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Soliton solutions of the massive Thirring model and the inverse scattering transform*

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Classical solutions of the massive Thirring model are found that describe the fermion constituent solitons of multisoliton solutions of the sine-Gordon equation. The construction of these fermion solutions is based on the inverse scattering transform for the sine-Gordon equation. It is shown that the equations of the inverse scattering formalism transform into the equations of the massive Thirring model through Coleman's correspondences between the two theories. The charge structure associated with multisoliton solutions is made very explicit by these correspondences. The 1-soliton, 2-soliton, and breather cases are discussed in detail.

I. INTRODUCTION

The sine-Gordon (SG) equation is an example of a nonlinear field theory that possesses particle-like solutions, the solitons and the breather, and can be solved exactly by the inverse scattering transform (IST) method.[1,2] The IST provides a powerful way of solving a wide class of nonlinear equations whereby the nonlinear equation is replaced by an equivalent linear problem.[1-4] Through the inverse scattering formalism a 2-component Dirac-type spinor is associated with each "constituent" soliton of a multisoliton solution of the SG equation. On the other hand, the quantum SG field corresponds to the fermion of the massive Thirring model.[5] One can ask then whether the IST formalism is in any way related to the massive Thirring model. In this paper we establish such a relationship and show that Coleman's correspondences[5] between the two theories can be formulated on the classical level in a natural way. The spinor associated with a "constituent" soliton of a multisoliton solution of the SG equation by the inverse scattering transform can be interpreted as the fermion wave function of that soliton in the presence of all the other solitons. Using Coleman's correspondences the equations of the IST formalism, for each fermion wave function, become the equations of motion of the massive Thirring model. A fermion field operator $\psi(x)$ can be defined within a given multisoliton sector. The charge structure and the Pauli principle associated with the corresponding multisoliton solution of the SG equation are thereby brought out in a very transparent way. The expectation values of all the bilinear covariants of ψ in the multisoliton state become sums over the individual "constituent" fermion contributions.

In Sec. II we formulate the IST method in a way that allows us to make contact with the massive Thirring model and show how to construct the solutions of the corresponding IST equations by means of the Bäcklund transformation. We discuss in detail the solutions of these equations for the 1-soliton, 2-soliton, and breather solutions of the SG equation. In Sec. III we discuss the correspondence of the IST with the massive Thirring model and explicitly construct the fermion wave functions for the 1-soliton, 2-soliton, and breather solutions, and establish Coleman's correspondences. In particular, the 1-soliton solution of the massive Thirring model turns out to be the same as the one previously found,[6,7] and we prove a uniqueness property for it. In the Appendix we discuss some aspects of the IST and the SG equation.

II. THE INVERSE SCATTERING METHOD

The SG equation in light-cone coordinates $x = (x^1 - x^0)/2$, $t = (x^1 + x^0)/2$ takes the form

$$\partial_x \partial_t \phi = \sin\phi . \tag{2.1}$$

The defining equations of the associated inverse scattering transform are[1]

$$\partial_x \chi_1 = \frac{a}{2}\chi_2 - \frac{i\phi_x}{2}\chi_1 , \tag{2.2a}$$

$$\partial_x \chi_2 = \frac{a}{2}\chi_1 + \frac{i\phi_x}{2}\chi_2 , \tag{2.2b}$$

$$\partial_t \chi_1 = \frac{1}{2a} e^{-i\phi}\chi_2 , \tag{2.2c}$$

$$\partial_t \chi_2 = \frac{1}{2a} e^{i\phi}\chi_1 , \tag{2.2d}$$

where we employ the notation $\phi_x = \partial_x \phi$. Equations (2.2a) and (2.2b) can be considered as a linear eigenvalue problem in x for any given t, where the parameter a is the eigenvalue and the spinor

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

is the corresponding eigenstate; the spectrum of a's depends on the "potential" ϕ_x. The integrability condition of the system of Eqs. (2.2) (i.e., $\partial_x\partial_t\chi = \partial_t\partial_x\chi$) plus the requirement that the eigenvalues a be t-independent implies the SG equation (2.1). We shall also employ the notation

