Conformal properties of a Yang-Mills pseudoparticle*

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The conformal transformation properties of the recently discovered pseudoparticle solution to a pure Yang-Mills theory are studied. It is shown that the solution is invariant under an O(5) subgroup of conformal transformations. A formalism is developed which renders this invariance explicit and which allows a very compact group-theoretical analysis of the propagation of fermions in the field of the pseudoparticle.

I. INTRODUCTION

A remarkable pseudoparticle solution to the SU(2) Yang-Mills theory in Euclidean four-space has been found by Belavin, Polyakov, Schwartz, and Tyupkin.[1] It has the property that although the gauge-covariant field strengths $F^{\mu\nu}=(\sigma^a/2i)F_a^{\mu\nu}$ vanish rapidly at large distance, the gauge potentials $A^\mu=(\sigma^a/2i)A_a^\mu$ decrease considerably more slowly—they tend to a pure gauge term

$$A^\mu \underset{x\to\infty}{\longrightarrow} g^{-1}\partial^\mu g. \tag{1.1}$$

As a consequence, the quantity

$$q=-\frac{1}{16\pi^2}\int d^4x \,\mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}, \tag{1.2}$$

where

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+[A^\mu,A^\nu]\ ,$$
$${}^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta},$$

is nonvanishing, even though the integrand is a total derivative

$$\mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}=2\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,\partial_\mu(A_\nu\partial_\alpha A_\beta+\tfrac{2}{3}A_\nu A_\alpha A_\beta). \tag{1.3}$$

Since $(1/8\pi^2)\,\mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}$ is the anomalous divergence of the U(1) axial-vector current,[2] q also measures the anomalous violation of chirality. Such behavior in Minkowski-space quantum field theory would have far-reaching physical consequences,[3] especially as regards the well-known U(1) problem.[4] Indeed it has been suggested that the pseudoparticle solution be used to dominate the functional-integral description of quantum field theory, continued to Euclidean space.[5]

We discuss here the behavior of the pseudoparticle under general conformal transformations, which form an O(5, 1) invariance group for the field equation in Euclidean space[6]

$$\partial_\mu F^{\mu\nu}+[A_\mu,F^{\mu\nu}]=0, \tag{1.4}$$

We show in Sec. II that the solution respects a subgroup of the conformal group—the O(5) group generated by $M^{\mu\nu}$, the rotation generator, and by a combination of K^μ and P^μ, the conformal and translation generators, respectively. This group has been previously considered by Adler[7] and Fubini.[8] Moreover, coordinate inversion corresponds to pseudoparticle conjugation.

The O(5) invariance can be made explicit by formulating the theory in a manifestly O(5)-covariant fashion. This we do in Sec. III and find a truly elegant expression for the pseudoparticle field. Furthermore, the particularly simple form offers remarkable computational advantages, as we show in Sec. IV, where we study the interaction of fermions with the pseudoparticle.

II. CONFORMAL TRANSFORMATIONS

The form of the pseudoparticle solution[1] is

$$A^\mu=\frac{x^2}{1+x^2}g^{-1}\partial^\mu g, \tag{2.1}$$

with $g=(x_4-i\vec{x}\cdot\vec{\sigma})(x_\mu x^\mu)^{-1/2}$. Translations and dilatations do not leave (2.1) invariant; they give equivalant solutions with x shifted (translations) or 1 rescaled (dilatations). The field strength $F^{\mu\nu}$ is

$$F^{\mu\nu}=\frac{4}{(1+x^2)^2}i\sigma^{\mu\nu}, \tag{2.2}$$

where we have defined

$$\sigma^{ij}=\frac{1}{4i}[\sigma^i,\sigma^j],$$
$$\sigma^{i4}=\tfrac{1}{2}\sigma^i. \tag{2.3}$$

The matrices $\sigma^{\mu\nu}$ and the field strengths are self-dual, ${}^*F^{\mu\nu}=F^{\mu\nu}$ and $q=1$.

Another solution is obtained by replacing g with $g^\dagger=g^{-1}=(x_4+i\vec{x}\cdot\vec{\sigma})(x_\mu x^\mu)^{-1/2}$. In this case

$$F^{\mu\nu}=\frac{4}{(1+x^2)^2}i\bar{\sigma}^{\mu\nu} \tag{2.4}$$

and

$$\bar{\sigma}^{ij}=\sigma^{ij},$$
$$\bar{\sigma}^{i4}=-\sigma^{i4}. \tag{2.5}$$

The matrices $\bar{\sigma}^{\mu\nu}$ and the field strengths are now self-antidual, $*F^{\mu\nu}=-F^{\mu\nu}$, and $q=-1$.

