons to serve as a useful starting point.

The last few years have seen a revival in making a different connection between field theory and hadron physics. One attempt has been to associate particle-like solutions of the classical nonlinear field equations with hadrons.[1] To make such a connection meaningful, these particle-like solutions should be carried over into quantum field theory and some calculational schemes must be developed. The quantum effects[2] have been studied by various techniques. These include among others a (a) functional-integral technique,[3] (b) Kerman-Klein method and Green's-function technique,[4] (c) variational technique,[5] and (d) canonical quantization approach.[6]

The model that we shall discuss is a two-dimensional space-time version of the four-fermion interaction studied by Nambu and Jona-Lasinio[7] in their classical work on chiral symmetry. Our motivations for considering this model are twofold.

A particle-like solution of a classical field equation has finite energy. It connects the vacuum field configurations at infinity. In most models that have been studied, the degeneracy of the vacuum is either finite or infinite but discrete in nature. It is natural to ask what would happen if we have continuous degeneracy of the vacuum.

Another reason to consider the present model is to try to understand the role of chiral symmetry in some quark confinement schemes (e.g., the SLAC bag[5]). Chang, Ellis, and Lee[8] have done work in this direction. They have succeeded in finding a confined solution of the classical nonlinear equation describing fermions. However, the connection of their results to the second-quantized version is unclear. It would be nice to find a model with chiral symmetry whose quantum effects we would be able to handle.

It turns out that the mass spectrum of the bound states in our model can be found explicitly in the semiclassical approximation. We believe that this is the first chiral model whose quantum effects can be treated in a consistent way. Since we know how to treat quantum effects, we can study the connections between the classical confined solutions of the nonlinear wave equation describing fermions and the quantum system. Let us describe briefly our results.

We find that the vacuum field configurations at infinity are related to each other by a chiral rotation. The chiral rotation angle is quantized and is proportioned to the fermion number of the bound state. The mass spectrum of the stable bound states is found explicitly. We also find that the solution of the classical nonlinear wave equation does not help us in understanding the quantum system in our model.

This paper is organized as follows: In Sec. II, we present the model and discuss the semiclassical method of Dashen, Hasslacher, and Neveu[9] to find the mass spectrum of the bound states. The semiclassical method of DHN[9] is a version of the stationary-phase approximation. Section III introduces the powerful technique of the inverse scattering method to find the static field configurations satisfying the stationary-phase conditions. We show that the stationary-phase conditions imply that the potentials be reflectionless. The reconstruction of potentials using the inverse scattering method is also presented. We discuss in Sec. IV how to quantize the system. We then obtain expressions for the mass spectrum of the bound states in our model. In the last section, we make several relevant remarks. One concerns the relevance of the classical solution of the nonlinear wave equation for the fermion. Another concerns the problem of spontaneous symmetry breaking and the Goldstone particle. Appendix A discusses an explicitly broken chiral model. This is one example which cannot be treated by the inverse scattering method in any simple way. The massive Thirring model in the large-N limit is discussed very briefly in Appendix B. This is yet another case in which the inverse scattering method does not provide any simplification. We present the derivation of the trace identities for the Dirac equation in Appendix C.

II. THE MODEL AND THE SEMICLASSICAL APPROXIMATION

The model that we shall consider is the famous model of Nambu and Jona-Lasinio[7] in one space and one time dimension. It is described by the Lagrangian

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \tfrac{1}{2} g^2 [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2], \tag{2.1}$$

where ψ is the N-component, massless fermion field. This Lagrangian is invariant under the chiral transformation

$$\psi \to e^{i\theta\gamma_5}\psi. \tag{2.2}$$

Equivalently, the theory can be described by the following Lagrangian:

$$\mathcal{L}(\sigma, \pi, \psi) = \bar{\psi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi - \tfrac{1}{2}(\sigma^2 + \pi^2). \tag{2.3}$$

Gross and Neveu[10] have investigated this model in the limit of large N with $\lambda = g^2 N$ fixed. They found that $\bar{\psi}\psi$ develops a nonvanishing vacuum expectation value so that chiral symmetry is broken spontaneously and the fermion becomes massive. In this paper we are interested in finding the mass spectrum of the bound states.

Our starting point is the functional integral representation of $\mathrm{tr}\,e^{-iHT}$,

$$\mathrm{tr}\,e^{-iHT} = \int [d\psi][d\bar{\psi}][d\sigma][d\pi] \times \exp\left[i \int_0^T dt \int_{-\infty}^{\infty} dx\, \mathcal{L}(\sigma, \pi, \psi) \right], \tag{2.4}$$

where the integrations run over fields with the periodic conditions

$$\begin{aligned} &\sigma(t+T) = \sigma(t), \\ &\pi(t+T) = \pi(t), \\ &\text{and} \\ &\psi(t+T) = -\psi(t). \end{aligned} \tag{2.5}$$

Following DHN we first integrate over ψ and $\bar{\psi}$ fields. The following result is useful:

$$\int [d\psi][d\bar{\psi}] \exp\left\{ i \int_0^T dt \int_{-\infty}^{\infty} dx\, \bar{\psi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi \right\} = \exp\left(\tfrac{1}{2} i \sum |\alpha_k|\right) \prod_k (1 + e^{-i|\alpha_k|}), \tag{2.6}$$

where α_k is the Floquet index defined by

$$\psi_i(x, t+T) = e^{-i\alpha_i}\psi_i(x,t), \tag{2.7}$$

and ψ_i is the solution of the Dirac equation with periodic σ and π:

$$[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi_i = 0. \tag{2.8}$$

Once the integrations over ψ and $\bar{\psi}$ are performed, Eq. (2.4) can be written as

$$\mathrm{tr}\,e^{-iHT} = \int [d\sigma][d\pi] \sum \frac{N!}{n_i^{(+)}!(N - n_i^{(+)})!} \frac{N!}{n_i^{(-)}!(N - n_i^{(-)})!} \times \exp[iS(\{n_i^{(+)}\}, \{n_i^{(-)}\})], \tag{2.9}$$

where

$$S(\{n_i^{(+)}\}, \{n_i^{(-)}\}) = -\tfrac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx (\sigma^2 + \pi^2) + \tfrac{1}{2} N \sum_i (\alpha_i^{(+)} + |\alpha_i^{(-)}|) - \sum n_i^{(+)}\alpha_i^{(+)} - \sum n_i^{(-)}|\alpha_i^{(-)}|, \tag{2.10}$$

and + and − refer to positive- and negative-energy solutions of the Dirac equation. We remark here that because π is odd under charge conjugation, the positive- and negative-energy solutions are not related to each other.

