Gauge fields on the null plane*

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The theory of gauge fields interacting with fermions is quantized on the null plane. The singularities of the gauge field propagator are related to the infrared divergences of the theory and cancel in the gauge-invariant sector. Ultraviolet divergences are due only to the high-transverse-momentum behavior and in the gauge-invariant sector are eliminated by coupling-constant renormalization. Equivalence to the covariant-gauge formulation is proved by explicitly performing the gauge transformation. Gauge-invariant ultraviolet and infrared cutoffs which preserve unitarity are used wherever necessary. Gauge-variant amplitudes are infrared divergent and non-Lorentz-invariant.

I. INTRODUCTION

The axial gauges,[1] defined by constraining one component of the gauge field to vanish, occupy a special place in that unitarity is manifest and no ghost, negative metric, or spurious degrees of freedom need be introduced into the quantization scheme. The null-plane gauge and the associated infinite-momentum-frame quantization offer the extra advantage (characteristic of this frame[2]) that the exact ground state of the field system is the bare Fock vacuum of the canonical quanta. Thus, if a viable nonperturbative approximation scheme is ever devised for this gauge, vacuum problems will be absent.

The formulation of perturbation theory in the null-plane gauge, besides being nonmanifestly covariant, is fraught with singularities. Specifically, the gauge-field free propagators are singular at $p\cdot n=0$, where n is the lightlike vector which defines the quantization null plane. Moreover, the counterterms needed to eliminate the ultraviolet divergences are in general noncovariant and singular. The purpose of what follows is to establish the legitimacy of the null-plane gauge in renormalized perturbation theory. This is achieved by using Hamiltonian (old-fashioned) perturbation theory, which is found to be more convenient than the Lagrangian version in analyzing the origin and cancellation of the singularities. Indeed, the propagator singularities are shown to have a simple Hamiltonian interpretation in that they are due to the divergence of certain matrix elements of the Hamiltonian. These divergences are canceled when the external states are restricted to be gauge-invariant. Once this has been achieved, the analysis of the ultraviolet behavior is straightforward and leads to the same results achieved by using the covariant gauges. In Sec. II the quantization scheme is reviewed in some detail. Section III deals with the classification and cancellation of the field propagator singularities. In Sec. IV the ultraviolet structure and renormalization are discussed. Finally, the proof of equivalence to the conventional covariant formulation is sketched in Sec. V.

II. NULL-PLANE-GAUGE QUANTIZATION

The null-plane gauge and the associated null-plane (or infinite-momentum) frame[2] are manifestly noncovariant, and it will prove convenient to use a notation which reflects this property. The preferred direction will be the three-axis and we designate

$$\tfrac{1}{4}(x_0+x_3)=\tau\,,\quad x_3-x_0=\zeta,\quad (x_1,x_2)=\vec{\mathrm{R}}, \tag{1}$$

$$\begin{aligned}&2(p_0-p_3)=\omega=i\partial_\tau\,,\\ &\tfrac{1}{2}(p_0+p_3)=\eta=\frac{1}{i}\partial_\zeta,\quad (p_1,p_2)=\vec{\mathrm{p}}=\frac{1}{i}\nabla\,.\end{aligned} \tag{2}$$

Four-vector components and scalar products are

$$a_+=\tfrac{1}{2}(a_0+a_3)\,,\quad a_-=2(a_0-a_3)\,,\quad \vec{\mathrm{A}}=(a_1,a_2)\,, \tag{3}$$

$$2a\cdot b=a_+b_-+a_-b_+-2\vec{\mathrm{A}}\cdot\vec{\mathrm{B}}\,. \tag{4}$$

Note also that owing to the various factors of 2,

$$dx_0dx_3=2d\tau d\zeta\,,\quad dp_0dp_3=\tfrac{1}{2}d\omega d\eta\,. \tag{5}$$

The system to be considered is that of a massless gauge field ϕ_μ^a interacting with a Dirac field ψ. The action is

$$I=\int d^4x[-\tfrac{1}{4}F_{\mu\nu}{}^2+\bar{\psi}(i\partial\!\!\!/-g\phi\!\!\!/^a t^a-M)\psi]\,. \tag{6}$$

The field strengths are defined by

$$F_{\mu\nu}^a=\partial_\mu\phi_\nu^a-\partial_\nu\phi_\mu^a+ig\phi_\mu^b T^{bac}\phi_\nu^c\,. \tag{7}$$