We consider first a discrete transformation of the conformal group, coordinate inversion, which takes x^μ into $(1/x)^\mu \equiv x^\mu/x^2$. The vector field transforms under inversion into $\bar{A}^\mu(x)$, given by

$$A^\mu(x)\to\bar{A}^\mu(x)=\frac{1}{x^2}I^{\mu\nu}(x)A_\nu(1/x),$$
$$I^{\mu\nu}(x)=g^{\mu\nu}-2\frac{x^\mu x^\nu}{x^2}. \tag{2.6}$$

It follows that $F^{\mu\nu}$ transforms similarly:

$$F^{\mu\nu}(x)\to\bar{F}^{\mu\nu}(x)=\frac{1}{x^4}I^{\mu\alpha}(x)I^{\nu\beta}(x)F_{\alpha\beta}(1/x). \tag{2.7}$$

Suppose $F^{\mu\nu}$ is self-dual or self-antidual:

$$*F^{\mu\nu}=\pm F^{\mu\nu}. \tag{2.8a}$$

It is easy to verify that the inverted field strengths satisfy an inverted relation

$$*(\bar{F}^{\mu\nu})=\mp(\bar{F}^{\mu\nu}). \tag{2.8b}$$

Consequently,

$$\begin{aligned} q&=-\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\,*F^{\mu\nu}F_{\mu\nu}\\ &=\mp\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}F^{\mu\nu}F_{\mu\nu}\to\bar{q},\\ \bar{q}&=-\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\,*(\bar{F}^{\mu\nu})(\bar{F}_{\mu\nu})\\ &=\pm\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\bar{F}^{\mu\nu}\bar{F}_{\mu\nu}. \end{aligned} \tag{2.9}$$

Since the action $-\frac{1}{8}\int d^4x\,\mathrm{Tr}F^{\mu\nu}F_{\mu\nu}$ is inversion-invariant, $\bar{q}=-q$. Hence coordinate inversion sends the pseudoparticle with $q=1$ into an antiparticle with $q=-1$.[9]

The explicit formula for the inverted potential is

$$A^\mu(x)\to\bar{A}^\mu(x)=\frac{1}{1+x^2}g^{-1}\partial^\mu g. \tag{2.10a}$$

This does not satisfy the boundary conditions (1.1) and is singular at the origin. But if $\bar{A}^\mu$ is subjected to a gauge transformation

$$\bar{A}^\mu\to U^{-1}\bar{A}^\mu U+U^{-1}\partial^\mu U,$$

with $U=g^{-1}$, then (2.10a) is seen to be equivalent to

$$\bar{A}^\mu=\frac{x^2}{1+x^2}g\partial^\mu g^{-1}, \tag{2.10b}$$

which is consistent with (1.1).

The particle and antiparticle solutions can be put together by extending the gauge group to SU(2) $\times$ SU(2) = O(4), which is a convenience already pointed out in Ref. 1, and is also adopted by us henceforth. The O(4) field strengths are now given by

$$F^{\mu\nu}=\frac{4}{(1+x^2)^2}i\Sigma^{\mu\nu}, \tag{2.11}$$

where $\Sigma^{\mu\nu}$ is constructed from the Dirac matrices

$$\Sigma^{\mu\nu}=\frac{1}{4i}[\alpha^\mu,\alpha^\nu]=\begin{pmatrix}\sigma^{\mu\nu} & 0\\ 0 & \bar{\sigma}^{\mu\nu}\end{pmatrix},$$
$$\alpha^i=\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix},\quad \alpha^4=i\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}. \tag{2.12}$$

These field strengths are obtained from the potentials

$$A^\mu=\frac{-2i}{1+x^2}\Sigma^{\mu\nu}x_\nu. \tag{2.13}$$

The matrices $\Sigma^{\mu\nu}$ are block diagonal, so that Eqs. (2.11) and (2.13) represent two separate solutions in the two SU(2)'s of the decomposition O(4) = SU(2) $\times$ SU(2). In terms of the O(4) formalism, we can say that the effect of the inversion is to produce solutions where, up to a gauge transformation, the upper and lower diagonal blocks of the matrices $\Sigma^{\mu\nu}$ are interchanged.