Our main interest is the mass spectrum of the bound states. Therefore, we can restrict ourselves to cases where $n_i^{(+)}$ and $n_i^{(-)}$ are nonvanishing only when they refer to the discrete states. From now on we use $n_{0i}^{(+)}, n_{0i}^{(-)}$ to remind us that they refer to the discrete states.

So far we have not taken into account the fact that S is divergent. A renormalization process has to be carried out. First we subtract the vacuum self-energy.[11] Next we have to renormalize the $\sigma^2 + \pi^2$ term. The result is

$$S_{\mathrm{eff}}(\{n_{0i}^{(+)}\}, \{n_{0i}^{(-)}\}) = -\tfrac{1}{2} Z \int_0^T dt \int_{-\infty}^{\infty} dx (\sigma^2 + \pi^2 - \sigma_0^2) + \tfrac{1}{2} N \sum_i [\alpha_i^{(+)} + |\alpha_i^{(-)}| - \alpha_i^{(+)}(\sigma_0) - |\alpha_i^{(-)}(\sigma_0)|] - \sum n_{0i}^{(+)}\alpha_{0i}^{(+)} - \sum n_{0i}^{(-)}|\alpha_{0i}^{(-)}|. \tag{2.11}$$

In the special case of time-independent solutions, we have the simplification

$$\frac{S_{\rm eff}}{T} = -\tfrac{1}{2}Z\int_{-\infty}^{\infty} dx(\sigma^2+\pi^2-\sigma_0^2) + \tfrac{1}{2}N\sum_i [\omega_i^{(+)}+|\omega_i^{(-)}| - \omega_i^{(+)}(\sigma_0) - |\omega_i^{(-)}(\sigma_0)|] - \sum n_{0i}^{(+)}\omega_{0i}^{(+)} - \sum n_{0i}^{(-)}|\omega_{0i}^{(-)}| . \quad (2.12)$$

The stationary-phase approximation (also referred to as the semiclassical approximation) to $\mathrm{tr}e^{-iHT}$ amounts to approximating $\int[d\sigma][d\pi]e^{iS_{\rm eff}}$ by $e^{i\bar{S}_{\rm eff}}$, where

$$\bar{S}_{\rm eff} = S_{\rm eff}\left(\text{evaluated at } \sigma, \pi \text{ where } \frac{\delta S}{\delta\sigma}=0,\ \frac{\delta S}{\delta\pi}=0\right). \quad (2.13)$$

The stationary-phase conditions $\delta S/\delta\sigma=0$, $\delta S/\delta\pi=0$ can be written as[12]

$$-\frac{Z\sigma}{g} = -\tfrac{1}{2}N\sum_i (\bar\psi_i^{(+)}\psi_i^{(+)} - \bar\psi_i^{(-)}\psi_i^{(-)}) + \sum(n_{0i}^{(+)}\bar\psi_{0i}^{(+)}\psi_{0i}^{(+)} - n_{0i}^{(-)}\bar\psi_{0i}^{(-)}\psi_{0i}^{(-)}), \quad (2.14)$$

$$-\frac{Z\pi}{g} = -\tfrac{1}{2}N\sum_i (\bar\psi_i^{(+)}i\gamma_5\psi_i^{(+)} - \bar\psi_i^{(-)}i\gamma_5\psi_i^{(-)}) + \sum(n_{0i}^{(+)}\bar\psi_{0i}^{(+)}i\gamma_5\psi_{0i}^{(+)} - n_{0i}^{(-)}\bar\psi_{0i}^{(-)}i\gamma_5\psi_{0i}^{(-)}). \quad (2.15)$$

The method of finding σ, π, satisfying the stationary-phase conditions, consists of the following processes:

(i) Guess a pair of σ and π.

(ii) Solve for the Dirac equation

$$[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi_i = 0.$$

(iii) Check if the conditions $\delta S/\delta\sigma = 0 = \delta S/\delta\pi$ [Eqs. (2.14) and (2.15)] are satisfied.

This is in general a very complicated and almost hopeless task. Fortunately, for the time-independent case, there is a better way. In the next section, we shall use the inverse scattering method to simplify the task of finding σ, π which satisfy the stationary-phase conditions.

III. THE INVERSE SCATTERING METHOD

The inverse scattering method for the Dirac equation with scalar and pseudoscalar potentials has been studied by Frolov.[13] He establishes a 1-1 correspondence between σ, π and the set of scattering data $\{r^{(\pm)}(k), k_{0i}^{(\pm)}, N_i^{(\pm)}\}$, where $r(k)$ is the reflection the coefficient, $k_{0i}^{(\pm)}$ is related to the energy of the discrete state by $E_{0i} = +(m^2 - k_{0i}^2)^{1/2}$, and N is the normalization factor for the wave function of the discrete state.

We can also establish the trace identities[14] connecting integrals of σ, π to the reflection coefficient $r(k)$ and k_{0i}. One relevant identity for our discussion of S is

$$-\tfrac{1}{2}\int_{-\infty}^{\infty}(\sigma^2+\pi^2-\sigma_0^2)dx = \frac{1}{2\pi g^2}\int_{-\infty}^{\infty}\ln[1-|r^{(+)}(k)|^2]dk + \frac{1}{2\pi g^2}\int_{-\infty}^{\infty}\ln[1-|r^{(-)}(k)|^2]dk + \frac{2}{g^2}\sum_i k_{0i}^{(+)} + \frac{2}{g^2}\sum_i k_{0i}^{(-)} . \quad (3.1)$$

In evaluating S, we have to know

$$\sum[\omega_i^{(+)}+|\omega_i^{(-)}| - \omega_i^{(+)}(\sigma_0) - |\omega_i^{(-)}(\sigma_0)|] .$$

It turns out that this can be expressed in terms of the phase shifts. One finds that

$$\sum[\omega_i^{(+)} - \omega_i^{(+)}(\sigma_0) + |\omega_i^{(-)}| - |\omega_i^{(-)}(\sigma_0)|] = -\int(\delta^{(+)}+\delta^{(-)})\frac{d\omega}{\pi} + \sum(\omega_{0i}^{(+)} - m) + \sum(|\omega_{0i}^{(-)}| - m). \quad (3.2)$$

It can be shown that $\delta^{(+)}+\delta^{(-)}$ is expressible in terms of $r(k)$ and k_{0i} through the dispersion relation[15]

$$\delta^{(+)}+\delta^{(-)} = \frac{1}{2\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\ln[1-|r^{(+)}(q)|^2]+\ln[1-|r^{(-)}(q)|^2]}{k-q}dq + \sum 2\tan^{-1}\frac{k_{0i}^{(+)}}{k} + \sum 2\tan^{-1}\frac{k_{0i}^{(-)}}{k} . \quad (3.3)$$

Equipped with Eqs. (3.1)–(3.3), we can express S in terms of $r(k)$ and k_{0i}. To find the stationary-phase point, all we have to do is look for $r(k)$ and k_{0i} such that $\delta S/\delta r = 0$ and $\delta S/\delta k_{0i} = 0$. It is easy to see that $\delta S/\delta r = 0$ implies $r(k) = 0$. Namely, the stationary-phase conditions require that the potentials σ, π be reflectionless. The inverse scattering method allows us to find σ and π once $k_{0i}^{(\pm)}$ are given.