$$\chi = \chi_a = \begin{pmatrix} \chi_{a1} \\ \chi_{a2} \end{pmatrix}$$

to show the dependence on the parameter a. We observe that χ_a transforms as a scalar under Lorentz transformations and not as a spin-$\frac{1}{2}$ spinor. The procedure by which $\phi = \phi(x, t)$ can be constructed from Eqs. (2.2) is described briefly in the Appendix. Here, we are rather interested in the physical interpretation of the solutions χ_a for every a. An arbitrary solution ϕ can have, in general, three kinds of eigenvalues a associated with it: (i) a finite number of real eigenvalues, say, a_i ($i = 1, \ldots, n$) corresponding to the soliton content of the solution, (ii) a finite number of complex pairs of eigenvalues, say, a_i, a_i^* ($i = 1, \ldots, m$) corresponding to the breather content, and (iii) a continuous range of imaginary eigenvalues, $a = ik$ (k real) corresponding to the so-called "radiation" part of ϕ. Asymptotically, as $t \to \infty$ the radiation part dies out and ϕ splits into n individual solitons moving with velocities $v_i = (a_i - a_i^{-1})/(a_i + a_i^{-1})$ ($i = 1, \ldots, n$) and into m individual breathers with velocities $v_i = (|a_i| - |a_i|^{-1})/(|a_i| + |a_i|^{-1})$ ($i = 1, \ldots, m$). In this paper we shall be interested only in multisoliton-multibreather solutions ϕ, i.e., solutions with no radiation part. Closed forms for such solutions have been given in the literature.[8] The IST equations are intimately related to the Bäcklund transformation[4] for the SG equation. This connection is brought out by defining[1] a function $\hat{\phi}$, for a given a and

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

through the equation

$$\frac{\chi_2}{\chi_1} = e^{i(\phi+\hat{\phi})/2} . \tag{2.3}$$

It follows, then, from Eqs. (2.2) that $\hat{\phi}$ is also a solution of the SG equation and ϕ is obtainable from $\hat{\phi}$ by the application of the Bäcklund transformation with parameter a; that is,

$$\begin{aligned} \phi_x &= \hat{\phi}_x + 2a \sin\left(\frac{\phi+\hat{\phi}}{2}\right), \\ \phi_t &= -\hat{\phi}_t + 2a^{-1} \sin\left(\frac{\phi-\hat{\phi}}{2}\right). \end{aligned} \tag{2.4}$$

Symbolically

$$\phi = B_a \hat{\phi} . \tag{2.5}$$

The meaning of this transformation is that, from a given solution $\hat{\phi}$, it defines a new solution ϕ that contains an additional soliton (with parameter a). It can be interpreted as a "linear superposition" of $\hat{\phi}$ with this soliton. Multisoliton solutions are obtained from the "vacuum" solution $\phi_0 \equiv 0$ by successive applications of B_a, e.g., $\phi = B_{a_1} \cdots B_{a_n} \times \phi_0$. The inverse of B_a is $B_a^{-1} = B_{-a}$.

We shall use Hirota's method[9] of constructing the solutions χ to the IST, Eqs. (2). This method utilizes a new form of the Bäcklund transformation and applies to solutions ϕ which are expressible in the form

$$\phi(x, t) = 4 \tan^{-1}\left(\frac{g(x, t)}{f(x, t)}\right), \tag{2.6}$$

where a sufficient (but not necessary) condition for ϕ to be a solution of (2.1) is that g and f satisfy

$$\begin{aligned} &fg_{xt} + gf_{xt} - f_x g_t - f_t g_x = 2gf, \\ &gg_{xt} - g_x g_t = ff_{xt} - f_x f_t . \end{aligned} \tag{2.7}$$

The Bäcklund transformation expressed in terms of g and f is defined by[9]

$$(f \pm ig)\overleftrightarrow{\partial}_x(\hat{f} \pm i\hat{g}) = \frac{a}{2}(f \mp ig)(\hat{f} \mp i\hat{g}), \tag{2.8a}$$

$$(f \pm ig)\overleftrightarrow{\partial}_t(\hat{f} \mp i\hat{g}) = \frac{a^{-1}}{2}(f \mp ig)(\hat{f} \pm i\hat{g}), \tag{2.8b}$$

where $F\overleftrightarrow{\partial}_x G \equiv FG_x - GF_x$. The $\hat{f}$ and $\hat{g}$ defined by (2.8) do satisfy Eqs. (2.7), and $\hat{\phi} = 4\tan^{-1}(\hat{g}/\hat{f})$ is related to ϕ by the usual Bäcklund transformation expressed by (2.4) and (2.5). The solution

$$\chi_a = \begin{pmatrix} \chi_{a1} \\ \chi_{a2} \end{pmatrix}$$

of the IST equations (2.2) can be expressed[9] now very simply in terms of f, g, $\hat{f}$, $\hat{g}$ satisfying (2.8) as

$$\chi_{a1} = \frac{\hat{f} - i\hat{g}}{f + ig}, \quad \chi_{a2} = \frac{\hat{f} + i\hat{g}}{f - ig} . \tag{2.9}$$

That these functions satisfy Eqs. (2.2) is shown in the Appendix.