Next we consider the effect on the pseudoparticle of continuous transformations of the conformal group. Of course the solution does not respect translations and dilatations. An infinitesimal rotation with parameters $\omega^{\alpha\beta}=-\omega^{\beta\alpha}$ takes x^α into $x^\alpha-\omega^{\alpha\beta}x_\beta$, and

$$\delta F^{\alpha\beta}=\frac{4i}{(1+x^2)^2}(\omega^\alpha{}_\nu\Sigma^{\nu\beta}-\omega^\beta{}_\nu\Sigma^{\nu\alpha}). \tag{2.14}$$

The appraent noninvariance of $F^{\alpha\beta}$ can be compensated by an O(4) gauge transformation $U=e^{-i\Theta}$, $\Theta=\frac{1}{2}\theta_{\alpha\beta}\Sigma^{\alpha\beta}$. The response of $F^{\alpha\beta}$ to an infinitesimal gauge rotation is

$$\begin{aligned}\delta F^{\alpha\beta}&=i[\Theta,F^{\alpha\beta}]\\ &=-\frac{4}{(1+x^2)^2}[\Theta,\Sigma^{\alpha\beta}],\end{aligned} \tag{2.15}$$

which compensates (2.14) provided $\theta^{\alpha\beta}=\omega^{\alpha\beta}$. Thus, as was already remarked in Ref. 1, the pseudoparticle is invariant under the combined space and isospace rotation generated by $J^{\mu\nu}$,

$$J^{\mu\nu}=M^{\mu\nu}+\Sigma^{\mu\nu}. \tag{2.16}$$

A conformal transformation, i.e., an inversion, a translation, and another inversion, will not leave the solution invariant, since translations do not. However, if we perform the infinitesimal transformation generated by[7,8]

$$R^\mu=\tfrac{1}{2}(K^\mu+P^\mu), \tag{2.17}$$

with parameters a_μ, we find

$$\delta F^{\alpha\beta}=\frac{4i}{(1+x^2)^2}(\Omega^\alpha{}_\nu\Sigma^{\nu\beta}-\Omega^\beta{}_\nu\Sigma^{\nu\alpha}),\qquad \Omega^{\alpha\nu}=a^\alpha x^\nu-a^\nu x^\alpha. \tag{2.18}$$

Comparison with Eqs. (2.14) and (2.15) shows that the noninvariance may be compensated by a gauge transformation parametrized by Θ of the form

$$\Theta=\tfrac{1}{2}\Omega_{\mu\nu}\Sigma^{\mu\nu}=a_\mu x_\nu\Sigma^{\mu\nu}. \tag{2.19}$$

The invariance is therefore generated by $\mathcal{R}^\mu$:

$$\mathcal{R}^\mu=R^\mu+\Sigma^{\mu\nu}x_\nu. \tag{2.20}$$

When it is recalled that the action of R^μ on an arbitrary field Φ is[6]

$$\delta\Phi=\tfrac{1}{2}(1+x^2)^{1-d}a^\mu\partial_\mu[(1+x^2)^d\Phi]+a_\mu x_\nu(x^\mu\partial^\nu-x^\nu\partial^\mu+iS^{\mu\nu})\Phi, \tag{2.21}$$

where d is the scale dimension and $S^{\mu\nu}$ the spin matrix of Φ, it is recognized that $\mathcal{R}^\mu$ takes a similar form, except that $S^{\mu\nu}$ is replaced by $S^{\mu\nu}+\Sigma^{\mu\nu}$, just as in Eq. (2.16). The combination law for the modified generators $J^{\mu\nu}$ and $\mathcal{R}^\mu$ follows that of $M^{\mu\nu}$ and R^μ; the algebra closes on O(5).[7,8]

III. YANG-MILLS THEORY ON A HYPERSPHERE

The O(5) invariance of the pseudoparticle solution suggests that the theory be formulated in a manifestly O(5)-covariant fashion. This is achieved by projecting 4-dimensional Euclidean space onto the surface of a unit hypersphere embedded in 5-dimensional Euclidean space.[7,8] We introduce 5-dimensional coordinates r_a, $a=1,\ldots,5$, $r_ar_a=1$ (in what follows Latin labels $a,b,c,\ldots$ run from 1 to 5; Greek labels $\mu,\nu,\zeta,\ldots$ run from 1 to 4):

$$r_\mu=\frac{2x_\mu}{1+x^2},\qquad r_5=\frac{1-x^2}{1+x^2}. \tag{3.1}$$

Rotations on the sphere are generated by M_{ab}, with $M_{5\mu}=R_\mu$, as is easily shown by defining the 5-dimensional orbital angular momentum tensor L_{ab}:

$$L_{ab}=-ir_a\frac{\partial}{\partial r_b}+ir_b\frac{\partial}{\partial r_a},\qquad L_{\mu\nu}=-ix_\mu\frac{\partial}{\partial x^\nu}+ix_\nu\frac{\partial}{\partial x^\mu},\qquad L_{5\mu}=-ix_\mu x^\nu\frac{\partial}{\partial x^\nu}-\tfrac{1}{2}i(1-x^2)\frac{\partial}{\partial x^\mu}. \tag{3.2}$$