The matrices t^a satisfy the gauge-group commutation rules

$$[t^a,t^b]=-T^{abc}t^c\,, \tag{8}$$

where the structure constants T^{abc} are totally antisymmetric and imaginary. Covariant derivatives will be designated by

$$D_\mu = \partial_\mu + ig\mathcal{T}^a\phi^a_\mu \,, \tag{9}$$

where $\mathcal{T}^a$ is either t^a or $(T^a)^{bc}$. In particular, the transverse part of D_μ is

$$\vec{\mathrm{D}} = \nabla - ig\mathcal{T}^a\vec{\phi}^a \,. \tag{10}$$

As will be seen, it is natural to use a particular representation of the Dirac matrices, in which α_3 is diagonal:

$$\alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1,2 \tag{11}$$

$$\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix}. \tag{12}$$

The null-plane gauge is specified by restricting the path-integral configurations to

$$\phi^a_+ = \tfrac{1}{2}(\phi_0 + \phi_3)^a = 0 \,. \tag{13}$$

While the action (6) is invariant under the full gauge group defined by the infinitesimal transformations

$$\begin{aligned} &\delta\phi_\mu = -D_\mu\omega(x) \,, \\ &\delta\psi = -i\omega^a(x)t^a\psi \,, \quad \delta\bar{\psi} = i\bar{\psi}t^a\omega^a(x) \,, \end{aligned} \tag{14}$$

the gauge condition (13) is violated by (14) and we have

$$\delta_\omega \phi^a_+ = \partial_\zeta \omega^a(\tau, \vec{\mathrm{R}}, \zeta) \,. \tag{15}$$

As is well known, the field independence of the gauge-condition variation (15) eliminates the necessity for a Feynman-Faddeev-Popov (FFP)[3,4] determinant and its associated ghost loops. Introducing the field component ϕ^a_-

$$\phi^a_- = 2(\phi_0 - \phi_3)^a \tag{16}$$

and the fermion gauge-charge density

$$\rho^a = \chi^\dagger t^a \chi = \bar{\psi}\gamma_+ t^a \psi \,, \tag{17}$$

the action assumes the form

$$\begin{aligned} I(\phi_+ = 0) = \int d\tau\, d\vec{\mathrm{R}}\, d\zeta \Big\{ &\tfrac{1}{4}(\partial_\zeta \phi_-)^2 - \phi_-(g\rho + \vec{\mathrm{D}}\cdot\partial_\zeta\vec{\phi}) - 2\partial_\tau\vec{\phi}\cdot\partial_\zeta\vec{\phi} \\ &- F_{12}{}^2 + \chi^\dagger i\partial_\tau \chi + 4\xi^\dagger \frac{1}{i}\partial_\zeta \xi - 2\left[\xi^\dagger\left(\frac{1}{i}\vec{\mathrm{D}}\cdot\vec{\sigma} + iM\right)\chi + \mathrm{H.c.}\right]\Big\}. \end{aligned} \tag{18}$$

The quadratic part of the gauge-field action is now an invertible 3×3 form and the momentum-space gauge-field propagators are

$$\begin{aligned} &\Delta_{--} = \frac{1}{k^2 + i0}\frac{4\omega}{\eta} \,, \\ &j = 1,2: \;\; \Delta_{-j} = \frac{1}{k^2 + i0}\frac{2K_j}{\eta} \,, \\ &\Delta_{jk} = \frac{1}{k^2 + i0}\delta_{jk} \,. \end{aligned} \tag{19}$$

The Fermi field propagator is the usual $(\not{p} - M + i0)^{-1}$, and in the representation (11) and (12) has the matrix form

$$\begin{aligned} S &= (\not{p} - M + i0)^{-1}\gamma_0 \\ &= \frac{1}{p^2 - M^2 + i0}\begin{pmatrix} 2\eta & \vec{\mathrm{P}}\cdot\vec{\sigma} - iM \\ \vec{\mathrm{P}}\cdot\vec{\sigma} + iM & \tfrac{1}{2}\omega \end{pmatrix}. \end{aligned} \tag{20}$$

