Yang-Mills particle in 't Hooft's gauge field*

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We investigate the classical motion of a Yang-Mills test particle in an external field given by 't Hooft's monopole solution. For large distances the space motion is that of a charged particle in a magnetic-monopole field. It is different at small distances though. We also find the asymptotic solution for the motion of the particle's isospin vector and discuss its interpretation.

Recently, 't Hooft[1] demonstrated that the system consisting of an SU(2) gauge field coupled to a scalar triplet has interesting magnetic-monopole-like classical static solutions. The Lagrangian density is given by

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \tfrac{1}{2}\pi^a_\mu \pi^a_\mu + \tfrac{1}{2}\mu^2 \phi^a\phi^a - \tfrac{1}{4}\lambda(\phi^a\phi^a)^2 ,\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu ,\\ \pi^a_\mu &= \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c ,\end{aligned} \tag{1}$$

and the solutions belong to the general type (of Wu and Yang[2] and Julia and Zee[3])

$$\begin{aligned}A^a_i &= \frac{1}{gr}\epsilon_{aij}\hat{x}_j[1-K(r)] ,\\ A^a_4 &= \frac{i}{gr}\hat{x}_a J(r) ,\\ \phi^a &= \frac{1}{gr}\hat{x}_a H(r) ,\end{aligned} \tag{2}$$

where $K(r)$, $J(r)$, and $H(r)$ are certain functions of the radius r. 't Hooft assumed $J=0$, while the original Wu-Yang solution for the pure gauge field case corresponds to $J=H=0$. Many authors[4-6] have discussed various aspects of this system. In particular, Arafune, Freund, and Goebel[4] have pointed out an interesting topological property of these solutions. Furthermore, the original solutions were found approximately, but then Prasad and Sommerfield[5] noted that very similar solutions existed for the $\mu=\lambda=0$ case in which the following exact answer could be gotten (c and χ are arbitrary constants):

$$\begin{aligned}K(r) &= cr/\sinh cr ,\\ J(r) &= \sinh\chi(cr\coth cr - 1) ,\\ H(r) &= \cosh\chi(cr\coth cr - 1) .\end{aligned} \tag{3}$$

Now, in order to discuss the physical meaning of the solutions (2), 't Hooft defined a gauge-invariant tensor which may be rewritten as

$$\mathcal{F}_{\mu\nu} = \partial_\mu(\hat{\phi}^a A^a_\nu) - \partial_\nu(\hat{\phi}^a A^a_\mu) - \frac{1}{g}\epsilon_{abc}\hat{\phi}^a \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c \tag{4}$$

with

$$\hat{\phi}_a = \phi_a/|\phi_a| = \hat{x}_a .$$

It is easy to see that (for $J=0$) only the second term of (4) receives a contribution when (2) is substituted in. This contribution is independent of K and has the form of a pure magnetic monopole: $B_i \equiv \frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk} = -\hat{x}_i/gr^2$. Thus an electrically charged test particle coupled to an Abelian vector potential corresponding to $\mathcal{F}_{\mu\nu}$ would move exactly as a charged particle in a pure monopole field.

Since we are dealing with a Yang-Mills theory a natural and related question is: How would a Yang-Mills test particle move in the field system (2) treated as a classical external field? We shall find that for large r the Yang-Mills test particle does indeed behave as an electrically charged particle in a monopole field. However, the motion is different at small distances.