The formulation of an Abelian gauge theory on a hypersphere has been given by Adler.[7] To generalize his results to the non-Abelian case, one introduces gauge fields $\hat{A}_a$, which are anti-Hermitian matrices in the space of infinitesimal generators of the gauge group and obey the constraint $r_a\hat{A}_a=0$. The relation between the 5-dimensional $\hat{A}_a$ and the conventional 4-dimensional A_μ is the following:

$$\frac{1+x^2}{2}A_\mu=\hat{A}_\mu-x_\mu\hat{A}_5,\qquad \hat{A}_\mu=\frac{1+x^2}{2}A_\mu-x_\mu x_\nu A^\nu,\qquad \hat{A}_5=-x_\mu A^\mu. \tag{3.3}$$

Under a gauge transformation U, $\hat{A}_a$ is changed into $\hat{A}'_a$,

$$\hat{A}_a\to\hat{A}'_a=U^{-1}\hat{A}_aU+U^{-1}ir_bL_{ba}U. \tag{3.4}$$

From the potentials $\hat{A}^a$ one constructs a totally antisymmetric field-strength tensor of rank three, $\hat{F}_{abc}$:

$$\hat{F}_{abc}=iL_{ab}\hat{A}_c+r_a[\hat{A}_b,\hat{A}_c]+\text{cyclic permutations of }a,b,c. \tag{3.5}$$

The invariant action is

$$I=-\tfrac{1}{48}\int d\Omega\,\mathrm{Tr}\hat{F}_{abc}\hat{F}_{abc}, \tag{3.6}$$

where the integral is over the angular hyperspherical variables. I is identical to the action $-\frac{1}{16}\int d^4x\,\mathrm{Tr}F^{\mu\nu}F_{\mu\nu}$ constructed with the 4-dimensional field strength. The variational principle $\delta I=0$ then gives the equation satisfied by $\hat{F}_{abc}$,

$$iL_{ab}\hat{F}_{abc}+r_a[\hat{A}_b,\hat{F}_{abc}]-r_b[\hat{A}_a,\hat{F}_{abc}]=0. \tag{3.7}$$

We have seen in Sec. II that O(5) invariance of the pseudoparticle is achieved by a combination of an O(5) rotation, $M_{ab}=\{M_{\mu\nu},R_\mu\}$, and a gauge transformation. This suggests that in the present O(5) formalism, the invariance can be made explicit by adding to M_{ab} some suitable generators of a gauge transformation isomorphic to O(5). Of course, the gauge group we are dealing with is O(4), which is only a subgroup of O(5). But the matrix representation of O(4) provided by Eq. (2.12) can be immediately extended to a representation of O(5) by defining the matrices $\Sigma_{\mu5}$,

$$\Sigma_{\mu5}=\tfrac{1}{2}\alpha_\mu. \tag{3.8}$$

Then the 10 matrices Σ_{ab} are isomorphic to the infinitesimal generators of O(5). We stress that the embedding of the O(4) gauge group in an O(5) gauge group does not alter the theory; rather it is a convenient device for exposing invariance under the O(5) subgroup of conformal transformations.

Having extended the gauge group to O(5), we can look for solutions which are explicitly invariant under the combined space and gauge rotations generated by $J_{ab}=M_{ab}+\Sigma_{ab}$. The most general form for $\hat{A}_a$ compatible with this invariance is

$$\hat{A}_a = i\alpha\Sigma_{ab}r_b, \tag{3.9}$$

where α is a constant. The field strength is determined from Eq. (3.5),

$$\hat{F}_{abc} = -i(\alpha^2+2\alpha)(r_a\Sigma_{bc}+r_b\Sigma_{ca}+r_c\Sigma_{ab}). \tag{3.10}$$

In order that the field equation (3.7) be satisfied, the following condition must be true:

$$-6i(\alpha+1)(\alpha^2+2\alpha)\Sigma_{ab}r_b = 0, \tag{3.11}$$

which requires that $\alpha = 0, -1, -2$. The two values $\alpha = 0, -2$ lead to vanishing field strengths, and correspond therefore to a pure gauge ansatz for the potentials, while $\alpha = -1$ gives

$$\hat{A}_a = -i\Sigma_{ab}r_b, \tag{3.12a}$$

$$\hat{F}_{abc} = i(r_a\Sigma_{bc}+r_b\Sigma_{ca}+r_c\Sigma_{ab}). \tag{3.12b}$$

This is the most general nontrivial solution to an O(5) Yang-Mills field theory on the hypersphere which is O(5)-invariant.