The interaction vertices defined by the action (18) are the conventional gauge theory ones. Note that since the field ϕ^a_+ is decoupled, its emission vertex Γ^a_- is missing, though it is still definable in terms of the fields $(\phi_-, \vec{\phi}, \psi, \psi^\dagger)$. The gauge-field propagators $(\Delta_{--}, \Delta_{-j})$ are singular at $\eta \to 0$. The presence of this singularity dictates the null plane $\tau = 0$ as the quantization hypersurface, with τ playing the role of development parameter:

$$\text{“time”} = \tau = \tfrac{1}{4}(x_0 + x_3) \,, \tag{21}$$

$$\text{“energy”} = \omega = 2(p_0 - p_3) \,. \tag{22}$$

With this choice, η is a spatial momentum (conjugate to the longitudinal coordinate ζ) so that a singular behavior at $\eta \to 0$ does not influence unitarity and no spurious propagating degrees of freedom are introduced. The unitarity singularities are determined by the denominators $(k^2 + i0)$ and $(p^2 - M^2 + i0)$ and indicate that the spectrum is that of positive-η–positive-ω excitations whose mass shell is defined by

$$\text{gauge field:} \quad \omega(\vec{\mathrm{K}}, \eta) = \frac{K^2}{\eta} \,, \tag{23}$$

$$\text{Fermi field:} \quad \omega(\vec{\mathrm{P}}, \eta, M) = \frac{P^2 + M^2}{\eta} \,. \tag{24}$$

In order to derive the τ-development operator (“Hamiltonian”) the propagators (19) and (20) should be diagonalized and the coupling to the eigenvectors expressed in terms of the state variables $(\vec{\mathrm{K}}, \eta)$. This is easily achieved by rewriting the explicit ω's in the numerators of Δ_{--} and $S_{\xi\xi^\dagger}$

as $(1/\eta)(k^2+K^2)$ and $(1/\eta)(p^2-M^2+P^2+M^2)$, respectively. The terms proportional to k^2 and (p^2-M^2) cancel the denominators and their exchange induces extra four-point vertices into the interaction Lagrangian $\mathcal{L}_I$. No higher-order contact terms are induced since the relevant fields ϕ_- and ξ couple only singly in the action (18). In fact, the propagators assume the forms

$$(\mu=-j):\quad \Delta_{\mu\nu}=\frac{1}{k^2+i0}(e^{\parallel}_{\mu}e^{\parallel}_{\nu}+e^{\perp}_{\mu}e^{\perp}_{\nu})+\frac{1}{\eta^2}e^{-}_{\mu}e^{-}_{\nu}\,, \tag{25}$$

where

$$e^{\parallel}=\left(2\frac{K}{\eta},\hat{K}\right),\quad e^{\perp}=(0,\hat{e})\,,\quad e^{-}=(2,0)\,, \tag{26}$$

$$\hat{K}=\frac{\vec{\mathrm{K}}}{\mathrm{K}}\,,\quad \hat{e}\cdot\hat{K}=0\,,\quad \hat{e}^2=1\,, \tag{27}$$

$$S=\sum_{\sigma=1,2}U_{\sigma}(\vec{\mathrm{P}},\eta,M)\frac{1}{p^2-M^2+i0}U^{\dagger}_{\sigma}(\vec{\mathrm{P}},\eta,M)+\frac{1}{2\eta}\begin{pmatrix}0&0\\0&2\end{pmatrix}, \tag{28}$$

where U_σ are the solutions of the on-shell Dirac equation. The Hamiltonian is now defined by subtracting the modified $\mathcal{L}_I$ from $\mathcal{H}_0$. A completely equivalent procedure, which is simpler, is to go back to the action (18) and quantize it by the usual canonical rules, regarding τ as the time parameter. The Lagrange equations supply two equal-τ constraint relations which define ϕ_- and $(\xi,\xi^\dagger)$ in terms of the canonical variables $(\vec{\phi},\chi,\chi^\dagger)$:

$$-\partial_\zeta{}^2\phi_-=2(g\rho+\vec{\mathrm{D}}\cdot\partial_\zeta\vec{\phi})\,, \tag{29}$$

$$2\partial_\zeta\xi=(\vec{\mathrm{D}}\cdot\vec{\sigma}-M)\chi \text{ and H.c.} \tag{30}$$

The canonical conjugates of $\vec{\phi}$ and χ are the coefficients of $\partial_\tau\vec{\phi}$ and $\partial_\tau\chi$ in I, and supply the equal-τ commutation rules

$$[\partial_\zeta\phi^a_j(\zeta,\vec{\mathrm{R}})\,,\phi^b_l(\zeta',\vec{\mathrm{R}}')]=\tfrac{1}{2}i\delta(\zeta-\zeta')\delta(\vec{\mathrm{R}}-\vec{\mathrm{R}}')\delta^{ab}\delta_{jl}\,, \tag{31}$$

$$\{\chi(\zeta,\vec{\mathrm{R}})\,,\chi^\dagger(\zeta',\vec{\mathrm{R}}')\}=\delta(\zeta-\zeta')\delta(\vec{\mathrm{R}}-\vec{\mathrm{R}}')\,, \tag{32}$$

$$\{\chi(\zeta,\vec{\mathrm{R}})\,,\chi(\zeta',\vec{\mathrm{R}}')\}=0\,. \tag{33}$$