Frolov[13] has shown how to reconstruct σ and π. One solves the Gel'fand-Levitan equation[16]

$$K(x,y)+F(x+y)+\int_x^\infty K(x,t)F(t+y)\,dt=0\,, \tag{3.4}$$

where

$$F(x)=\sum_n \frac{1}{N_n}A_n\exp(-k_n x),\quad \text{when } r=0 \tag{3.5}$$

and

$$A_n=\begin{bmatrix}\left[\dfrac{m+(m^2-k_n^2)^{1/2}}{k_n}\right]^2 & \dfrac{m+(m^2-k_n^2)^{1/2}}{k_n}\\[2ex] \dfrac{m+(m^2-k_n^2)^{1/2}}{k_n} & 1\end{bmatrix}\quad \text{for positive-energy solutions}$$

$$=\begin{bmatrix}1 & \dfrac{m+(m^2-k_n^2)^{1/2}}{k_n}\\[2ex] \dfrac{m+(m^2-k_n^2)^{1/2}}{k_n} & \left[\dfrac{m+(m^2-k_n^2)^{1/2}}{k_n}\right]^2\end{bmatrix}\quad \text{for negative-energy solutions} \tag{3.6}$$

in the representation

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&i\\i&0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}0&-i\\i&0\end{pmatrix}. \tag{3.7}$$

The summation in Eq. (3.5) is over all discrete states. The Dirac equation in the representation (3.7) can be written as

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\frac{df}{dx}+\begin{pmatrix}m&0\\0&-m\end{pmatrix}f+g\begin{pmatrix}\sigma-\sigma_0&\pi\\ \pi&-(\sigma-\sigma_0)\end{pmatrix}f=\omega f, \tag{3.8}$$

where $m\equiv g\sigma_0$. The potentials σ, π are then given by

$$g\begin{pmatrix}\sigma-\sigma_0&\pi\\ \pi&-(\sigma-\sigma_0)\end{pmatrix}=\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix},K(x,x)\right]. \tag{3.9}$$

For illustration, let us restrict ourselves to the case where there is only one term in the summation in Eq. (3.5). The solution of the Gel'fand-Levitan equation[16] [Eq. (3.4)] can be written down explicitly:

$$K(x,y)=2k_0\left[\frac{1}{1+(A/2k_0N)\exp(-2k_0x)}-1\right]\times e^{k_0(x-y)}. \tag{3.10}$$

Consider the following two subcases:

(1) The potentials σ and π allow only one discrete state which has positive energy. In this case we have

$$A=\begin{pmatrix}a^2&a\\a&1\end{pmatrix},\quad \text{with } a=\frac{m+(m^2-k_0^2)^{1/2}}{k_0}.$$

A simple calculation gives us the following results:

$$g(\sigma-\sigma_0)=g\sigma-m=-\frac{2k_0^2}{m}\,\frac{1}{1+\exp[2k_0(x-x_0)]} \tag{3.11}$$

and

$$g\pi=\frac{2k_0(m^2-k_0^2)^{1/2}}{m}\,\frac{1}{1+\exp[2k_0(x-x_0)]}\,. \tag{3.12}$$

We remark that although $g(\sigma-\sigma_0)$ and $g\pi$ approach 0 when $x\to+\infty$, they approach nonvanishing values

$$g(\sigma-\sigma_0)\to-\frac{2k_0^2}{m},$$

$$g\pi\to\frac{2k_0(m^2-k_0^2)^{1/2}}{m}\quad \text{when } x\to-\infty.$$

In other words, the potentials $g(\sigma-\sigma_0)$, $g\pi$ are not localized in space. At first sight this seems puzzling. However, when we consider the combination

$$g^2(\sigma^2+\pi^2)=m^2-\frac{k_0^2}{\cosh^2k_0(x-x_0)}, \tag{3.13}$$

we see that it approaches m^2 at $\pm\infty$. We realize what happens is that we indeed have a free Dirac equation with mass m at $x=\pm\infty$ except that the mass matrix has been rotated by a finite amount at $x=-\infty$.

The solutions in the continuum can be obtained from the equation[13]

$$f(x,\lambda)=e(x,\lambda)+\int_x^\infty K(x,t)e(t,\lambda)\,dt\,, \tag{3.14}$$

where $\lambda=\pm 1$ for positive and negative solutions, respectively, and

$$e(x,+1)=\begin{pmatrix} i\dfrac{m+(m^2+k^2)^{1/2}}{k} \\ 1\end{pmatrix} e^{ikx}\equiv\begin{pmatrix} e^{(1)} \\ e^{(2)}\end{pmatrix} e^{ikx}\,,$$

$$e(x,-1)=\begin{pmatrix} 1 \\ i\dfrac{m+(m^2+k^2)^{1/2}}{k}\end{pmatrix} e^{ikx}\,. \tag{3.15}$$

From Eq. (3.14) one obtains

$$\begin{aligned} f(x,+1)=&\frac{1}{a^2+1}\begin{pmatrix} e^{(1)}-ae^{(2)} \\ -ae^{(1)}+a^2e^{(2)}\end{pmatrix} e^{ikx} \\ &+\frac{1}{a^2+1}\begin{pmatrix} a^2e^{(1)}+ae^{(2)} \\ +ae^{(1)}+e^{(2)}\end{pmatrix} \\ &\times\left[1+\frac{2k_0}{ik-k_0}\,\frac{1}{1+\exp[2k_0(x-x_0)]}\right]e^{ikx}\,. \end{aligned} \tag{3.16}$$

There is no reflection in the scattering states. Therefore, the potentials σ,π in Eqs. (3.1) and (3.2) are indeed reflectionless. We would like to point out that, besides the physical cut, f has a kinematical cut in the complex k plane owing to the presence of the function $(k^2+m^2)^{1/2}$.