The solution (3.12) is also equivalent to the pseudoparticle solution (2.11) and (2.13). To show this, we first eliminate the matrices $\Sigma_{\mu 5}$ from (3.12) with a gauge transformation (3.4), thereby rotating the fields into the O(4) subspace. Then we use Eq. (3.3) to construct A_μ from $\hat{A}_a$. The required gauge transformation is

$$U = \exp[if(r_5)\Sigma_{\mu 5}r^\mu]. \tag{3.13}$$

One verifies that the two choices for $f(r_5)$

$$f(r_5) = \frac{\cos^{-1}r_5}{(1-r_5^2)^{1/2}}, \tag{3.14a}$$

$$f(r_5) = \frac{\cos^{-1}r_5 - \pi}{(1-r_5^2)^{1/2}} \tag{3.14b}$$

lead respectively to

$$A^\mu = \frac{-2i}{1+x^2}\Sigma^{\mu\nu}x_\nu, \tag{3.15a}$$

$$A^\mu = \frac{-2i}{(1+x^2)x^2}\Sigma^{\mu\nu}x_\nu, \tag{3.15b}$$

which are precisely the two gauge-equivalent forms of the potentials that characterize the pseudoparticle solution.

To summarize, the invariance of the pseudoparticle solution under the O(5) subgroup of conformal transformations generated by $M^{\mu\nu}$ and $R^\mu = \frac{1}{2}(K^\mu + P^\mu)$ can be made manifest by projecting Euclidean 4-space onto a hypersphere and embedding the O(4) gauge group in an O(5) gauge group. Using a gauge transformation, we can elegantly express the solution as in (3.12)—a formula invariant under the combined space and gauge transformations generated by $J_{ab}=M_{ab}+\Sigma_{ab}$. Note also that on the hypersphere the solution (3.12) is not concentrated around any definite point, and the action density $\mathcal{L} = -\frac{1}{48}\mathrm{Tr}\hat{F}_{abc}\hat{F}_{abc}$ is uniformly distributed, even though, by a dilatation transformation, it is of course possible to obtain solutions where $\mathcal{L}$ is not constant on the unit hypersphere.

IV. FERMIONS IN THE FIELD OF THE PSEUDOPARTICLE

In this section we analyze the fermion-pseudoparticle system. The power of the O(5) formalism allows for a complete solution of the equations. To describe Fermi fields on the hypersphere one must introduce a set of five anticommuting matrices Γ_a,

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \tag{4.1}$$

We take these to be 4-dimensional,[10]

$$\Gamma_\mu = i\alpha_\mu\alpha_5,$$
$$\Gamma_5 = \alpha_5, \tag{4.2}$$
$$\alpha_5 = \alpha_1\alpha_2\alpha_3\alpha_4,$$

and we define

$$S_{ab} = \frac{1}{4i}[\Gamma_a, \Gamma_b]. \tag{4.3}$$

The Fermi fields $\hat{\psi}$ have four components[10]; they are related to the 4-dimensional, Euclidean Fermi fields ψ by

$$\hat{\psi} = \frac{(1+x^2)}{2}(1+i\alpha_\mu x^\mu)\psi. \tag{4.4}$$

It is possible to show that

$$\int d^4x\psi^\dagger i\alpha^\mu \frac{\partial}{\partial x^\mu}\psi = \int d\Omega\, \hat{\psi}^\dagger(S_{ab}L_{ab}+2)\hat{\psi}, \tag{4.5}$$

so that the free Dirac equation $i\alpha^\mu(\partial/\partial x^\mu)\psi = 0$ is equivalent to the O(5)-covariant equation

$$(S_{ab}L_{ab}+2)\hat{\psi} = 0. \tag{4.6}$$

In the presence of gauge fields, Eq. (4.6) becomes

$$[(S_{ab})_{ij}\delta_{mn}L_{ab} - 2(S_{ab})_{ij}r_a\hat{A}_b^k T_{mn}^k + 2\delta_{mn}\delta_{ij}]\hat{\psi}_{jn} = 0, \tag{4.7}$$

where the matrices T_{mn}^i are the infinitesimal generators of the internal-symmetry group in the representation chosen for the fermions, and we have also put into evidence the spinor indices of $\hat{\psi}$. The equation may now be analyzed by expanding $\hat{\psi}$ in terms of O(5) harmonics appropriate to conservation of J_{ab}.

As an example of the computational simplification that our formalism effects, we consider the special case of fermions transforming as isospinors in the two SU(2)'s of the reduction O(4) = SU(2) × SU(2). We choose therefore for the fermions the 4-dimensional representation of O(4), where the infinitesimal generators are represented by the matrices $(\Sigma^{\mu\nu})_{mn}$. We notice then that the Fermi fields also span a representation of O(5), obtained simply by enlarging the set of infinitesimal generators to the matrices $(\Sigma_{ab})_{mn}$. We are thus led to consider the equation

$$[(S_{ab})_{ij}L_{ab}\delta_{mn} - 2(S_{ab})_{ij}r_a(\Sigma_{bc})_{mn}r_c + 2\delta_{mn}\delta_{ij}]\hat{\psi}_{jn} = 0, \qquad (4.8)$$

which is obtained from Eq. (4.7) by substituting for $(1/i)\hat{A}_b^k T_{mn}^k$ the O(5)-invariant expression of Eq. (3.12a).