The relations (31)–(33) can be realized by a Fourier decomposition which attaches annihilation (creation) operators to $e^{i\eta\zeta}$ ($e^{-i\eta\zeta}$) where $\eta>0$:

$$\vec{\phi}(\zeta)=\int_0^\infty\frac{d\eta}{(4\pi\eta)^{1/2}}[\vec{\mathrm{a}}(\eta)e^{i\eta\zeta}+\vec{\mathrm{a}}^\dagger(\eta)e^{-i\eta\zeta}]\,, \tag{34}$$

$$\chi(\zeta)=\int_0^\infty\frac{d\eta}{(2\pi)^{1/2}}[b(\eta)e^{i\eta\zeta}+d^\dagger(\eta)e^{-i\eta\zeta}]\,. \tag{35}$$

The $\vec{\mathrm{R}}$ dependence has been suppressed in (34) and (35). The Hamiltonian[5] $\mathcal{H}$ may now be read off the action by omitting the ∂_τ terms and reversing the sign,

$$\mathcal{H}=\int d\vec{\mathrm{R}}\,d\zeta\left[\tfrac{1}{4}(\partial_\zeta\phi_-)^2+(F_{12})^2+4\xi^\dagger\frac{1}{i}\partial_\zeta\xi\right]. \tag{36}$$

The operators (ϕ_-,F_{12},ξ) in Eq. (36) are defined as functions of the canonical degrees of freedom (31)–(33) by Eqs. (29), (7), and (30), respectively. When this has been done, it is straightforward to verify that $(1/i)[\chi,\mathcal{H}]$ and $(1/i)[\partial_\zeta\vec{\phi},\mathcal{H}]$ indeed reproduce the Lagrange equation for $\partial_\tau\chi$ and $\partial_\tau\partial_\zeta\vec{\phi}$. $\mathcal{H}$ describes an interaction between the transverse gauge field $\vec{\phi}$ and the (two-component) Fermi field χ whose free τ propagation is given by the mass-shell equations (23) and (24). The three- and four-point interaction vertices generated by $\mathcal{H}_I$ are depicted in Fig. 1 and their momentum-space expressions are listed in Eq. (37):

$$\frac{g}{\sqrt{\pi}}t^a\left[\frac{\vec{\mathrm{K}}}{\eta}-\frac{1}{2}\left(\frac{\vec{\mathrm{P}}_f\cdot\vec{\sigma}-iM}{\eta_f}\vec{\sigma}+\vec{\sigma}\frac{\vec{\sigma}\cdot\vec{\mathrm{P}}_i+iM}{\eta_i}\right)\right], \tag{37a}$$

$$\frac{g}{\sqrt{\pi}}T^{a_1a_2a_3}\frac{1}{2}\left\{\left[\frac{K_1}{\eta_1}(\eta_2-\eta_3)+(K_3-K_2)\right]_{j_1}\delta_{j_2j_3}+\text{cyc perm}\right\}, \tag{37b}$$

$$\frac{g^2}{\pi}\frac{t^a_{12}t^a_{34}}{(\eta_1-\eta_2)^2}\,, \tag{37c}$$

$$\frac{g^2}{\pi}T^{ba_1a_2}T^{ba_3a_4}\frac{(\eta_1-\eta_2)(\eta_3-\eta_4)}{4(\eta_1+\eta_2)^2}\delta_{j_1j_2}\delta_{j_3j_4}\,, \tag{37d}$$

$$\frac{g^2}{\pi}T^{ba_1a_2}t^b\frac{\eta_1-\eta_2}{2(\eta_1+\eta_2)^2}\delta_{j_1j_2}\,, \tag{37e}$$

$$\frac{g^2}{\pi}T^{ba_1a_2}T^{ba_3a_4}(\delta_{j_1j_3}\delta_{j_2j_4}-\delta_{j_1j_4}\delta_{j_2j_3})+\text{perm}\,, \tag{37f}$$

$$\frac{g^2}{4\pi}\left[\frac{t^{a_1}t^{a_2}}{\eta_f+\eta_1}\alpha_{j_1}\alpha_{j_2}+(1\leftrightarrow2)\right]. \tag{37g}$$