To simplify the interpretation, it is useful to make a local chiral rotation such that

$$g\begin{pmatrix} \sigma & \pi \\ \pi & \sigma\end{pmatrix}\to g\begin{pmatrix} \dfrac{m}{g}+\sigma' & 0 \\ 0 & -\left(\dfrac{m}{g}+\sigma'\right)\end{pmatrix}.$$

The necessary local chiral transformation is

$$\tilde f=Gf\,,\quad G=\begin{pmatrix} \cos\frac{1}{2}\alpha & -\sin\frac{1}{2}\alpha \\ \sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha\end{pmatrix}, \tag{3.17}$$

where

$$\begin{aligned}\tan\alpha&=-\frac{\pi}{\sigma}\\ &=-\frac{2k_0(m^2-k_0{}^2)^{1/2}}{m^2\{1+\exp[2k_0(x-x_0)]\}-2k_0{}^2}\,.\end{aligned} \tag{3.18}$$

The new Dirac equation is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\frac{d\tilde f}{dx}+\begin{pmatrix} m+g\sigma' & 0 \\ 0 & -(m+g\sigma')\end{pmatrix}\tilde f=\left(\omega+\frac{1}{2}\,\frac{d\alpha}{dx}\right)\tilde f\,. \tag{3.19}$$

From Eq. (3.13), we have

$$(m+g\sigma')^2=m^2-\frac{k_0{}^2}{\cosh^2 k_0(x-x_0)}\,.$$

Therefore, $\sigma'\to 0$ at $x=\pm\infty$.

We find, after some algebra, that the wave function for the rotated scattering state has the following asymptotic behaviors:

$$\tilde f(x,+1)e^{-ikx}\big|_{x=-\infty}=\frac{[1+(k/m)^2]^{1/2}k_0+ik[1-(k_0/m)^2]^{1/2}}{ik-k_0}\begin{pmatrix} e^{(1)} \\ e^{(2)}\end{pmatrix} \tag{3.20}$$

and

$$\tilde f(x,+1)e^{-ikx}\big|_{x=+\infty}=\begin{pmatrix} e^{(1)} \\ e^{(2)}\end{pmatrix}. \tag{3.21}$$

Thus, the phase shift is given by

$$e^{-i\delta^{(+)}}=\frac{[1+(k/m)^2]\,k_0+ik[1-(k_0/m)^2]^{1/2}}{ik-k_0}\,,$$

or

$$\delta^{(+)}=\tan^{-1}\frac{k_0}{k}+\tan^{-1}\frac{k_0}{k}\left(\frac{m^2+k^2}{m^2-k_0{}^2}\right)^{1/2}. \tag{3.22}$$

We would like to remark that here $\delta^{(+)}\not\to 0$ as $k\to\infty$ in contrast to the nonrelativistic potential scattering problem where $\delta\to 0$ as $k\to\infty$ when the potential is square-integrable.[15]

By exactly the same method, the phase shift for the negative-energy solution can be found. It is given by

$$\delta^{(-)}=\tan^{-1}\frac{k_0}{k}-\tan^{-1}\frac{k_0}{k}\left(\frac{m^2+k^2}{m^2-k_0{}^2}\right)^{1/2}. \tag{3.23}$$

This differs from $\delta^{(+)}$ only by the sign of the second term on the right-hand side. It is a reflection of the fact that the scattering amplitude in the complex k plane has a two-sheet structure corresponding to the double-valued structure of $(k^2+m^2)^{1/2}$. Although $\delta^{(+)}$ and $\delta^{(-)}$ do not approach zero individually, nevertheless their sum approaches zero in the limit of $k\to\infty$.

The wave function for the discrete state can also be found. It is, up to a normalization factor,

$$\psi_0=\begin{pmatrix} \dfrac{m+(m^2-k_0{}^2)^{1/2}}{k_0} \\ 1\end{pmatrix}\frac{1}{\cosh k_0(x-x_0)}\,. \tag{3.24}$$

(2) We consider here the case where the potentials allow both positive- and negative-energy bound states such that

$$k_0^{(+)}=k_0^{(-)}=k_0,\quad n_0^{(+)}=n_0^{(-)}\,. \tag{3.25}$$

We can proceed as before to find the potentials using the inverse scattering method. It is found that the potentials are

$$\sigma = \sigma_0 - \sigma_0 \frac{y}{(1-y^2)^{1/2}} \times \frac{1}{\cosh^2 k_0(x-x_0) + \frac{1}{2}[1/(1-y^2)^{1/2} - 1]}, \tag{3.26}$$

where

$$y = \frac{k_0}{m}$$

and

$$\pi = 0. \tag{3.27}$$

Equation (3.26) has the same form as the σ in the Gross-Neveu model. The fact that $\pi = 0$ agrees with the prediction of the charge conjugation. The phase shifts in this case are given by

$$\delta^{(+)} = \delta^{(-)} = 2\tan^{-1}\frac{k_0}{k}. \tag{3.28}$$

We shall not attempt to solve for the potentials σ, π for the general cases. In the next section, we shall show that the results we obtained in the special cases considered in this section are sufficient for the purpose of finding the mass spectrum of the bound states in our model.

IV. QUANTIZATION AND MASS SPECTRUM

In the preceding section, we showed that the stationary-phase conditions require that the potentials σ, π be reflectionless and we indicated how to solve for σ, π using the inverse scattering method. Up to now we have not used the second stationary-phase condition $\delta S/\delta k_{0i} = 0$. We shall show that this leads to the quantization condition on k_0. This in turn gives us the mass spectrum of the bound states.

The calculation of k_{0i} such that $\delta S/\delta k_{0i} = 0$ proceeds exactly as in DHN. We shall not repeat their calculation here. It suffices to write down the results in the following cases:

(1) The potentials σ and π allow only one discrete state with positive energy.

It is clear that $\sigma = \sigma_0$ and $\pi = 0$ is a solution of the stationary-phase conditions. The renormalization constant Z can be determined by Eq. (2.14). One finds that

$$Z = \frac{Ng^2}{\pi}\int_0^\Lambda \frac{dk}{(k^2+m^2)^{1/2}}, \tag{4.1}$$

which also makes S finite. In terms of θ ($k_0 \equiv m\sin\theta$), the action S is

$$\frac{S}{TNm} = -\frac{n_0}{N}\cos\theta - \frac{1}{\pi}(\sin\theta - \theta\cos\theta). \tag{4.2}$$

The stationary-phase condition $\delta S/\delta k_0 = 0$ can be expressed as $dS/d\theta = 0$, and this implies that $\theta = n_0\pi/N$. The mass spectrum of the bound states corresponding to this case is

$$E_{n_0} = -\frac{S}{T} = \frac{Ng\sigma_0}{\pi}\sin\left(\frac{n_0\pi}{N}\right) = \frac{Nm}{\pi}\sin\left(\frac{n_0\pi}{N}\right), \tag{4.3}$$

where n_0 is the number of fermions occupying the state k_0. The condition that $0 \leqslant \theta \leqslant \pi/2$ requires $n_0 \leqslant \frac{1}{2}N$. The chiral rotation angle introduced in the preceding section is also quantized. In fact

$$\phi = 2\theta = \frac{2n_0\pi}{N}. \tag{4.4}$$

We would like to remind you that n_0 is the fermion number of the bound state which is a conserved quantum number. The stability of the bound state against decaying into two lower-lying states is guaranteed by the elementary inequality

$$\sin\left(\frac{n_1+n_2}{N}\pi\right) < \sin\left(\frac{n_1}{N}\pi\right) + \sin\left(\frac{n_2}{N}\pi\right). \tag{4.5}$$

Here $\sin[(n_1/N)\pi]$ and $\sin[(n_2/N)\pi]$ are positive.