The preceding formulation of the Bäcklund transformation can be applied to all multisoliton-multibreather solutions. We now use this formalism to construct the solutions to the IST for the 1-soliton and 2-soliton solutions of the SG equation.

A. The 1-soliton solution

This solution is characterized only by one velocity parameter a and is given by

$$\begin{aligned}
&\phi_a = 4\tan^{-1}\left(\frac{g_a}{f_a}\right),\\
&g_a = e^{\theta_a/2}, \quad f_a = e^{-\theta_a/2},\\
&\theta_a \equiv ax + a^{-1}t = \gamma_a(x^1 - v_a x^0),\\
&v_a = \frac{a-a^{-1}}{a+a^{-1}}, \quad \gamma_a = \frac{a+a^{-1}}{2}.
\end{aligned} \tag{2.10}$$

According to (2.9), in order to construct the corresponding χ_a solution to the IST, we must undo the Bäcklund transformation to find $\hat{g}$, $\hat{f}$, and $\hat{\phi} = B_a^{-1}\phi$. But since $\phi_a = B_a\phi_0$ with $\phi_0 \equiv 0$ it follows that $\hat{\phi} = \phi_0 = 0$, and for $\hat{f}$, $\hat{g}$ we can choose $\hat{f} = 1$, $\hat{g} = 0$. Then, the two components of χ_a are

$$\chi_{a1} = (f_a + ig_a)^{-1}, \quad \chi_{a2} = (f_a - ig_a)^{-1}. \tag{2.11}$$

We also find the following relations:

$$\begin{aligned}
&a\chi_{a2}^{\dagger}\chi_{a2} = \tfrac{1}{4}\partial_x\phi_a, \quad a^{-1}\chi_{a1}^{\dagger}\chi_{a1} = \tfrac{1}{4}\partial_t\phi_a,\\
&-i\chi_{a1}^{\dagger}\chi_{a2} = \tfrac{1}{4}(1 - e^{i\phi_a}).
\end{aligned} \tag{2.12}$$

These will be used in Sec. III to establish Coleman's correspondences. The parameter a can be positive or negative, corresponding respectively to the soliton or the antisoliton; it can also be complex, leading to a complex-valued solution ϕ_a; such solutions are used to build the soliton-antisoliton bound state, i.e., the breather. Complex conjugation defines a different solution, i.e., $(\phi_a)^* = \phi_{a^*}$; therefore, in order to have quantities referring to the same eigenvalue in the left-hand sides of (2.12), we must define in this case $\chi_{a1}^{\dagger} = (\chi_{a^*1})^*$, $\chi_{a2}^{\dagger} = (\chi_{a^*2})^*$.

B. The 2-soliton solution

This solution is characterized by two real eigenvalues a and b and is built from $\phi_0 \equiv 0$ by the Bäcklund transformations

$$\phi = B_a B_b \phi_0 . \tag{2.13}$$

When a and b have opposite signs this solution contains two solitons (or two antisolitons), and it contains a soliton and an antisoliton when a and b have the same sign. This remark implies a certain Pauli exclusion principle associated with the multisoliton solutions of the SG equation: Two solitons cannot be put together if they have the same velocity. In fact, choosing $b = -a$ in (2.13) we obtain

$$\phi = B_a B_{-a}\phi_0 = B_a B_a^{-1}\phi_0 = \phi_0 = 0 .$$

For definiteness we may assume that $a^2 > b^2$ (this is a Lorentz-invariant statement). We find for the functions g and f

$$\begin{aligned}
&\phi = 4\tan^{-1}\left(\frac{g}{f}\right),\\
&g = \frac{2}{\kappa}(a+b)\sinh\left(\frac{\theta_a - \theta_b}{2}\right),\\
&f = \frac{2}{\kappa}(a-b)\cosh\left(\frac{\theta_a + \theta_b}{2}\right),\\
&\theta_a = ax + a^{-1}t, \quad \theta_b = bx + b^{-1}t, \quad \kappa \equiv (a^2 - b^2)^{1/2}.
\end{aligned} \tag{2.14}$$

We choose g, f so that (2.8) are satisfied (see Appendix). To construct the two solutions χ_a and χ_b of the IST equations (2.2) we need to find the corresponding $\hat{\phi}$, $\hat{g}$, and $\hat{f}$. For χ_a we must undo, according to (2.9), the Bäcklund transformation B_a, and we obtain $\hat{\phi} = B_a^{-1}\phi = B_b\phi_0 = \phi_b$, i.e., the 1-soliton solution with parameter b, and we have already found the corresponding $\hat{f}$, $\hat{g}$. Similarly, for the χ_b we have $\hat{\phi} = B_b^{-1}\phi = B_a\phi_0 = \phi_a$. We have then