The Yang-Mills test particle may be formally added to the theory by including in (1) the additional terms

$$-\bar{\psi}(\gamma_\mu\partial_\mu - ig\gamma_\mu A^a_\mu \tau^a/2 + m)\psi , \tag{5}$$

where ψ is for definiteness an isospinor fermion. The τ^a are the Pauli matrices. A test particle with an additional Yukawa-type coupling to the Higgs particle can also be contemplated but we shall not do that here. To get the classical equations of motion one can go to the one-particle sector of the theory represented by (1) and (5) and then proceed as in the usual treatment of the Dirac equation with an external field. This was done by Wong[7] who got the following equations of motion for a particle at position x_μ:

$$m\ddot{x}_\mu = gF^a_{\mu\nu}\dot{x}_\nu I^a , \tag{6}$$

$$\dot{I}^a = -g\epsilon^{abc}A^b_\mu \dot{x}_\mu I^c . \tag{7}$$

In these equations an overdot indicates differentiation with respect to proper time, but for simplicity we shall consider nonrelativistic motions here. The quantity I^a is a classical isospin vector (equal to $\frac{1}{2}\tau^a$ in the quantum case). The fields on the right-hand sides of (6) and (7) will be taken from

(2) (with $J=0$, initially). We will then examine the resulting motion in space and determine from the spatial (as opposed to the isospin) motion whether the test particle is moving in the same way as a charge in a pure magnetic-monopole field.

First, however, we must find a convenient way of treating the time evolution of the isospin vector since I^a also appears in (6). [Note from (7) that I^aI^a is constant in time.] Define at each point along the particle's trajectory an orthogonal set of vectors $\vec{x}$, $\vec{w}=\vec{x}\times\vec{v}$, $\vec{z}=\vec{x}\times\vec{w}$, with $\vec{v}=\dot{\vec{x}}$. Then we may without any loss of generality parametrize I^a by

$$I^a=\alpha\hat{x}_a+\beta\hat{w}_a+\gamma\hat{z}_a. \tag{8}$$

The coefficients α,β,γ evidently satisfy

$$\alpha^2+\beta^2+\gamma^2=\text{const}. \tag{9}$$

Substituting (8) and (2) into (7) gives three relatively simple equations for α, β, and γ:

$$\dot{\alpha}=-\gamma\rho K/r^2, \tag{10}$$

$$\dot{\beta}-\frac{\gamma r}{\rho^2}\dot{\vec{v}}\cdot\vec{x}\times\vec{v}=-\gamma J/r, \tag{11}$$

$$\dot{\gamma}+\frac{\beta r}{\rho^2}\dot{\vec{v}}\cdot\vec{x}\times\vec{v}=\alpha\rho K/r^2+\beta J/r, \tag{12}$$

where $r=|\vec{x}|$ and $\rho=|\vec{w}|$. Next, substituting (8) and (2) into (6) gives the ordinary equation of motion:

$$m\dot{\vec{v}}=\frac{1}{r^2}\left\{(\vec{v}\times\hat{x})\left[\alpha(K^2-1)+\frac{\gamma rK'}{\rho}\,\vec{x}\cdot\vec{v}\right]+(\vec{x}v^2-\vec{v}\vec{x}\cdot\vec{v})\,\frac{\beta rK'}{\rho}\right\}+\frac{1}{r^2}\left[\alpha\hat{x}(-J+J'r)+JK(\beta\hat{w}+\gamma\hat{z})\right]. \tag{13}$$

In (13) a prime means differentiation with respect to r and $v=|\vec{v}|$.

Now consider the above equations for large r. Also take $J=0$ for the time being. Then $K(r)$ is expected to fall off exponentially with r [see (3) for example] so we may set

$$K(r)=K'(r)=0\quad(\text{large } r).$$

This simplifies (10) to give

$$\alpha=\alpha_0=\text{const}$$

and simplifies (13) to give

$$m\dot{\vec{v}}=-\frac{\alpha_0}{r^2}\vec{v}\times\vec{x}. \tag{14}$$

Equation (14) is identical to the equation for a charged particle moving in the field of a pure magnetic monopole. We shall discuss the motion of the isospin vector later.