Likewise, for σ, π allowing only one discrete state with negative energy, the energy of the bound state corresponding to the fermion number $-n_0$ is also given by Eq. (4.3). In other words, the energy of the bound state is invariant under charge conjugation.

(2) We consider the case that the potentials allow both positive- and negative-energy discrete states such that $k_0^{(+)} = k_0^{(-)} = k_0$ and $n_0^{(+)} = n_0^{(-)}$.

In this case the mass spectrum of the bound state is

$$E_n = \frac{2g\sigma_0 N}{\pi}\sin\left(\frac{\pi n}{2N}\right), \tag{4.6}$$

where $n = n_0^{(+)} + n_0^{(-)} = 2n_0^{(+)}$. These states have zero fermion number. This result is similar to the result of DHN. It is worthwhile to point out that although the mass spectrum in Eq. (4.6) is of the same form as in DHN, the quantum number content as well as the degeneracy of states are different.

(3) The general time-independent case. Here, although the potentials σ, π are more difficult to find explicitly, nevertheless, we have the following relations:

$$\int_{-\infty}^{\infty}(\sigma^2 + \pi^2 - \sigma_0^2)\,dx = -\frac{4}{g^2}\left(\sum k_{0i}^{(+)} + \sum k_{0i}^{(-)}\right) \tag{4.7}$$

and

$$\delta^{(+)}+\delta^{(-)}=2\sum \tan^{-1}\frac{k_{0i}^{(+)}}{k}+2\sum \tan^{-1}\frac{k_{0i}^{(-)}}{k}. \quad (4.8)$$

One sees readily that the action is given by

$$S=\sum S(k_{0i}^{(+)})+\sum S(k_{0i}^{(-)}). \quad (4.9)$$

In other words, it is additive. This implies that the mass spectrum of the bound is additive. To find the mass spectrum for the general cases, all we have to do is find the mass spectrum of case (1).

Up to now we have restricted ourselves to the time-independent solutions in most parts of our paper. It would be nice if we can say something about cases corresponding to time-dependent solutions. Unfortunately, we no longer have the powerful tool of the inverse scattering method. We have to find the potentials σ, π by trial and error. For case (2) which allows both positive and negative solutions with $k_0^{(+)}=k_0^{(-)}$, $n^{(+)}=n^{(-)}$, the solutions can be found along the same line as in Sec. IV of DHN. However, we are unable to make any progress for the other cases.

V. CONCLUDING REMARKS

Before we conclude this paper, we would like to make some remarks:

(1) Relevance of classical solutions of the nonlinear wave equations corresponding to fermions. For the boson fields, it is well known that the classical solutions of the nonlinear wave equations provide first approximations to the quantum system. For example, to calculate the energies of the quantum system, we calculate the energies corresponding to the classical solutions and then calculate the quantum corrections. For fermion fields in general we do not know whether the classical solutions of the nonlinear wave equations play any role in our understanding of the quantum system.

In the Gross-Neveu model discussed in DHN, the field σ corresponding to the stationary-phase point of S is given by

$$\sigma-\sigma_0=-\sigma_0\frac{y}{(1-y^2)^{1/2}} \times\frac{1}{\cosh^2 k_0(x-x_0)+\frac{1}{2}[1/(1-y^2)^{1/2}-1]} \quad (y=k_0/m).$$

This happens to agree exactly with $-g\bar{\psi}_c\psi_c$,[17] where ψ_c is the static confined solution of the nonlinear wave equation

$$i\partial\!\!\!/\psi-m\psi_c+g^2(\bar{\psi}_c\psi_c)\psi_c=0. \quad (5.1)$$

In this sense the classical confined solution of the nonlinear wave equation for the fermion may be relevant to the understanding of the quantum system including the problem of bound states. It would be nice if this connection were a general one.

In our model we find that the fields σ and π corresponding to case (1), where only one positive-energy discrete state is allowed, are

$$g(\sigma-\sigma_0)=-\frac{2k_0^2}{m}\frac{1}{1+\exp[2k_0(x-x_0)]},$$

$$g\pi=\frac{2k_0(m^2-k_0^2)^{1/2}}{m}\frac{1}{1+\exp[2k_0(x-x_0)]}.$$

It is obvious that

$$\sigma-\sigma_0\neq-g\bar{\psi}_c\psi_c,$$

$$\pi\neq-g\bar{\psi}_c i\gamma_5\psi_c,$$

where ψ_c is the classical confined solution. This is because ψ_c is localized, therefore, $\bar{\psi}_c\psi_c$ and $\bar{\psi}_c i\gamma_5\psi_c\to 0$ at $\pm\infty$ contradicting Eqs. (3.11) and (3.12). In fact, Lee and Gavrielides[18] have solved ψ_c explicitly. According to them

$$-g\bar{\psi}_c\psi_c=\frac{-2(m-E)}{g}\frac{1-\alpha^2\tanh^2\beta x}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)^2}, \quad (5.2)$$

$$-ig\bar{\psi}_c\gamma_5\psi_c=\frac{4(m-E)\alpha}{g}\frac{\tanh\beta x}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)^2}, \quad (5.3)$$

where

$$\alpha=\left(\frac{m-E}{m+E}\right)^{1/2}, \quad \beta=(m^2-E^2)^{1/2}.$$

These bear no resemblance to $\sigma-\sigma_0$, π obtained in our model. We therefore conclude that, in general, the classical confined solutions of the nonlinear wave equations for the fermion are not useful in understanding the mass spectrum of the quantum system in contrast to the boson case.