$$\begin{aligned}
&\chi_{a1} = \frac{f_b - ig_b}{f + ig}, \quad \chi_{a2} = \frac{f_b + ig_b}{f - ig},\\
&\chi_{b1} = \frac{f_a - ig_a}{f + ig}, \quad \chi_{b2} = \frac{f_a + ig_a}{f - ig},
\end{aligned} \tag{2.15}$$

where f_a and g_a are given by (2.10), and similarly for f_b and g_b with the replacement $\theta_a \to \theta_b$. It is straightforward (see Appendix) to show the relations analogous to (2.12), namely

$$\begin{aligned}
&a\chi_{a2}^{\dagger}\chi_{a2} - b\chi_{b2}^{\dagger}\chi_{b2} = \tfrac{1}{4}\partial_x\phi,\\
&a^{-1}\chi_{a1}^{\dagger}\chi_{a1} - b^{-1}\chi_{b1}^{\dagger}\chi_{b1} = \tfrac{1}{4}\partial_t\phi,\\
&-i\chi_{a1}^{\dagger}\chi_{a2} + i\chi_{b1}^{\dagger}\chi_{b2} = \tfrac{1}{4}(1 - e^{i\phi}).
\end{aligned} \tag{2.16}$$

Asymptotically, as $x^0 \to \pm\infty$ the right-hand sides split into the sum of individual single-soliton terms, which are matched term by term by the left-hand sides. For example, since $a^2 > b^2$, the b soliton is going to be left behind by the a soliton as $x^0 \to +\infty$. Moving along with the b soliton ($x^1 = v_b x^0$) corresponds then to the limit $\theta_b = 0$, $\theta_a \to -\infty$ and we get in this case

$$\begin{aligned}
&a\chi_{a2}^{\dagger}\chi_{a2} \to 0,\\
&b\chi_{b2}^{\dagger}\chi_{b2} \to \frac{b(a^2 - b^2)}{8(a^2 + b^2)},
\end{aligned}$$

and

$$\tfrac{1}{4}\phi_x \to -\frac{b(a^2 - b^2)}{8(a^2 + b^2)} .$$

By choosing a complex and $b = a^*$ we can also describe by the same equations (2.15) and (2.16) the solutions χ for the breather which is defined by $\phi = B_a B_{a*}\phi_0$.

We have now analyzed the solutions to the IST problem sufficiently to turn to the connections with the massive Thirring model.

III. CORRESPONDENCE WITH THE MASSIVE THIRRING MODEL

This model is defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \tfrac{1}{2} g J_\mu^2 ,$$
$$J_\mu = \bar{\psi}\gamma_\mu \psi$$

with equations of motion

$$i\gamma^\mu \partial_\mu \psi = m\psi - g J_\mu \gamma^\mu \psi . \tag{3.1}$$

Since we are interested in the classical equations it is convenient for our purposes to choose $m = \frac{1}{2}$, $g = 1$. Then, in the basis $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, Eqs. (3.1) become, for

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

the following:

$$\begin{aligned} -i\partial_x \psi_1 &= \tfrac{1}{2}\psi_2 - 2\psi_2^\dagger \psi_2 \psi_1 , \\ i\partial_t \psi_2 &= \tfrac{1}{2}\psi_1 - 2\psi_1^\dagger \psi_1 \psi_2 . \end{aligned} \tag{3.2}$$

Coleman's correspondences[5] between the SG theory and the massive Thirring model can be expressed as

$$\begin{aligned} &:\psi_2^\dagger \psi_2: \to \tfrac{1}{4}\partial_x \phi , \\ &:\psi_1^\dagger \psi_1: \to \tfrac{1}{4}\partial_t \phi , \\ &:\psi_1^\dagger \psi_2: \to \tfrac{1}{4}(1 - e^{i\phi}) . \end{aligned} \tag{3.3}$$

If we naively substituted these correspondences into the equation of motion (3.2) we would obtain

$$\begin{aligned} -i\partial_x \psi_1 &= \tfrac{1}{2}\psi_2 - \tfrac{1}{2}\phi_x \psi_1 , \\ i\partial_t \psi_2 &= \tfrac{1}{2}\psi_1 - \tfrac{1}{2}(1 - e^{i\phi})\psi_1 \\ &= \tfrac{1}{2} e^{i\phi}\psi_1 . \end{aligned}$$

