Chiral-charge conservation and gauge fields*

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An example of a gauge field theory with classical dyon solutions which violates anomalous chiral-charge conservation is presented. We also show that the pseudoparticle solution to the four-dimensional pure Yang-Mills equations is equivalent to a kink solution in one dimension. The net change in chirality, the Pontryagin index, and kink number are equivalent. This suggests an intimate connection between the spherically symmetric four-dimensional Euclidean gauge field theory and one-dimensional scalar field theories.

I. INTRODUCTION

Recently there has been a revived interest in classical solutions to scalar and gauge field theories. Among the most carefully examined solutions are the kink,[1] vortex,[2] magnetic monopole,[3] and pseudoparticle.[4] These exhibit the common quality of topological stability for one, two, three, and four spatial dimensions, respectively. In addition, all are static, finite-energy, dynamically stable, nondissipative solutions to classical field equations. A comprehensive review of these developments has been given by Coleman.[5] Here, we will make a few remarks concerning some not-so-well-known aspects of these solutions.

II. DYONS AND THE $U_A(1)$ PROBLEM

One of us[6] has recently proposed as a solution to the $U_A(1)$ problem the existence of gauge-theory magnetic monopoles. As an illustration of this possibility, we will consider an SU(2) gauge field interacting with a massless fermion isodoublet. The symmetry of this model is $SU(2)\times U(1)\times U_A(1)$, corresponding to the gauge symmetry, baryon-number conservation, and axial-vector baryon-number conservation, respectively.

The gauge-invariant axial-vector current, $A_\mu=\bar{\psi}\gamma_\mu i\gamma_5\psi$, associated with $U_A(1)$ will exhibit an anomalous divergence in the quantum theory,

$$\partial_\mu A_\mu=\frac{g^2}{8\pi^2}\,\mathrm{Tr}(F_{\mu\nu}\,{}^*F_{\mu\nu}), \tag{1}$$

where $F_{\mu\nu}$ is the gauge-covariant curl of the gauge field $G_\nu(x)=G^a_\nu(x)\tau^a/2$ and g is the coupling constant of the model. Equation (1) also specifies the rate of change of chirality, since

$$\dot{Q}_A=\int d^3x\,\partial_0 A_0 = \int d^3x\,\partial_\mu A_\mu, \tag{2}$$

assuming that there is no contribution from the surface terms due to A_i. To understand how a dyon solution can lead to a nonvanishing $\dot{Q}_A$ and thus overcome the $U_A(1)$ problem, consider the decomposition identity

$$\mathrm{Tr}(F_{\mu\nu}\,{}^*F_{\mu\nu})=\partial_\mu\xi_\mu-2\,\mathrm{Tr}(G_\mu\,{}^*J_\mu)\ , \tag{3}$$

where

$$\xi_\mu=2\epsilon_{\mu\nu\lambda\delta}\,\mathrm{Tr}\left(G_\nu\partial_\lambda G_\delta+\frac{2g}{3i}G_\nu G_\lambda G_\delta\right),\qquad {}^*J_\mu=D_\nu\,{}^*F_{\nu\mu}\ . \tag{4}$$

For Abelian dyons, ${}^*J_\mu$ is necessarily nonvanishing because of a string singularity in the potential. However, in the case of non-Abelian theories, the existence of string singularities in the fields depends on our choice of gauge.[7] For the model under discussion, this can be clearly seen by making the string-free classical ansatz

$$G^a_0=\frac{x_a}{gr^2}J(r),\quad G^a_i=\epsilon_{aij}\frac{x_j}{gr^2}[1-K(r)],\quad r^2=\sum x_i^2 \tag{5}$$

where J and K are determined from the field equations.[8] Putting this ansatz into Eqs. (1) and (2), we find

$$\dot{Q}_A=\frac{1}{\pi}\int_0^\infty dr\,\frac{d}{dr}\left(\frac{J(1-K^2)}{r}\right) =\frac{1}{\pi}\,\frac{J(1-K^2)}{r}\bigg|_0^\infty\ . \tag{6}$$

The classical field equations provide a nontrivial family of solutions

$$J(r)=\text{const},\quad K(r)=0. \tag{7}$$

Unfortunately, the singularity of these solutions at the origin manifests itself by yielding infinity for both the energy and $\dot{Q}_A$, just like the dyon of Abelian electromagnetism.