(2) *Spontaneous symmetry breaking and the Goldstone theorem.* The Lagrangian in our model is manifestly chiral-symmetric. The symmetry is broken spontaneously so that the fermion becomes massive. The Goldstone theorem[19] implies the existence of a massless particle. Unfortunately, in one space one time dimension, the infrared divergences forbid the existence of a spin-0 meson (Coleman's theorem[20]). Since our results depend crucially on the idea of spontaneous symmetry breaking, we have to under-

stand what happens in our model.

By looking at the expression for the energies of the bound states, we realize that we cannot get a massless particle because in the expression for E, all n have to be greater than zero.

We must answer the following questions: (a) Is there any massless particle? (b) If such a massless particle exists, how can we reconcile it with Coleman's theorem?

It turns out that the massless particle decouples from the rest of the system. Therefore, we evade the conclusion of Coleman's theorem.

The following argument concerning the decoupling of the Goldstone boson is due to Dashen.[21] We include it here for completeness. For this purpose, it is convenient to express our model in terms of boson fields. This can be done along the lines of Halpern.[22] The result is

$$\mathcal{L}=\sum_a \tfrac{1}{2}(\partial_\mu \phi_a)^2 + C\sum_{a,b}\cos[2\sqrt{\pi}(\phi_a-\phi_b)], \quad (5.4)$$

where $a=1,2,\ldots,N$, and C has the dimension of (mass)^2. We can express ϕ_a in terms of other bases $\{\Phi_+,\Phi_b\}$:

$$\phi_a=\frac{1}{\sqrt{N}}\Phi_+ + D_{ab}\Phi_b, \quad (5.5)$$

with

$$\sum_a D_{ab}=0, \qquad \sum_a D_{ab}D_{ab'}=\delta_{bb'}. \quad (5.6)$$

Now the kinetic terms can be expressed as

$$\begin{aligned}\sum(\partial_\mu\phi_a)^2 &= \sum_a\left(\frac{1}{\sqrt{N}}\partial_\mu\Phi_+ + \sum_b D_{ab}\partial_\mu\Phi_b\right)^2\\ &= (\partial_\mu\Phi_+)^2+\frac{2}{\sqrt{N}}\partial_\mu\Phi_+\sum_b\left(\sum_a D_{ab}\right)\partial^\mu\Phi_b\\ &\quad+\sum_{a,b}D_{ab}\partial^\mu\Phi_b D_{ac}\partial_\mu\Phi_c\\ &=(\partial_\mu\Phi_+)^2+\sum_b(\partial^\mu\Phi_b)^2. \end{aligned} \quad (5.7)$$

Therefore, the Lagrangian can be written as

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\Phi_+)^2+\mathcal{L}', \quad (5.8)$$

where

$$\mathcal{L}'=\tfrac{1}{2}\sum_a(\partial^\mu\Phi_b)^2+C\sum_{a,b}\cos[2\sqrt{\pi}(D_{ac}-D_{bc})\Phi_c] \quad (5.9)$$

is independent of Φ_+.

It is now obvious that Φ_+ corresponds to the massless Goldstone particle. However, it decouples from the rest of the system and causes no infrared divergence.

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APPENDIX A: AN EXPLICITLY BROKEN CHIRAL MODEL

We have been discussing a model with manifest chiral symmetry. In this appendix we discuss briefly a model with partial symmetry. The simplest way to break the chiral symmetry explicitly in the Lagrangian [Eq. (2.1)] is by inclusion of a mass term

$$\mathcal{L}_m=\bar\psi(i\partial\!\!\!/-m)\psi+\tfrac{1}{2}g^2[(\bar\psi\psi)^2-(\bar\psi\gamma_5\psi)^2], \quad (A1)$$

or, equivalently,

$$\mathcal{L}_m(\sigma,\pi,\psi)=\bar\psi[i\partial\!\!\!/-m-g(\sigma+i\pi\gamma_5)]\psi-\tfrac{1}{2}(\sigma^2+\pi^2). \quad (A2)$$

We no longer have conservation of the axial-vector current. Instead, the following partial conservation law for the axial-vector current (PCAC) obtains:

$$\partial^\mu J^5_\mu=m\bar\psi i\gamma_5\psi. \quad (A3)$$

This model has been discussed by Lee and Gavrielides.[18] They start with the classical confined solution of the nonlinear Dirac equation and use the semiclassical method to find the mass spectrum of bound states. As will become clear from our discussions, we believe that their conclusions and results are, unfortunately, wrong.

It is instructive to see whether we can say anything about the mass spectrum of the bound states in the theory described by $\mathcal{L}_m$.

To answer these we proceed as before. We begin with the effective action

$$\begin{aligned}\frac{S_{\text{eff}}(m)}{T}&=-\tfrac{1}{2}Z\int_{-\infty}^{\infty}dx(\sigma^2+\pi^2+2b\sigma)\\ &\quad+\tfrac{1}{2}N\sum_i[\omega_i^{(+)}+|\omega_i^{(-)}|-\omega_i^{(+)}(\sigma_0)-|\omega_i^{(-)}(\sigma_0)|]\\ &\quad-\sum n_{0i}^{(+)}\omega_{0i}^{(+)}-\sum n_{0i}^{(-)}|\omega_{0i}^{(-)}|. \end{aligned} \quad (A4)$$

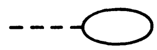

FIG. 1. The tadpole diagram.

FIG. 2. Two possible one-loop divergent diagrams in the massive Thirring model.

The linear counterterm $-bZ\sigma$ is needed here because, in the $\psi, \bar{\psi}$ integrations, there is a divergent term corresponding to the tadpole diagram in Fig. 1. The renormalization constants Z and b can be determined from the effective potential

$$V_{\text{eff}} = \tfrac{1}{2}Z(\sigma^2 + \pi^2 + 2b\sigma) - \frac{N}{\pi}\int_0^\infty dk[k^2 + (m+g\sigma)^2 + g^2\pi^2]^{1/2} + c\,, \tag{A5}$$

where c is an infinite constant. We do the renormalization on the mass shell. The requirements that

$$\frac{\partial V_{\text{eff}}}{\partial\sigma}\bigg|_{\sigma=0} = 0$$

and

$$\frac{\partial^2 V_{\text{eff}}}{\partial\sigma^2}\bigg|_{\sigma=0} = 1 \tag{A6}$$

determine Z and b uniquely. One finds that

$$Z = 1 + \frac{Ng^2}{\pi}\left(\ln\frac{2\Lambda}{m} - 1\right), \tag{A7}$$

$$bZ = \frac{Ng^2}{\pi} m\left(\ln\frac{2\Lambda}{m}\right), \tag{A8}$$

and

$$V_{\text{eff}} = \tfrac{1}{2}(\sigma^2 + \pi^2) + \frac{Ngm}{\pi}\sigma + \frac{N}{4\pi}[(m+g\sigma)^2 + g^2\pi^2] \times\left[\ln\frac{(m+g\sigma)^2 + g^2\pi^2}{m^2} - 3\right], \tag{A9}$$

to be compared with the results of Lee and Gavrielides.[18] They found no need of renormalization for the $\sigma^2 + \pi^2$ term, i.e., $Z = 1$. The contradicts the result obtained from the simple one-loop calculation of Z. One source of error is the mistaken belief that the σ and π they obtained were reflectionless. Indeed, it can be shown that their σ and π are not reflectionless.