Equation (4.8) is the equation of motion for a Dirac field in the potential of the pseudoparticle, after a projection has been made from Euclidean space onto the surface of the hypersphere and a gauge transformation has been used to express the gauge fields in a convenient form. By means of an inverse gauge rotation and projection, it is always possible to transform a solution of Eq. (4.8) into a solution of the Dirac equation in Euclidean space with the SU(2) × SU(2) gauge group.

To analyze Eq. (4.8), it is convenient to perform a unitary transformation acting on internal-symmetry indices

$$\hat{\psi}_m \to \Psi_m = U_{mn}\hat{\psi}_n, \quad U_{mn} = (i\alpha_1\alpha_3)_{mn}, \qquad (4.9a)$$

with

$$U^\dagger \Sigma_{ab} U = -(\Sigma_{ab})^{\mathrm{tr}}. \qquad (4.9b)$$

The transformed Dirac equation becomes

$$[(S_{ab})_{ij}L_{ab}\delta_{mn} + 2(S_{ab})_{ij}r_a(\Sigma_{bc})_{nm}r_c + 2\delta_{ij}\delta_{mn}]\Psi_{jn} = 0, \qquad (4.10a)$$

or, more compactly,

$$S_{ab}L_{ab}\Psi + 2r_aS_{ab}\Psi S_{bc}r_c + 2\Psi = 0, \qquad (4.10b)$$

where we consider Ψ to be a 4 × 4 matrix, and since $\Sigma_{ab} = S_{ab}$ it is no longer necessary to distinguish between these two matrices in order to keep track of spin and isospin indices.[11]

The left-hand side of Eq. (4.10) is of the form $\mathfrak{D}\Psi$, where $\mathfrak{D}$ is a linear Hermitian operator with respect to the inner product $\int d\Omega\, \mathrm{Tr}\Psi_1^\dagger\Psi_2$. In the rest of this section we shall find eigenvalues and eigenfunctions of this operator, as well as its inverse, which gives the fermion propagator in the field of the pseudoparticle. The relevance of these quantities for an analysis of the field-theoretical implications of the pseudoparticle solution is obvious.[12]

It is important to notice that $\mathfrak{D}$ has definite symmetry properties under two unitary transformations of the field Ψ. These two transformations, which we shall call chiral transformation and chiral gauge transformation, are obtained by left-multiplying and right-multiplying the fields with the unitary matrix $\Gamma_a r_a$:

$$U_C:\ \Psi \to \Psi' = \Gamma_a r_a \Psi, \qquad (4.11a)$$

$$U_{CG}:\ \Psi \to \Psi' = \Psi\Gamma_a r_a. \qquad (4.11b)$$

[The chiral transformation U_C is the projection over the hypersphere of the chiral transformation $\psi \to \psi' = \alpha_5\psi$. The chiral gauge transformation is a useful addition symmetry of the system. It can be viewed as a gauge transformation that leaves the gauge fields unchanged, but to realize U_{CG} as a gauge transformation one must consider the gauge group O(5) further embedded into an SU(4) gauge group generated by the 15 matrices Σ_{ab} and $\frac{1}{2}\Gamma_a$.] The transformation properties of $\mathfrak{D}$ are the following:

$$U_C\mathfrak{D}U_C^{-1} = -\mathfrak{D}, \qquad (4.12a)$$

$$U_{CG}\mathfrak{D}U_{CG}^{-1} = \mathfrak{D}. \qquad (4.12b)$$

The eigenvalue equation which we solve is

$$\mathfrak{D}\Psi = \mu\Psi. \qquad (4.13)$$

Note that U_C takes a solution of (4.13) into another solution with μ replaced by $-\mu$; this symmetry is analogous to Fermi-number conjugation. U_{CG} takes a solution of (4.13) into another solution with μ unchanged. Thus we expect to find doubly degenerate solutions both for $\mu > 0$ and $\mu < 0$, and a self-conjugate solution for $\mu = 0$.[13]

To solve (4.13) we expand Ψ into a complete set of 4 × 4 matrices,

$$\Psi = AI + B_a\Gamma_a + C_{ab}S_{ab}, \qquad (4.14)$$

separate from B_a and C_{ab} components parallel and perpendicular to r^a, and obtain a set of coupled first-order differential equations, which converts easily to a set of uncoupled second-order differential equations. These equations are trivial to solve since they are simply free wave equations on the hypersphere, involving the wave operator L^2 whose eigenvectors are O(5) harmonics with eigenvalues $2n(n+3)$, $n = 0, 1, \ldots$.[7,8]