These equations look very much like Eqs. (2.2a) and (2.2d) of the IST problem. To make this connection more precise, we consider the IST equations for some multisoliton solution ϕ of the SG equation, and for every eigenvalue a and eigenstate χ_a we define a new function

$$\psi_a = \begin{pmatrix} \psi_{a1} \\ \psi_{a2} \end{pmatrix}$$

by

$$\begin{aligned} \psi_{a1} &= i a^{-1/2} \chi_{a1} , \\ \psi_{a2} &= a^{1/2} \chi_{a2} , \end{aligned} \tag{3.4}$$

where we define $a^{1/2} = i(-a)^{1/2}$ if a is negative; in general, if $a = |a| e^{i\theta}$ we choose $a^{1/2} = |a|^{1/2} e^{i\theta/2}$. Then, (2.2a) and (2.2d) become

$$\begin{aligned} -i\partial_x \psi_{a1} &= \tfrac{1}{2}\psi_{a2} - \tfrac{1}{2}\phi_x \psi_{a1} , \\ i\partial_t \psi_{a2} &= \tfrac{1}{2}\psi_{a1} - \tfrac{1}{2}(1 - e^{i\phi})\psi_{a1} . \end{aligned} \tag{3.5}$$

The spinor $\psi_a(x,t)$ transforms as a spin-$\frac{1}{2}$ object under the Lorentz group. This follows from the definition (3.4) and the fact that χ_a transforms as a scalar. To find the connection of (3.5) with the massive Thirring model let us consider first the 1-soliton solution for ϕ. The relations in (2.12) now become

$$\begin{aligned} \epsilon_a \psi_{a2}^\dagger \psi_{a2} &= \tfrac{1}{4}\partial_x \phi_a , \\ \epsilon_a \psi_{a1}^\dagger \psi_{a1} &= \tfrac{1}{4}\partial_t \phi_a , \\ \epsilon_a \psi_{a1}^\dagger \psi_{a2} &= \tfrac{1}{4}(1 - e^{i\phi_a}) , \end{aligned} \tag{3.6}$$

where $\epsilon_a = +1$ or $\epsilon_a = -1$ respectively as $a > 0$ or $a < 0$, i.e., for a soliton or an antisoliton. These are Coleman's correspondences, and their substitution into (3.5) leads to the equations of motion (3.2) for the massive Thirring model. This solution coincides with the one found previously.[7] Using (2.10) and (2.11) it can be written in the form of Ref. 7:

$$\begin{aligned} \psi_{a1} &= i a^{-1/2}\left(\tfrac{1}{2}\sin\frac{\phi_a}{2}\right)^{1/2} e^{-i\phi_a/4} , \\ \psi_{a2} &= a^{1/2}\left(\tfrac{1}{2}\sin\frac{\phi_a}{2}\right)^{1/2} e^{i\phi_a/4} . \end{aligned} \tag{3.7}$$

This solution is unique up to a constant gauge transformation in the sense that if it satisfies (i) the equation of motion and (ii) the correspondences (3.6) with some solution ϕ of the SG equation, then ϕ must necessarily be the 1-soliton solution and ψ must be as given above. What really restricts ψ to the 1-soliton sector are the correspondences (3.6). To prove this uniqueness we require the correspondence equations $\psi_2^\dagger \psi_2 = \phi_x/4$, $\psi_1^\dagger \psi_1 = \phi_t/4$, and $\psi_1^\dagger \psi_2 = (1 - e^{i\phi})/4$, where ϕ satisfies the SG equation, and, anticipating the answer, we look for solutions of the form $\psi_1 = i\rho_1 e^{i\omega_1 - i\phi/4}$, $\psi_2 = \rho_2 e^{i\omega_2 + i\phi/4}$, where $\rho_1 = |\psi_1|$, $\rho_2 = |\psi_2|$, and ω_1, ω_2 are to be determined. Now from $\psi_1^\dagger \psi_2 = (1 - e^{i\phi})/4$ it follows that

$$\rho_1 \rho_2 = \tfrac{1}{2}\sin(\phi/2)$$

and

$$\omega_2(x,t) = \omega_1(x,t) + 2\pi n(x,t) ,$$

where $n(x,t)$ is an integer and can be set equal to zero since it would not contribute to ψ_2. From the other correspondences we have $\rho_1^2 \rho_2^2 = \frac{1}{16}\phi_x \phi_t$. Thus, $\phi_x \phi_t = 4\sin^2(\phi/2)$. This equation plus the

SG equation imply that ϕ_x/ϕ_t is a constant. In fact, differentiation with respect to x gives

$$\begin{aligned}\phi_t\phi_{xx}+\phi_x\phi_{xt} &= 4\phi_x \sin\frac{\phi}{2}\cos\frac{\phi}{2}\\ &= 2\phi_x \sin\phi\\ &= 2\phi_x\phi_{xt}\ .\end{aligned}$$