Since $\sigma^2 + \pi^2 + 2b\sigma$ is not equal to $\sigma^2 + \pi^2 + 2(m/g)\sigma$, we cannot use trace identities to evaluate $\sigma^2 + \pi^2 + 2b\sigma$. The condition $\partial S/\partial r = 0$ no longer implies $r = 0$. Therefore, we cannot follow Secs. III and IV to determine the mass spectrum using the inverse scattering method. In other words, the inverse scattering method does not give us any simplification. We have to resort to solving Eqs. (2.14) and (2.15) directly. We shall not attempt to do it here.

APPENDIX B: MASSIVE THIRRING MODEL IN LARGE-N LIMIT

In this appendix, we mention another model which cannot be treated by the inverse scattering method in any simple way. The model is the massive Thirring model with N species of fermions. The theory is described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \tfrac{1}{2}\lambda^2 j^\mu j_\mu\,, \tag{B1}$$

where

$$j^\mu = \sum_{a=1}^N \bar{\psi}_a\gamma^\mu\psi_a\,. \tag{B2}$$

An equivalent description is by

$$\mathcal{L}(A,\psi) = \bar{\psi}(i\partial\!\!\!/ - m + \lambda A\!\!\!/)\psi - \tfrac{1}{2}A^2. \tag{B3}$$

We can proceed as before. When the integrations of ψ and ψ are performed, we obtain

$$\int d\psi\, d\bar{\psi} \exp\left[i\int d^2x\, \bar{\psi}(i\partial\!\!\!/ - m + \lambda A\!\!\!/)\psi\right] = \frac{\text{Det}[\gamma^0(i\partial\!\!\!/ - m + \lambda A\!\!\!/)]}{\text{Det}[\gamma^0(i\partial\!\!\!/ - m)]}$$
$$= \sum C^N_{n_{0i}^{(+)}} C^N_{n_{0i}^{(-)}} \exp\left(iT\left\{\tfrac{1}{2}N\sum[\omega_i^{(+)} + |\omega_i^{(-)} - \omega_i^{(+)}(0) - |\omega_i^{(-)}(0)|] - \sum n_{0i}^{(+)}\omega_{0i}^{(+)} - \sum n_{0i}^{(+)}|\omega_{0i}^{(-)}|\right\}\right). \tag{B4}$$

It is easy to see that the only possible divergences in Eq. (B4) are from the diagrams in Fig. 2. A little thought convinces us that they are in fact finite. The finiteness of diagram 2(a) is related to the fact that $\langle j_\mu\rangle_{\text{vac}}=0$. The finiteness of diagram 2(b) is due to the current conservation. These can also be verified by direct calculation.

Equation (B4) tells us that

$$\sum[\omega_i^{(+)}+|\omega_i^{(-)}|-\omega_i^{(+)}(0)-|\omega_i^{(-)}(0)|]=-\int(\delta^{(+)}+\delta^{(-)})\frac{d\omega}{\pi}+\sum(\omega_{0i}^{(+)}-m)+\sum(|\omega_{0i}^{(-)}|-m)$$

is finite. This implies that $k(\delta^{(+)}+\delta^{(-)})\to 0$ as $k\to\infty$. We recall that $\delta^{(+)}+\delta^{(-)}$ satisfies the dispersion relation (3.3). We are led immediately to the following conclusions:

(i) It is impossible to have $r^{(+)}\equiv 0$, $r^{(-)}\equiv 0$ simultaneously when these are discrete states.

(ii) The condition $k(\delta^{(+)}+\delta^{(-)})\to 0$ as $k\to\infty$ implies that $r^{(+)}(q)$ and $r^{(-)}(q)$ can no longer be treated as independent variables. In other words, to find the stationary-phase condition, we cannot have independent conditions

$$\frac{\delta S}{\delta r}=0 \text{ and } \frac{\delta S}{\delta k_{0i}}=0.$$

The inverse scattering method is no longer a useful tool in the massive Thirring model.

APPENDIX C: TRACE IDENTITIES FOR THE DIRAC EQUATION

Let us derive the trace identities for the Dirac equation in Sec. III. We start with

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\dfrac{df_1}{dx}\\[2mm] \dfrac{df_2}{dx}\end{pmatrix}+\begin{pmatrix}m+g\sigma' & 0\\ 0 & -(m+g\sigma')\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}=\left[(k^2+m^2)^{1/2}+\frac{1}{2}\frac{d\alpha}{dx}\right]\begin{pmatrix}f_1\\ f_2\end{pmatrix},$$

or

$$\frac{df_2}{dx}+\left\{m+g\sigma'-\left[(k^2+m^2)^{1/2}+\frac{1}{2}\frac{d\alpha}{dx}\right]\right\}f_1=0, \tag{C1}$$

$$\frac{df_1}{dx}+\left[m+g\sigma'+(k^2+m^2)^{1/2}+\frac{1}{2}\frac{d\alpha}{dx}\right]f_2=0, \tag{C2}$$

where σ', $d\alpha/dx\to 0$ when $x\to\pm\infty$. We define

$$\theta(x)\equiv\frac{d}{dx}\ln f_1-ik, \tag{C3}$$

$$\omega(x)\equiv\frac{f_2}{f_1}-\frac{(k^2+m^2)^{1/2}-m}{ik}. \tag{C4}$$

The phase shift δ is related to θ by

$$\delta=-i\int_{-\infty}^{\infty}\theta(x)dx. \tag{C5}$$

It is obvious that $\theta(x)=\omega=0$ if $\sigma'=d\alpha/dx=0$.