In this way the following solutions are found: $\mu = 0$,

$$\Psi_{(1)} = \tfrac{1}{2}Y_0 = \frac{1}{4\pi}\left(\tfrac{3}{2}\right)^{1/2}, \qquad (4.15a)$$

$$\Psi_{(2)} = \Psi_{(1)}\Gamma_a r_a,$$

$\mu^2 = n(n+3), \quad n = 1, 2, \ldots,$

$$\Psi_{(1)} = \frac{1}{\sqrt{5}}\left(1 + \frac{1}{\mu} S_{ab} L_{ab}\right) Y_n, \tag{4.15b}$$

$$\Psi_{(2)} = \Psi_{(1)} \Gamma_a r_a,$$

$\mu^2 = (n+1)(n+2), \quad n = 0, 1, \ldots,$

$$\Psi_{(1)} = \frac{S_{cd}}{\sqrt{2}}\left[\left(1 - \frac{1}{\mu}\right) r_c + \frac{i}{\mu} r_a L_{ac}\right] \mathcal{Y}_{dn}, \tag{4.15c}$$

$$\Psi_{(2)} = \Psi_{(1)} \Gamma_a r_a,$$

where Y_n and $\mathcal{Y}_{an}$ are scalar and vector spherical harmonics. Their properties, already given by Adler,[7] are as follows. Each is an eigenfunction of L^2 with eigenvalue $2n(n+3)$. Y_n spans the $(n, 0)$ representation of O(5) with dimensionality $\frac{1}{6}(n+1) \times (n+2)(2n+3)$. [There is a "magnetic" label m, which we have suppressed, ranging from 1 to $\frac{1}{6}(n+1)(n+2)(2n+3)$.] The normalization is

$$\int d\Omega\, Y^*_{nm} Y_{n'm'} = \delta_{nn'} \delta_{mm'}.$$

Y_n may be constructed from

$$Y_n = r_{a_1} \cdots r_{a_n} M^{a_1 \cdots a_n},$$

where M is a constant, totally symmetric, and traceless tensor. $\mathcal{Y}_{an}$ is a vector harmonic, spanning the $(n, 1)$ representation of O(5) with dimensionality $\frac{1}{2}n(n+3)(2n+3)$. It satisfies the subsidiary conditions $r_a \mathcal{Y}_{an} = 0$, $iL_{ab}\mathcal{Y}_{bn} = \mathcal{Y}_{an}$, and is normalized by

$$\int d\Omega\, \mathcal{Y}^*_{anm} \mathcal{Y}_{an'm'} = \delta_{nn'} \delta_{mm'}.$$

One may construct $\mathcal{Y}_{an}$ from the overcomplete set of functions $(2L^2\delta_{ac} + 2L_{ab}L_{bc} + i6L_{ac})(u_l)_c Y_n$; $(u_l)_a$ is a constant unit vector with components δ_{la}.

The eigensolutions (4.15) are orthogonal to each other, and normalized to $\int d\Omega\, \mathrm{Tr}\Psi^\dagger\Psi = 1$. When nonvanishing, μ takes on positive and negative values of equal magnitude. There are the expected zero-eigenvalue solutions which can be arranged into eigenstates of $\Gamma^a r_a$. Evidently they also solve the Dirac equation (4.10).

We turn our attention now to the propagator associated with the operator $\mathfrak{D}$. It is obvious, from the existence of zero eigenvalues, that $\mathfrak{D}^{-1}$ does not exist. The propagator G should be rather defined through the equation

$$\mathfrak{D} G = I - P_0, \tag{4.16}$$

where I is the identity, and P_0 projects onto the space of eigenfunctions with $\mu = 0$. To make the expression of the propagator more explicit, we notice that it will be a matrix with indices i_1 (Dirac degrees of freedom), n_1 (internal degrees of freedom) referring to the initial configuration, and similarly i_2, n_2 for the final configuration. Furthermore, G depends on two position vectors r_1 and r_2. The symmetry properties of $\mathfrak{D}$ under chiral and chiral gauge transformations restrict G. It can be easily shown that these restrictions are satisfied when G is of the form

$$G = P^+_{\mathrm{C}} P^+_{\mathrm{CG}} \mathcal{G} P^-_{\mathrm{C}} P^+_{\mathrm{CG}} - P^+_{\mathrm{C}} P^-_{\mathrm{CG}} \mathcal{G} P^-_{\mathrm{C}} P^-_{\mathrm{CG}} + P^-_{\mathrm{C}} P^+_{\mathrm{CG}} \mathcal{G} P^+_{\mathrm{C}} P^+_{\mathrm{CG}} - P^-_{\mathrm{C}} P^-_{\mathrm{CG}} \mathcal{G} P^+_{\mathrm{C}} P^-_{\mathrm{CG}}, \tag{4.17}$$