Thus $\phi_x\phi_{xt}=\phi_t\phi_{xx}$, implying $\partial_x(\phi_x/\phi_t)=0$ and similarly with respect to t. We set $\phi_x = a^2\phi_t$ (a^2 must be positive from the correspondences). This can be expressed simply as $\phi_\eta = 0$ in terms of new variables $\xi = ax + a^{-1}t$, $\eta = ax - a^{-1}t$. Then $\phi_x = a\phi_\xi$, $\phi_t = a^{-1}\phi_\xi$, and $\phi_\xi{}^2 = \phi_x\phi_t = 4\sin^2(\phi/2)$. The solution to this equation is nothing but the 1-soliton solution ϕ_a. Thus, we establish that ϕ must be the 1-soliton solution by using only the correspondences. The equations of motion for ψ imply, in addition, that ω_1 must be a constant independent of x and t. Thus, ψ must be given by (3.7) up to a constant gauge transformation.

Next, we consider the 2-soliton solution ϕ given by Eqs. (2.14). Then, with the definition (3.4), the relations (2.16) become

$$\begin{aligned}&\epsilon_a\psi^\dagger_{a2}\psi_{a2}+\epsilon_b\psi^\dagger_{b2}\psi_{b2}=\tfrac{1}{4}\partial_x\phi\ ,\\ &\epsilon_a\psi^\dagger_{a1}\psi_{a1}+\epsilon_b\psi^\dagger_{b1}\psi_{b1}=\tfrac{1}{4}\partial_t\phi\ ,\\ &\epsilon_a\psi^\dagger_{a1}\psi_{a2}+\epsilon_b\psi^\dagger_{b1}\psi_{b2}=\tfrac{1}{4}(1-e^{i\phi})\ ,\end{aligned}\tag{3.8}$$

where $\epsilon_a = \pm 1$ depending on whether the a constituent is a soliton or an antisoliton, and similarly for the b constituent.[10] Thus, the charge structure of the 2-soliton solution ϕ is made explicit through the above correspondence. Substitution of (3.8) into (3.5) leads to a classical formulation of the massive Thirring model where $\psi_a(x)$ can be interpreted as the fermion wave function describing the a-constituent soliton in the presence of the other. Each constituent fermion interacts with the sum of the currents of all constituents. We obtain

$$\begin{aligned}-i\partial_x\psi_{a1}&=\tfrac{1}{2}\psi_{a2}-2\Big(\sum_{c=a,b}\epsilon_c\psi^\dagger_{c2}\psi_{c2}\Big)\psi_{a1}\ ,\\ i\partial_t\psi_{a2}&=\tfrac{1}{2}\psi_{a1}-2\Big(\sum_{c=a,b}\epsilon_c\psi^\dagger_{c1}\psi_{c2}\Big)\psi_{a1}\end{aligned}\tag{3.9}$$

and similar equations for the ψ_b wave functions. There is a certain asymmetric treatment of these two equations. One might have expected the interaction term in the second of Eqs. (3.9) to be of the form

$$\Big(\sum_{c=a,b}\psi^\dagger_{c1}\psi_{c1}\Big)\psi_{a2}\ ,$$

as was suggested for example in Ref. 6. However, the above formulation is necessary if one wants to have the correspondence between the massive Thirring model and the SG theory. These equations reflect a certain normal-ordering prescription of the interaction terms in the field equations (3.2). To see this, we define a fermion field operator acting in the 2-fermion sector by

$$\psi(x)=c_a\psi_a(x)+c_b\psi_b(x)$$

if a and b are both solitons and by

$$\psi(x)=c_a\psi_a(x)+d_b^\dagger\psi_b(x)$$

if a is a soliton and b is an antisoliton. The c_a and d_b are annihilation operators for solitons and antisolitons satisfying canonical anticommutation relations. Then, by taking the expectation values of the normal-ordered bilinear covariants of ψ in the 2-fermion state, i.e. in $c_a^\dagger c_b^\dagger|0\rangle$ or $c_a^\dagger d_b^\dagger|0\rangle$, we can rewrite (3.8) as

$$\begin{aligned}\sum_{c=a,b}\epsilon_c\overline{\psi}_c\gamma^\mu\psi_c&=\langle:\overline{\psi}\gamma^\mu\psi:\rangle\\ &=\tfrac{1}{2}\epsilon^{\mu\nu}\partial_\nu\phi\ ,\\ \sum_{c=a,b}\epsilon_c\overline{\psi}_c\Big(\frac{1+\gamma_5}{2}\Big)\psi_c&=\Big\langle:\overline{\psi}\Big(\frac{1+\gamma_5}{2}\Big)\psi:\Big\rangle\\ &=\tfrac{1}{4}(1-e^{i\phi})\ ,\end{aligned}\tag{3.10}$$

where $\epsilon^{01}=1$. These are Coleman's correspondences. If we rewrite the field equations (3.2) with the normal-ordering prescription