Next consider the above equations for arbitrary r (and $J=0$). It is almost immediately apparent that the motion cannot be given by an equation like (14). Assume that it is and note that we arrive at a contradiction. For in order for the right-hand side of (13) to contain no terms which are not proportional to $\vec{v}\times\hat{x}$ we must require $\beta=0$. From (11) we conclude $\gamma=0$ and then from (12) that $\alpha=0$. Substituting $\alpha=\beta=\gamma=0$ back in (13) gives the equation of motion $m\dot{\vec{v}}=0$, which is the desired contradiction. Thus, while the Yang-Mills test particle moves in the same way as a charge in a monopole field at large distances it has a different motion at small distances. The significance of this appears to be that 't Hooft's tensor (4) defines a topological property of the solution, putting it into a general class corresponding to a given net magnetic-pole strength. This seems analogous to the Wu-Yang[8] concept of a gauge which can accommodate many different physical fields so long as the net magnetic-pole strength is specified.

To complete the picture it is desirable to study the motion of the isospin vector and to verify that a consistent interpretation can be found. We shall give an exact scattering solution for the $K=0$ case where the space motion is described by (14). Alternatively, our solution may be regarded as an approximate (asymptotic) solution to (13). From (14) one easily verifies that the speed $v=|\vec{v}|$, the magnitude of the angular momentum $l=m|\vec{w}|$, the vector $\vec{j}=m\vec{w}+\alpha_0\hat{x}$, and $j\cdot\hat{x}\equiv\alpha_0$ are constants of motion. In addition, $j^2=l^2+\alpha_0^{\,2}$, where $j\equiv|\vec{j}|$. These lead to the well-known result[9] that the particle moves on the surface of a cone whose axis (through the origin) is parallel to $\vec{j}$ and whose half-angle is $\cos^{-1}(\alpha_0/j)$. At time $t=-\infty$ the particle is entering the scene, heading towards the origin practically along a radial direction. It winds its way in on the cone until $t=0$ when it reaches the minimum distance

$$r_{\min}=l/mv$$

and then winds its way back out, approaching at $t=+\infty$ another radial asymptote. The r motion for both positive and negative t is given by

$$r=(r_{\min}{}^2+v^2t^2)^{1/2}. \tag{15}$$

Now note that if the quantity $\dot{\vec{v}}\cdot\vec{x}\times\vec{v}$ is evaluated using (14) and if the complex variable

$$\mathfrak{z}=\beta+i\gamma \tag{16}$$

is introduced, the isospin-motion equations (11) and (12) can be combined to yield (remembering $J=K=0$)

$$\dot{\mathfrak{z}}=-\frac{i\alpha_0}{mr^2}\mathfrak{z}. \tag{17}$$

Substituting (15) into (17) and carrying out a straightforward integration gives the solution

$$\mathfrak{z}=\mathfrak{z}_0\exp\left(-\frac{i\alpha_0}{mvr_{\min}}\tan^{-1}\frac{vt}{r_{\min}}\right), \tag{18}$$

where $\mathfrak{z}_0=\beta_0+i\gamma_0$ is a constant. Equation (18) represents [see (8)] a precession of the isospin vector (in isospace) around the direction of the particle's radius vector. For small t the angular frequency of precession is $\alpha_0/mr_{\min}{}^2$, while as $t\to\pm\infty$ the precession slows so that asymptotically I^a points in a fixed direction. From (18) we see that as $t\to\pm\infty$

$$\beta\to\beta_0\cos\frac{\alpha_0\pi}{2l}\pm\gamma_0\sin\frac{\alpha_0\pi}{2l},$$

$$\gamma\to\gamma_0\cos\frac{\alpha_0\pi}{2l}\mp\beta_0\sin\frac{\alpha_0\pi}{2l}, \tag{19}$$