Substituting (C2) into (C3) and using (C4), one finds that

$$\theta+\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)\omega+\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)\frac{(k^2+m^2)^{1/2}-m}{ik}+[m+(k^2+m^2)^{1/2}]\omega=0. \tag{C6}$$

Another relation between θ and ω can be obtained by differentiating (C4) with respect to x and using (C1) and (C2):

$$\omega_x=\frac{1}{f_1}\frac{df_2}{dx}-\frac{f_2}{f_1^2}\frac{df_1}{dx},$$

$$\omega_x+\omega(\theta+ik)+\left(g\sigma'-\frac{1}{2}\frac{d\alpha}{dx}\right)+\frac{(k^2+m^2)^{1/2}-m}{ik}\theta=0. \tag{C7}$$

We recall that ω and θ have two-sheet structures. Let us use $\pm$ to denote quantities related to the positive- and negative-energy solutions. Equations (C6) and (C7) can be written in terms of $\theta^{(+)}+\theta^{(-)}$, $\theta^{(+)}-\theta^{(-)}$, $\omega^{(+)}+\omega^{(-)}$, and $\omega^{(+)}-\omega^{(-)}$:

$$(\theta^{(+)}+\theta^{(-)})+\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}+m\right)(\omega^{(+)}+\omega^{(-)})+(k^2+m^2)^{1/2}(\omega^{(+)}-\omega^{(-)})+\frac{2im}{k}\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)=0\,, \tag{C8}$$

$$(\theta^{(+)}-\theta^{(-)})+\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}+m\right)(\omega^{(+)}-\omega^{(-)})+(k^2+m^2)^{1/2}(\omega^{(+)}+\omega^{(-)})+2\frac{(k^2+m^2)^{1/2}}{ik}\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)=0\,, \tag{C9}$$

$$(\omega_x^{(+)}+\omega_x^{(-)})+\tfrac{1}{2}[(\omega^{(+)}+\omega^{(-)})(\theta^{(+)}+\theta^{(-)})+(\omega^{(+)}-\omega^{(-)})(\theta^{(+)}-\theta^{(-)})]$$
$$+ik(\omega^{(+)}+\omega^{(-)})+2\left(g\sigma'-\frac{1}{2}\frac{d\alpha}{dx}\right)+\frac{(k^2+m^2)^{1/2}}{ik}(\theta^{(+)}-\theta^{(-)})+\frac{im}{k}(\theta^{(+)}+\theta^{(-)})=0\,, \tag{C10}$$

$$(\omega_x^{(+)}-\omega_x^{(-)})+\tfrac{1}{2}[(\omega^{(+)}+\omega^{(-)})(\theta^{(+)}-\theta^{(-)})+(\omega^{(+)}-\omega^{(-)})(\theta^{(+)}+\theta^{(-)})]$$
$$+ik(\omega^{(+)}-\omega^{(-)})+\frac{(k^2+m^2)^{1/2}}{ik}(\theta^{(+)}+\theta^{(-)})+\frac{im}{k}(\theta^{(+)}-\theta^{(-)})=0\,. \tag{C11}$$

It can be easily established that the following combinations are analytic and can be expanded in powers of $1/k$ in the upper half plane in the limit of $k\to\infty$:

$$\omega^{(+)}+\omega^{(-)}=\sum\frac{\omega_n^{(1)}(x)}{k^n}\,,$$
$$\frac{k}{(k^2+m^2)^{1/2}}(\omega^{(+)}-\omega^{(-)})=\sum\frac{\omega_n^{(2)}(x)}{k^n}\,,$$
$$\theta^{(+)}+\theta^{(-)}=\sum\frac{\theta_n^{(1)}(x)}{k^n}\,,$$
$$\frac{k}{(k^2+m^2)^{1/2}}(\theta^{(+)}-\theta^{(-)})=\sum\frac{\theta_n^{(2)}(x)}{k^n}\,. \tag{C12}$$

Substituting these into Eqs. (C8) to (C11) and picking up terms of order k^0 and k^{-1}, we obtain the following eight equations:

$$\theta_0^{(1)}+\omega_1^{(2)}=0\,, \tag{C13}$$

$$\theta_1^{(1)}+\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}+m\right)\omega_1^{(1)}+\omega_2^{(2)}+2im\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)=0\,, \tag{C14}$$

$$\theta_0^{(2)}+\omega_1^{(1)}-2i\left(g\sigma'+\frac{1}{2}\frac{d\alpha}{dx}\right)=0\,, \tag{C15}$$

$$\theta_1^{(2)}+\omega_2^{(1)}=0\,, \tag{C16}$$

$$i\omega_1^{(1)}+2\left(g\sigma'-\frac{1}{2}\frac{d\alpha}{dx}\right)-i\theta_0^{(2)}=\theta\,, \tag{C17}$$

$$\frac{d}{dx}\omega_1^{(1)}+\tfrac{1}{2}\omega_1^{(2)}\theta_0^{(2)}+\tfrac{1}{2}\omega_1^{(1)}\theta_0^{(1)}+i\omega_2^{(1)}-i\theta_1^{(2)}+im\theta_0^{(1)}=0\,, \tag{C18}$$

$$i\omega_1^{(2)}-i\theta_0^{(1)}=0\,, \tag{C19}$$

$$\frac{d}{dx}\omega_1^{(2)}+\tfrac{1}{2}\omega_1^{(1)}\theta_0^{(2)}+\tfrac{1}{2}\omega_1^{(2)}\theta_0^{(1)}+i\omega_2^{(2)}-i\theta_1^{(1)}+im\theta_0^{(2)}=0\,. \tag{C20}$$

From (C11) and (C19), one finds

$$\omega_1^{(2)}=\theta_0^{(1)}=0\,. \tag{C21}$$

From (C15) and (C17), one finds

$$\omega_1^{(1)}=2ig\sigma' \tag{C22}$$

and

$$\theta_0^{(2)}=i\frac{d\alpha}{dx}\,. \tag{C23}$$

Combining Eqs. (C13), (C19), and (C21)–(C23), one obtains

$$\theta_1^{(1)}=-i(g^2\sigma'^2+2mg\sigma')\,. \tag{C24}$$

As our final step, we use the dispersion relation for the phase shift Eq. (3.3) and the equation relating δ to θ [Eq. (C5)] to obtain the trace identity we used in Sec. III:

$$-\tfrac{1}{2}\int_{-\infty}^{\infty}(\sigma^2+\pi^2-\sigma_0{}^2)dx$$
$$=-\tfrac{1}{2}\int_{-\infty}^{\infty}\left(\sigma'^2+2\frac{m}{g}\sigma'\right)dx$$
$$=\frac{1}{2\pi g^2}\int_{-\infty}^{\infty}\{\ln[1-|r^{(+)}(k)|^2]+\ln[1-|r^{(-)}(k)|^2]\}dk$$
$$+\frac{2}{g^2}\left(\sum_i k_{0i}^{(+)}+\sum_i k_{0i}^{(-)}\right).$$

The other trace identities can be derived in exactly the same way by considering terms of higher order in $1/k$ in Eqs. (C8)–(C11).

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