$$\begin{aligned}-i\partial_x\psi_1&=\tfrac{1}{2}\psi_2-2:\psi_2^\dagger\psi_2:\psi_1\ ,\\ i\partial_t\psi_2&=\tfrac{1}{2}\psi_1-2:\psi_1^\dagger\psi_2:\psi_1\ ,\end{aligned}$$

then to obtain (3.9) we replace the normal-ordered factors by their expectation values given above, i.e., $:\psi_2^\dagger\psi_2:\to\langle:\psi_2^\dagger\psi_2:\rangle$, $:\psi_1^\dagger\psi_2:\to\langle:\psi_1^\dagger\psi_2:\rangle$. This procedure represents a self-consistent Hartree-Fock type of approximation which linearizes the equations of motion. Our explicit construction shows how self-consistency is established through the correspondences (3.10).

IV. CONCLUSION

We have shown how to formulate the classical correspondence between the SG equation and the massive Thirring model through the inverse scattering transform. The fermion character of the soliton is thus clarified and provides an explanation for the charge structure and the Pauli principle associated with multisoliton solutions of the SG equation. Although we have considered only the 1- and 2-soliton solutions, the generalization to higher-multisoliton solutions is fairly obvious and we expect again the correspondences

$$\sum_{i=1}^{n} \epsilon_{a_i}\bar{\psi}_{a_i}\gamma^\mu\psi_{a_i} = \tfrac{1}{2}\epsilon^{\mu\nu}\partial_\nu\phi ,$$

$$\sum_{i=1}^{n} \epsilon_{a_i}\bar{\psi}_{a_i}\left(\frac{1+\gamma_5}{2}\right)\psi_{a_i} = \tfrac{1}{4}(1-e^{i\phi})$$

for an n-soliton solution ϕ. Other nonlinear field theories which are soluble by means of the IST with Dirac-type equations might correspond to some fermion field theory in much the same way.

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APPENDIX

In this appendix we consider some aspects of the inverse scattering method. A complete discussion can be found in Ref. 1.

(a) In matrix form Eqs. (2.2) are expressed as

$$\chi_x = M\chi \equiv \tfrac{1}{2}(a\sigma_1 - i\phi_x\sigma_3)\chi , \tag{A1}$$

$$\chi_t = N\chi \equiv \frac{a^{-1}}{2}(\cos\phi\sigma_1 + \sin\phi\sigma_2)\chi . \tag{A2}$$

Differentiation of the first with respect to t and of the second with respect to x leads to the integrability condition

$$M_t - N_x + [M,N] = -i\sigma_3(\phi_{xt} - \sin\phi) = 0 . \tag{A3}$$

Thus, the SG equation implies the consistency of the system (A1), (A2). Usually,[1] the IST equations are formulated in terms of the variables v_1, v_2, and ζ given by

$$\chi_1 = v_1 + iv_2, \quad \chi_2 = v_1 - iv_2, \quad \zeta = ia/2 \tag{A4}$$

and by replacing $\phi \to -\phi$ in (2.2).

The procedure by which $\phi(x,t)$ is constructed via the IST is as follows: (i) Given the initial data $\phi(x,0)$ the eigenvalue problem in (A1) is solved, thereby determining the corresponding range of eigenvalues a and the corresponding scattering data (e.g. reflection coefficients). The scattering data are determined by the asymptotic behavior of χ as $x \to \pm\infty$. The function $\phi(x,0)$ uniquely determines the scattering data and can be reconstructed from them. (ii) The t equations (A2) are used next to find the time development of the scattering data. (iii) Finally, from the knowledge of the scattering data at time t the function $\phi(x,t)$ can be reconstructed. Solitons correspond to "bound states" of the scattering problem (A1), while the radiation part of ϕ corresponds to the "scattering states." The real advantage of this method lies in the fact that only the asymptotic values (as $x \to \pm\infty$) of ϕ and of the solutions χ are needed, so that the t equations (A2) lead to very simple equations for the t evolution of the scattering data.