and

$$\alpha=\alpha_0\quad(\text{all } t).$$

From the standpoint of giving a physical interpretation, it is encouraging that asymptotically the particle's isospin vector has a fixed orientation since we interpret this orientation as the "identity" of the particle. Equation (19) shows that by adjusting α_0, β_0, and γ_0 we may choose any initial identity. Note, however, that α,β,γ are components along a coordinate system which travels along with the particle. The interpretation of this result may be facilitated by making use of a gauge transformation which has been exploited by several authors.[1,4,8] We rotate, at each point (r,θ,ϕ) in real space, the *isospin* direction $\hat{x}$ into the third isospin direction with the transformation

$$S^{-1}=\cos\frac{\theta}{2}+i\vec{n}\cdot\vec{\tau}\sin\frac{\theta}{2},$$

$$n_i=-\frac{\epsilon_{ij3}\hat{x}_j}{\sin\theta}.$$

The field transforms to

$$A'^a_i\tau^a=S^{-1}\tau^aSA^a_i-\frac{2i}{g}\ \partial_iS^{-1}S.$$

For the case $K=0$ (and only for this case) being considered now, the new field A'^a_i lines up along the isospin three-direction and is a pure monopole type:

$$A'^a_i=\frac{\delta^{a3}}{g}\ \frac{\epsilon_{3ij}\hat{x}_j}{r+x_3}.$$

It is clear that under this transformation the test particle's isospin I^a must also be transformed. If $\alpha_0\neq0$ while $\beta_0=\gamma_0=0$, I^a will be transformed [see Eq. (8)] to lie along the three-direction and the situation will be exactly that (remembering $K=0$) of a charged particle in a pure monopole field. β and γ will be zero at all times. At the other extreme, if β_0 and γ_0 are nonzero while $\alpha_0=0$, it is easy to see from (14) that the particle will proceed with uniform rectilinear motion. This corresponds, in the new gauge, to a particle whose classical isospin vector has no component along the three-direction scattering in a field which has components only in the isospin three-direction. For general $\alpha_0,\beta_0,\gamma_0$ the particle will move as a charge in a monopole field but its isospin vector will precess in accordance with (18).

Finally, we briefly comment on the $J\neq0$ case, expected to correspond to a dyonlike structure. It is convenient to work in the framework of the Prasad-Sommerfield solution (3). This has the property that the hyperbolic rotation angle χ relating ϕ^a and A^a_4 appears directly in the relation between the non-Abelian magnetic and electric fields, namely

$$E^a_i=-\sinh\chi\,B^a_i, \tag{20}$$

where $B^a_i\equiv\frac{1}{2}\epsilon_{ijk}F_{jk}$ and $E^a_i=iF^a_{4i}$. Substituting the large-r limit of (20), $B^a_i\sim-\hat{x}_i\hat{x}_a/gr^2$, as well as (8) into (6) gives the large-r equation of motion:

$$m\dot{\vec{v}}=-\frac{\alpha_0}{r^2}\left(\vec{v}\times\hat{x}-\sinh\chi\,\hat{x}\right), \tag{21}$$

where we have also used $\alpha=\alpha_0=\text{const}$ from (10). This is, of course, the same as the equation for a charged particle moving in a dyon field.

Note added in proof. We may see that for $K\neq0$ (and $J=0$) the motion is not that of a pure monopole in another way. Using (10)–(13) we verify that

$$\vec{J}=m(\vec{x}\times\vec{v})+K\vec{I}+\alpha(1-K)\hat{x}$$

is a constant of motion. Note that $\vec{J}\to\vec{j}$ for $K=0$. Hasenfratz and 't Hooft [Phys. Rev. Lett. 36, 1119 (1976)] have recently shown that the second and third terms correspond to the angular momentum associated with the fields of the "monopole" and the test charge, in analogy to the $\vec{E}\times\vec{B}$ contribution in the usual monopole case. We next observe that $\vec{J}\cdot\hat{x}=\alpha$. For $K=0$ (10) shows that $\alpha=\alpha_0=\text{const}$ so that the motion is confined to the cone discussed previously. On the other hand, for $K\neq0$ (10) shows that α is not in general constant and the test particle will depart from the cone.

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