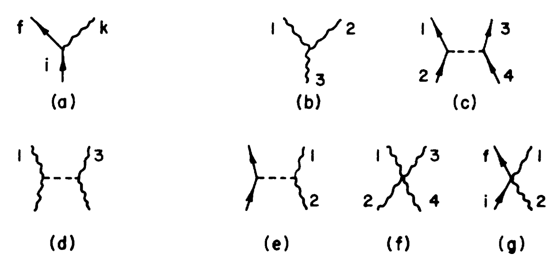

FIG. 1. Interaction vertices due to the Hamiltonian (36). Wavy lines are gauge fields and straight lines are fermions. The corresponding expressions are listed in Eq. (37).

In Eq. (37), all boson momenta are outgoing, while fermion momenta follow the Fermi arrow. Also, the boson states have been normalized to η by $a(\eta)\to\sqrt{\eta}\,a(\eta)$. The terms (37c)–(37e) are the contact terms induced by the e^- term in $\Delta_{\mu\nu}$ [Eq. (25)], while (37g) is the result of the equal-τ exchange of ξ [Eq. (28)]. Note that (37c)–(37e) are simply an η^{-2} exchange between the matrix elements of the gauge charge density J_+:

$$J_+^a=\chi^\dagger t^a\chi+\vec{\phi}^b\cdot T^{abc}\frac{1}{i}\partial_\xi\vec{\phi}^c\,. \tag{38}$$

This static exchange is the analog of the Coulomb interaction, and in fact describes a one-dimensional Coulomb potential $|\xi-\xi'|$ between two gauge charges occupying the same transverse point $\vec{R}$. The main features of the Hamiltonian defined by Eqs. (36) and (37) will now be briefly discussed.

1. $\mathcal{H}$ is explicitly invariant under the Lorentz transformations which leave the null plane invariant. As is well known[2] the latter comprise the following infinitesimal transformations:

$$\text{longitudinal boost: }\delta\eta=\lambda\eta\,,\quad \delta\omega=-\lambda\omega\,, \tag{39}$$

$$\text{two-dimensional rotations: }\delta R_j=\theta\epsilon_{jk}R_k\,, \tag{40}$$

$$\text{Galilei transformations: }\delta\vec{P}=\vec{V}\eta\,,\quad \delta\omega=2\vec{V}\cdot\vec{P}\,. \tag{41}$$

It is clear that $\mathcal{H}_0$, defined by Eqs. (23) and (24), is invariant under (39)–(41). The rotational invariance (40) holds for $\mathcal{H}_I$ when $\vec{\phi}$ and χ transform as a two-vector and spinor, respectively. The boost invariance (39) is verified by noting that if the fermion normalization in (37) is also changed to η all matrix elements depend only on ratios of η. Finally, the Galilean invariance is verified by observing that the momentum-dependent vertices (37a) and (37b) actually depend only on invariant velocity differences $(\vec{K}/\eta-\vec{K}'/\eta')$, etc.

2. The three-point couplings (37a) and (37b) are linear in the transverse momenta, while the four-point couplings are $\vec{K}$-independent. This fact will supply the basis for ultraviolet power counting.

3. Since all positive-energy (ω) excitations carry a positive longitudinal momentum (η), $\mathcal{H}_I$ does not contain pure creation terms and leaves the Fock vacuum of (a,b,d) invariant.[2] Moreover, by rewriting the ξ integration in Eq. (36) as an η integral over positive η, it may be verified that $\mathcal{H}$ is positive-definite (up to some delicate operator-ordering issues which will not be discussed).

4. The gauge-group current satisfies a conservation law which reads in momentum space

$$\omega J_+^a+\eta J_-^a-2\vec{K}\cdot\vec{J}^a=0\,, \tag{42}$$

where

$$J_\mu^a=\bar{\psi}\gamma_\mu t^a\psi+i\phi_\lambda^b T^{abc}F_{\lambda\mu}^c\,. \tag{43}$$

Recall that (ψ,ϕ,F) should be regarded as the appropriate functions of $(\chi,\chi^\dagger,\vec{\phi})$. The (+) component of Eq. (43) is J_+^a as defined in Eq. (38). As a consequence of Eq. (42) a Ward identity is derived,[1]

$$2k\cdot(J^a(k)\cdots A(q)\cdots)_+=i\sum_A(\cdots A^a(k+q)\cdots)_+\,, \tag{44}$$

where $(\,)_+$ is the τ-ordering symbol and A^a is the equal-τ commutator

$$A^a(\tau,\vec{Q}+\vec{K},\xi+\eta)=[J^a(\tau,\vec{K},\eta),A(\tau,\vec{Q},\xi)]\,. \tag{45}$$

$(\vec{K},\eta)$ and $(\vec{Q},\xi)$ are the (transverse, longitudinal) momenta carried by J and A, respectively.