These deficiencies can be overcome if we are willing to amend to this model an isotriplet of scalar fields which spontaneously break the gauge symmetry. The addition of these scalar fields leads to a new family of solutions free of singularities and for which both the energy and $\dot{Q}_A$ are finite and calculable. These classical solutions are the Julia-Zee dyons,[9] generalizations of the topologically stable 't Hooft-Polyakov magnetic monopole.[3] Asymptotically and at the origin they are of the form[9]

$$J(r) \underset{r\to 0}{\sim} \text{const}\times r^2, \quad J(r) \underset{r\to\infty}{\sim} Mr + b + \cdots, \quad \text{such that } M < \mu g/\sqrt{\lambda}\,,$$
$$K(r) \underset{r\to 0}{\sim} 1 + \text{const}\times r^2, \quad K(r) \xrightarrow[r\to\infty]{} 0, \tag{8}$$

where the mass M sets the scale for J and b determines the electric charge of the dyon. From Eq. (6), we find for these classical solutions

$$\dot{Q}_A = \frac{1}{\pi} M\,. \tag{9}$$

It is important to notice that the large asymptotic behavior of $J(r)$ leads to a surface term in this example and thus provides the nonvanishing result of Eq. (9). We shall return to this point later, but now we briefly discuss the quantum version of this model. Since the anomalous divergence of Eq. (1) is a quantum result, the legitimacy of our nonvanishing $\dot{Q}_A$ requires the existence of dyon solutions in the quantized theory. Such an occurrence is indeed the case; their appearance and stability have been rigorously verified for this model.[10] As expected, they exhibit only discrete electric charges $q_{\text{dyon}} = \frac{1}{2}ng$, n = integer. (This result is intuitively seen by applying the Saha-Wilson[11] argument of angular momentum conservation to two widely separated dyons.)

In the limit where the scalar mass $\mu \to 0$ and the quartic coupling $\lambda \to 0$ [i.e., $V(\phi) = 0$], Prasad and Sommerfield[12] have obtained exact analytic solutions to the classical field equations studied by Julia and Zee. They found

$$J(r) = \sinh\gamma[Cr\coth(Cr) - 1],$$
$$K(r) = Cr/\sinh(Cr)\,, \tag{10}$$

where C sets the mass scale and γ is an arbitrary constant. These exact solutions are convenient for examining the general properties of the dyons; however, because of the required limit [$V(\phi) = 0$], they exhibit neither topological nor dynamical stability. The dynamical instability of their monopole is easily seen, since the energy is proportional to the mass scale C and goes into the vacuum solution in the limit $C \to 0$. (This is just the Derrick-Coleman argument.[5,13]) One may wonder what happens to the monopole if the solution is unstable. To define the monopole one requires a closed surface beyond which the gauge field is asymptotic. In the limit $t \to \infty$, the energy dissipates (i.e., $C \to 0$) and this surface becomes infinite; the monopole occupies all of space.

Let us in any case list the mass and electric charge of the dyon implied by their solution:

$$M_{\text{dyon}} = \frac{4\pi}{g^2} C\cosh^2\gamma\,,$$
$$q_{\text{dyon}} = \frac{4\pi}{g}\sinh\gamma\,. \tag{11}$$

Calculating $\dot{Q}_A$ for this solution, we find

$$\dot{Q}_A = \frac{g^2}{4\pi^2} C\left(\frac{1}{g}\right) q_{\text{dyon}}\,. \tag{12}$$

This result explicitly demonstrates the dependence of $\dot{Q}_A$ on both the magnetic $(1/g)$ and electric charges of the dyon.

Before leaving these exact solutions, let us discuss another quality they possess. Because they were found under the assumption $V(\phi) = 0$, they are also static solutions to field equations corresponding to a pure SU(2) Yang-Mills theory in four spatial dimensions and one time dimension, with cylindrical symmetry about the additional spatial dimension. This is easily seen by making the identification $G_4^a = \phi^a$. Then because the fields are assumed independent of x_4, the equations obtained for this pure Yang-Mills theory are identical to those solved by Prasad and Sommerfield.[12] Unfortunately, the same instability arguments given above also carry over to this cylindrically symmetric theory.

Several remarks about the above development are in order. We have shown that the existence of dyon solutions to the classical theory implies a nonconserved Q_A. This is evidently a general feature if monoples are present. A characteristic of gauge theories supplemented with symmetry-breaking scalar fields is the existence of gauges in which the fields are singularity free and yet one has monopole solutions. In such gauges $D_\lambda\, {}^*F_{\lambda\mu} = 0$ and the integral

$$\int d^3x\, \mathrm{Tr}(F_{\mu\nu}\, {}^*F_{\mu\nu}) = 2\int d^3x\, E_i^a H_i^a \tag{13}$$

can be written as a surface integral. The dyon is an example for which this surface integral does not vanish. However, the integrand of the surface integral is gauge dependent and indeed can be gauged to zero, but such a gauge transformation introduces singularities into the gauge field and $D_\lambda\, {}^*F_{\lambda\mu}$ becomes singular $\sim \delta^3(x)$. Of course, the integral of Eq. (13) is gauge independent and its value will not be influenced by our method of evaluation.