$$P^\pm = \frac{1 \pm \Gamma_a r_a}{2}.$$

$P^\pm_{\mathrm{C}}$ ($P^\pm_{\mathrm{CG}}$) are the projection operators on the subspaces of positive and negative chirality (gauge chirality). The notation in (4.17) is schematic; matrix multiplication is not indicated. The projection operators standing to the left (right) of $\mathcal{G}$ refer to the first (second) configuration and are functions of r_1 (r_2). $P^\pm_{\mathrm{C}}$ ($P^\pm_{\mathrm{CG}}$) act on the spin (internal) indices. Further, $P^\pm_{\mathrm{C}}$ ($P^\pm_{\mathrm{CG}}$) of the first configuration act on the left (right), while those of the second configuration act on the right (left). For example, the first term on the right-hand side of (4.17) is explicitly

$$G^{(1)}_{i_1 n_1; n_2 i_2} = \tfrac{1}{2}(1 + \Gamma_a r_1^a)_{i_1 j_1} \tfrac{1}{2}(1 + \Gamma_a r_1^a)_{m_1 n_1} \mathcal{G}_{j_1 m_1; m_2 j_2} \times \tfrac{1}{2}(1 + \Gamma_a r_2^a)_{n_2 m_2} \tfrac{1}{2}(1 - \Gamma_a r_2^a)_{j_2 i_2}.$$

$\mathcal{G}$ can be expanded as

$$\mathcal{G}_{i_1 n_1; n_2 i_2}(r_1, r_2) = f(r_1^a r_2^a)(\delta_{i_1 n_1} r_1^b \Gamma^b_{n_2 i_2} + r_2^b \Gamma^b_{i_1 n_1} \delta_{n_2 i_2}) + ig(r_1^a r_2^a)(S^{ab}_{i_1 n_1} r_2^b \Gamma^a_{n_2 i_2} - \Gamma^a_{i_1 n_1} r_1^b S^{ab}_{n_2 i_2}), \tag{4.18}$$

with f and g to be determined. The form of G given by (4.17) and (4.18) satisfies a crossing property

$$G_{i_1 n_1; n_2 i_2}(r_1, r_2) = G^*_{i_2 n_2; n_1 i_1}(r_2, r_1). \tag{4.19}$$

In principle, f and g could be found from the eigenfunctions of $\mathfrak{D}$, Eqs. (4.15), but it is simpler to determine the propagator directly from (4.16). Inserting into this equation the expansion provided by (4.17) and (4.18), we obtain four equations for f and g

$$\begin{aligned} &(x^2 - 1)f' + 4xf = -\frac{3}{16\pi^2}, \\ &(x - 1)g' + 2g = 0, \\ &-f' + \tfrac{1}{2}xg' + \tfrac{3}{2}g = 0, \\ &f + \frac{x^2 - 1}{2}g' + \tfrac{3}{2}xg = 0, \end{aligned} \tag{4.20a}$$

where we have set $x = r_1^a r_2^a$. These equations are of course compatible and fix f and g completely:

$$\begin{aligned} f &= \frac{1}{16\pi^2}\frac{2-x}{(1-x)^2} \\ &= \frac{1}{4\pi^2}\frac{1}{|r_1-r_2|^4}+\frac{1}{8\pi^2}\frac{1}{|r_1-r_2|^2}, \\ g &= \frac{1}{8\pi^2}\frac{1}{(1-x)^2} \\ &= \frac{1}{2\pi^2}\frac{1}{|r_1-r_2|^4}. \end{aligned} \tag{4.20b}$$

Since we have thus satisfied (4.16), G is determined up to solutions of the homogeneous equation $\mathfrak{D}G_0=0$, that is, up to terms proportional to the zero-eigenvalue solutions (4.15a). Our propagator is the unique function which is orthogonal to these.

V. CONCLUSION

The Yang-Mills pseudoparticle is distinguished by its topological properties which give it a nonvanishing Pontryagin index q; this in turn makes it relevant to questions of anomalous nonconservation of symmetries. We have here demonstrated that the pseudoparticle is further distinguished by possessing a large kinematical invariance group, possibly important in future developments of the theory. We have already put into evidence the computational simplification that the O(5) formalism affords in analyzing the Dirac equation, which displays a peculiar zero-frequency mode, previously encountered in the spectrum of fermions in topologically interesting external fields, and having novel physical consequences.[5,11,12] Moreover, the O(5) invariance may possess further implications—in connection with spontaneous breakdown of space-time symmetries, as Fubini[8] has recently proposed.

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