5. $\mathcal{H}$ is invariant under the restricted gauge transformations defined by setting $\partial_\xi\omega^a=0$ in Eq. (14). These gauge transformations leave the null-plane (NP) gauge condition (15) invariant. Moreover, all $\eta\neq0$ canonical operators transform homogeneously since $\omega(\vec{R})$ contains no $\eta\neq0$ Fourier components. The conserved canonical generator of the restricted gauge group is

$$G^a(\vec{R})=\int d\xi[gJ_+^a(\vec{R},\xi)+\nabla\cdot\partial_\xi\vec{\phi}^a(\vec{R},\xi)]$$
$$=\tfrac{1}{2}(\eta^{-2}\phi_-^a)(\eta=0)\,. \tag{46}$$

6. $\mathcal{H}$ is a singular operator in that the $\eta\to0$ singularities of $\Delta_{\mu\nu}$ have been transformed into the matrix elements of $\mathcal{H}$. The "Coulomb" interaction (37c)–(37e) diverges as η^{-2} at $\eta\to0$. Moreover, the emission amplitude for an $\eta\to0$ vector meson polarized along $\vec{K}$ behaves as K/η [(37a) and (37b)]. In order to control these infrared (IR) divergences and define the action of $\mathcal{H}$ on the whole Hilbert space an IR cutoff will be introduced. We require the cutoff procedure to be Galilei-invariant and gauge-invariant, and preserve the equations of motion and canonical commutation rules. The key to satisfying these conditions is supplied by the observation that they all involve operations which are local in coordinate space. The simplest (probably the only) procedure is thus to enclose the system in a finite longitudinal box, $|\xi|<\tfrac{1}{2}L$, and impose periodic boundary conditions. The length L in fact drops out of the problem if all longitudinal momenta are rescaled by boosting the total (positive) momentum to 1. Thus, the IR cutoff is achieved by discretizing the η axis:

$$\eta_n = \frac{n}{N}, \quad n = 0, \pm 1, \ldots, \pm N. \tag{47}$$

The singular $n=0$ mode is eliminated by simply decoupling it *a priori*. Formally, this may be achieved by noting that the singular part of the Hamiltonian arises from the emission of the field $\phi_-(\eta=0)$, which in turn couples in the action (18) only to the conserved generator G. By a suitable choice of the wave function associated with $\nabla\cdot\vec{\phi}(\eta=0)$, $G(\vec{R})$ can always be made to vanish. Clearly the condition $G(\vec{R})=0$ is Galilei-invariant and commutes with $\mathcal{H}$. It should be emphasized that this procedure can yield a sensible theory only if it is applied to amplitudes which are IR-finite and contain no $\eta\to 0$ singularities.

7. Though not manifestly so, $\mathcal{H}$ is in fact Lorentz-invariant even under the null-plane rotating transformations, and generates a Lorentz-invariant perturbation expansion. Note, however, that the relevant transformations mix η and $\vec{K}$, and are therefore sensitive to the IR issue and violate the cutoff procedure. The only Lorentz-invariant amplitudes are those which are infrared-nonsingular. The Lorentz transformations are discussed in Sec. IV and Appendix B.

8. The perturbation expansion generated by $\mathcal{H}$ is ultraviolet- (UV-) divergent owing to the behavior at $\vec{K}\to\infty$. These divergences are logarithmic and will be controlled by introducing a transverse dimensional regularization[6]

$$d\vec{K} \to d^{2-\epsilon}K. \tag{48}$$

The UV behavior will be discussed in Sec. IV. We remark here only that (48) affects only the density of transverse-momentum states, so that Galilei invariance, equal-τ commutation rules, and the equations of motion, conservation laws, and Ward identities are not disturbed so long as the operators considered are not explicitly ϵ-dependent (the latter possibility actually occurs for the axial baryon current).