III. THE PSEUDOPARTICLE AND THE KINK

In this section we will discuss some interesting properties of the pseudoparticle.[4,14] This is a topologically stable classical solution to the four-dimensional Euclidean SU(2) Yang-Mills equations (the exceptional dimension of the Derrick-Coleman instability argument). Whereas Ref. 4 stressed the topological aspects of the pseudoparticle, we will point out some of its analytic properties. It turns out the pseudoparticle is equivalent to a one-dimensional kink.

The classical ansatz[15]

$$G_i^a = (\epsilon_{aij} x_j - x_4 \delta_{ai}) \frac{1-K(r)}{gr^2}, \qquad G_4^a = \frac{x_a}{gr^2}[1-K(r)], \tag{14}$$

$$i,j=1,2,3, \quad a = \mathrm{SU}(2) \text{ index} = 1,2,3, \quad r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2$$

for the SU(2) gauge fields in Euclidean four-space reduces the field equations

$$D_\lambda F^a_{\lambda\mu} = 0 \tag{15}$$

to one nonlinear second-order differential equation

$$r^2 K'' + rK' = 2K(K^2-1), \tag{16}$$

where a prime means d/dr. This equation admits the solutions

$$K(r) = \pm \frac{\lambda^2 - r^2}{\lambda^2 + r^2}, \tag{17}$$

where λ is an arbitrary scale parameter. Asymptotically $r \to \infty$ and at the origin $r=0$ (17) is a pure gauge corresponding to different vacua. This solution is called the pseudoparticle (the solution with the minus sign is the antiparticle.) It implies a self-dual solution

$$F^a_{\mu\nu} = {}^*F^a_{\mu\nu} \tag{18}$$

so that the energy

$$E = \tfrac{1}{2} \int d^4x \, \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) \tag{19}$$

and the topologically invariant Pontryagin index

$$q = \frac{g^2}{16\pi^2} \int d^4x \, \mathrm{Tr}(F_{\mu\nu} {}^*F_{\mu\nu}) = 1 \tag{20}$$

are related:

$$q = \frac{g^2}{8\pi^2} E. \tag{21}$$

The change of variable

$$r = \lambda e^y \tag{22}$$

transforms (16) into

$$\frac{d^2K(y)}{dy^2} = 2K(K^2-1), \tag{23}$$

which is the kink equation. Its solutions are the vacua (pure gauge) $K = \pm 1$ and the pseusoparticle (or kink)

$$K(y) = \mp \tanh(y - y_0), \tag{24}$$

which interpolates between the vacua as $y \to \pm\infty$. Scale invariance of the four-dimensional pseudoparticle manifests itself as translational invariance for the kink. For the spherically symmetric ansatz (14) the energy of the gauge field (19) is equivalent to the energy of the kink

$$E = E_{\text{kink}} = \frac{3\pi^2}{g^2} \int_{-\infty}^{+\infty} dy \left[\left(\frac{dK}{dy}\right)^2 + (1-K^2)^2 \right]. \tag{25}$$

Because of the scale parameter λ introduced by the pseudoparticle solution it can provide a mass gap in quantum chromodynamics.[16] The pseudoparticle is also an example of a topological solution to the $\mathrm{U}_A(1)$ problem, like the dyon; however, in this case scalar fields are not required. 't Hooft[17] has observed that for the pseudoparticle the net change in chirality is related to the Pontryagin index,

$$\Delta Q_A = \int dx_4 \frac{\partial Q_A}{\partial x_4} = \frac{g^2}{8\pi^2} \int d^4x \, \mathrm{Tr}(F_{\mu\nu} {}^*F_{\mu\nu}) = 2q = 2, \tag{26}$$

and is therefore integer. Evidently the net change in chirality, the Pontryagin index, and kink number are all equivalent. In 't Hooft's treatment the violation of chiral charge conservation due to the pseudoparticle is explicitly in the boundary condition placed on the gauge-dependent vector ξ_μ, given by Eq. (4), as $r \to \infty$. Alternately, by a singular gauge transformation one may transform $r^3 \xi_\mu \to 0$ as $r \to \infty$ but then the gauge field becomes singular at $r=0$ and $\mathrm{Tr}(G_\mu {}^*J_\mu) \sim \delta^4(x)$.

In the four-dimensional classical theory one may add the fermion doublet and this corresponds (again with spherical symmetry) to a fermion-kink coupling problem in one dimension. Exact classical solutions are known[18] and one finds a zero-energy nondegenerate ground state. Properties of the one-dimensional kink-fermion quantum theory have also been studied.[19] The possible relevance of the one-dimensional quantized kink-fermion system for the four-dimensional pseudoparticle-fermion system is presently under investigation.

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