(b) Next, we illustrate how the functions χ given by (2.9) satisfy the IST equations (2.2). For example, using (2.8) we obtain

$$\begin{aligned}\partial_t \chi_2 &= \partial_t\left(\frac{\hat{f}+i\hat{g}}{f-ig}\right)\\ &= \frac{(f-ig)\overleftrightarrow{\partial}_t(\hat{f}+i\hat{g})}{(f-ig)^2}\\ &= \frac{a^{-1}}{2}\frac{f+ig}{(f-ig)^2}(\hat{f}-i\hat{g})\\ &= \frac{a^{-1}}{2}\left(\frac{f+ig}{f-ig}\right)^2\chi_1\\ &= \frac{a^{-1}}{2}e^{i\phi}\chi_1 .\end{aligned}$$

(c) Here, we demonstrate that the functions g and f of Eqs. (2.14) for the 2-soliton ϕ satisfy Eqs. (2.8) for the corresponding $\hat{f}$'s and $\hat{g}$'s, so that χ_a and χ_b are indeed solutions of the IST equations. For example, we must have

$$(f-ig)\overleftrightarrow{\partial}_t(f_b + ig_b) = \frac{1}{2a}(f+ig)(f_b - ig_b) \tag{A5}$$

with f, g, f_b, g_b given by (2.14) and (2.10). Equation (A5) is equivalent to

$$gg_{bt} + ff_{bt} - g_t g_b - f_t f_b = \frac{1}{2a}(ff_b + gg_b) , \tag{A6}$$

$$fg_{bt} - gf_{bt} - f_t g_b + g_t f_b = \frac{1}{2a}(gf_b - fg_b) . \tag{A7}$$

Since

$$g_{bt} = \frac{1}{2b}g_b = \frac{1}{2b}e^{\theta_b/2}, \quad f_{bt} = \frac{-1}{2b}f_b = \frac{-1}{2b}e^{-\theta_b/2} ,$$

Eq. (A6) becomes

$$f_t f_b + g_t g_b = \frac{a-b}{2ab}gg_b - \frac{a+b}{2ab}ff_b . \tag{A8}$$

Since

$$f_t = \frac{\kappa}{2ab}\left[e^{(\theta_a+\theta_b)/2} - e^{-(\theta_a+\theta_b)/2}\right] ,$$

$$g_t = -\frac{\kappa}{2ab}\left[e^{(\theta_a-\theta_b)/2} + e^{(\theta_b-\theta_a)/2}\right]$$

we assert that

$$f_t f_b + g_t g_b = -\frac{\kappa}{ab}e^{-\theta_a/2}\cosh\theta_b ,$$

$$\frac{a-b}{2ab}gg_b - \frac{a+b}{2ab}ff_b = -\frac{\kappa}{ab}e^{-\theta_a/2}\cosh\theta_b ,$$

which verifies (A8). The rest of these equations are shown in a similar way.

(d) Here we illustrate Eqs. (2.16), which form the basis of Coleman's correspondences. Since the 2-soliton solution is given by $\phi = B_a B_b \phi_0$ we have

$$\tan\left(\frac{\phi - \phi_0}{4}\right) = \frac{a+b}{a-b}\tan\left(\frac{\phi_a - \phi_b}{4}\right), \tag{A9}$$

where $\phi_a = B_a\phi_0$, $\phi_b = B_b\phi_0$, $\phi_0 \equiv 0$. This is the basic equation for Bäcklund transformations.[4] From it we obtain

$$\frac{\phi_x}{4} = \frac{a+b}{a-b}\left(\frac{\phi_{ax}}{4} - \frac{\phi_{bx}}{4}\right)\frac{(1+T_a^2)(1+T_b^2)}{(1+T^2)(1+T_a T_b)^2}, \tag{A10}$$

where $T \equiv g/f = \tan(\phi/4)$ and $T_c \equiv g_c/f_c = \tan(\phi_c/4)$ $(c = a, b)$. Using (2.10) and (2.14) we can write

$$f = f_a f_b \frac{a-b}{\kappa}(1 + T_a T_b)$$

and this implies

$$\frac{a+b}{a-b}\frac{1}{(1+T^2)(1+T_a T_b)^2} = \frac{f_a^2 f_b^2}{f^2+g^2}. \tag{A11}$$

From Eqs. (2.10) we have also

$$\begin{aligned}&\frac{\phi_{ax}}{4}(1+T_a^2) = aT_a ,\\ &T_a f_a^2 = g_a f_a = 1 .\end{aligned} \tag{A12}$$

Then, (A11) and (A12) imply

$$\begin{aligned}\left(\frac{a+b}{a-b}\right)\frac{\phi_{ax}}{4}\frac{(1+T_a^2)(1+T_b^2)}{(1+T^2)(1+T_a T_b)^2} &= a\frac{f_b^2+g_b^2}{f^2+g^2}\\ &= a\chi_{a2}^\dagger\chi_{a2}\end{aligned}$$

and a similar equation for $\phi_{ax} \to \phi_{bx}$ and $\chi_a \to \chi_b$. Finally, (A10) becomes

$$\tfrac{1}{4}\phi_x = a\chi_{a2}^\dagger\chi_{a2} - b\chi_{b2}^\dagger\chi_{b2} .$$

The rest of Eqs. (2.16) can be shown in a similar way.

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