We conclude this section by recalling the main formulas of Hamiltonian perturbation theory. As already shown, all Feynman amplitudes are τ-ordered products of the external legs. All Fourier transforms of τ-ordered products are computable as matrix elements of and insertions into the resolvent

$$(\nu-\mathcal{H}+i0)^{-1} = (\nu-\mathcal{H}_0+i0)^{-1} + (\nu-\mathcal{H}_0+i0)^{-1}V(\nu-\mathcal{H}_0+i0)+\cdots, \tag{49}$$

where

$$V=\mathcal{H}_I+\Delta, \tag{50}$$

and Δ is the diagonal self-mass operator needed to make the spectra of $\mathcal{H}_0$ and $\mathcal{H}$ coincide. In particular, the absorptive part of a given amplitude is computed by calculating (50) to a given order, replacing each $(\nu-\mathcal{H}_0+i0)^{-1}$ in turn by $\delta(\nu-\mathcal{H}_0)$, changing $(\nu-\mathcal{H}_0+i0)^{-1}$ to $\mathcal{P}(\nu-\mathcal{H}_0)^{-1}$ and summing. Thus the Nth-order spectral function of the two-point function $\langle (A(\tau)B(0))_+\rangle_0$ is given by

$$R_{AB}^{(N)}(\nu) = \sum_{n=0}^{N} \left\langle A\left(\frac{\mathcal{P}}{\nu-\mathcal{H}_0}V\right)^n \delta(\nu-\mathcal{H}_0)\left(V\frac{\mathcal{P}}{\nu-\mathcal{H}_0}\right)^{N-n}B\right\rangle_0. \tag{51}$$

For any given set of intermediate states in (51) the factor (say) $[V/(\eta-\mathcal{H}_0+i0)]^n|B\rangle$ is the nth-order on-shell renormalized emission amplitude

$$\lim_{k^2,\,p^2-M^2\to 0} \prod(k^2 Z_3^{-1/2})\prod((p^2-M^2)Z_2^{-1/2}) \times\left\langle\left(\prod\vec{\phi}(k)\chi(p)B\right)_+\right\rangle_0^{(n)}. \tag{52}$$

Δ is also computable as a power series in $\mathcal{H}_I$. The most important feature of the latter for our purposes is that in the absence of degeneracy this power series contains only off-shell energy denominators [principal-value parts of $(\nu-\mathcal{H}_0)^{-1}$]. In the Feynman rules this corresponds to the irreducibility of the self-mass. Note that Eq. (51) is schematic in that the power series for Δ should be substituted in V in order to obtain an expansion in $\mathcal{H}_I$. We end by making the obvious remark that the wave-function renormalization constants (Z's) cancel out of Eq. (51) and its generalization to many external lines. Only external lines (A, B) renormalizations are necessary.

III. INFRARED STRUCTURE

The perturbative expansion represented by Eq. (51) contains in general four potential sources of infrared singularities:

(a) Owing to the masslessness of the vector field the off-shell energy denominators may cause degenerate-perturbation-theory divergences.

(b) The off-shell exchange of longitudinal ($\hat{K}$-polarization) vector lines and the static Coulomb exchange behaves as η^{-2} at $\eta\to 0$.

(c) The on-shell emission amplitudes of longitudinal vector fields are singular and behave as $\eta^{-1}K$ as $\eta\to 0$.

(d) The expression of the external operators [A and B in Eq. (51)] in terms of the canonical variables may contain explicit η^{-1} factors.

First, possibility (a) will be shown not to occur. To see this, consider an intermediate off-shell state which contains N "soft" vector lines and a set of other lines with total mass μ (Fig. 2). The

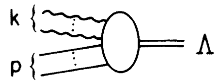

FIG. 2. Emission of a subsystem of mass μ and a set of soft gauge mesons.

Galilean invariance allows the total transverse momentum to be equated to 0. The energy denominator associated with this state is

$$(\nu-\mathcal{H}_0)^{-1}\to\left[\nu-\frac{\mu^2+(\sum\vec{K})^2}{1-\sum\eta}-\sum\frac{K^2}{\eta}\right]^{-1}. \qquad (53)$$

The phase-space element associated with this configuration is calculated in Appendix A and shown to behave at $\eta\to 0$ as

$$d\Omega[\mu,\{k\}]\underset{\eta\to 0}{\simeq} d\Omega[\mu]\prod\tfrac{1}{2}d\theta d\eta d\omega\,, \qquad (54)$$

where $\omega=K^2/\eta$ and θ is the polar angle associated with $\vec{K}$. The denominator (53) becomes in this limit

$$(\nu-\mathcal{H}_0)^{-1}\underset{\eta\to 0}{\sim}[\nu-\mu^2-\textstyle\sum\omega-(\sum\eta)(\nu-\sum\omega+\lambda)]^{-1}\,, \qquad (55)$$

where $\lambda(\{\omega,\eta/\sum\eta,\theta\})>0$. The degeneracy region is $\nu\to\mu^2$, which necessitates

$$\omega\to 0,\quad \eta\to 0\,, \qquad (56)$$

since (55) becomes under these conditions

$$(\nu-\mathcal{H}_0)^{-1}\underset{\nu\to\mu^2}{\sim}-[\mu^2\textstyle\sum\eta+\sum\omega]^{-1}. \qquad (57)$$

It is clear that the phase element (54) which contains $d\omega d\eta$ for every soft gluon eliminates the (logarithmic) divergence associated with the region (56). Moreover, the matrix elements for the emission and absorption of the soft mesons are all $O(1)$, $O(K^2/\eta^2)$, or $O(K)$ and nonsingular in the degeneracy region. Thus degenerate perturbation theory is irrelevant in the NP gauge. Next we consider the infrared behavior of an off-shell-exchanged longitudinal vector line. From Eqs. (37a) and (37b) we find the emission amplitude to be ($\omega=K^2/\eta$)

$$\eta\to 0:\ M_{fi}^{\parallel a}(\omega,\eta)\simeq\frac{g}{\sqrt{\pi}}\left(\frac{\omega}{\eta}\right)^{1/2}J^a_{+fi}(\omega,\eta)\,, \qquad (58)$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the emitter. Thus the exchange amplitude behaves as ($\eta\to 0^+$)

$$\frac{g^2}{\pi}J^a_{+(1)}(\omega,\eta)\frac{d\omega d\eta}{(\nu_1-\omega)\cdots(\nu_n-\omega)}\frac{\omega}{\eta}J^a_{+(2)}(\omega,\eta)\,, \qquad (59)$$

where $(\nu_1-\omega)\cdots(\nu_n-\omega)$ are the energy denominators of the (off-shell) state traversed by the exchanged lines (Fig. 3). It is clear that for $n\geq 2$ only the region $\omega=$ finite contributes to the IR divergence ($\eta\omega\to\infty$ is the UV region). Translated into the original $\vec{K}$ variable, the "IR corner" is

$$(K,\eta)\to 0\,,\ \frac{K^2}{\eta}\to\omega\neq 0\,. \qquad (60)$$

In this limit, $J^a_{+(2)}$ (say) is the conserved total global gauge-group charge Q associated with the state which stands to the right of the point of emission (2):

$$\lim_{(\vec{K},\eta)\to 0} J^a_+(\vec{K},\eta)=Q^a\,. \qquad (61)$$

Thus if the external operator(s) $\{B\}$ which produce this state are neutral under the gauge group, the IR-divergent contribution (59) will vanish. It should be emphasized that what is required is *not* just global neutrality which is a property associated with the (decoupled) point $\eta=0$, $\vec{K}=0$; rather, Eq. (61) should hold. This in turn requires

$$\lim_{(K,\eta)\to 0}[J^a_+(\vec{K},\eta),B]=0\,. \qquad (62)$$

Equation (62) should be satisfied in the tree-graph approximation without $\eta\to 0$ divergences (type d). Moreover, when moving the emission point (2) from the point B to the left, the state on which J_+ acts receives new perturbative contributions. Again, for Eq. (61) to be fulfilled it is necessary that the latter not cause the wave function to diverge.

We now turn to the case $n=1$ (Fig. 5). In this case the vector exchange has to be considered in conjunction with the topologically similar Coulomb exchange. Note that a Coulomb "self-energy" has been introduced [Fig. 5(d)]. In the Hamiltonian approach this corresponds to a particular choice of the operator ordering in the term $(\partial_\xi\phi_-)^2$ of Eq. (36). If Hamiltonian perturbation theory is derived through the Feynman rules, this term appears automatically and represents the contribution of the semicircle at infinity when ω integra-

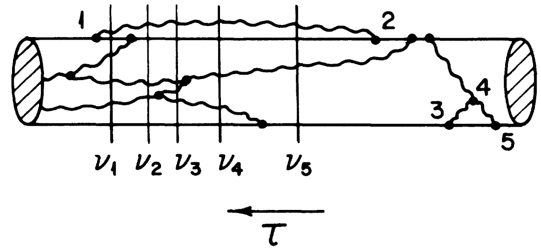

FIG. 3. An off-shell exchange of a soft gluon emitted at (2) and absorbed at (1). Dots represent $\mathcal{H}_I$. ($\nu_1\ldots\nu_5$) are the energy denominators traversed.

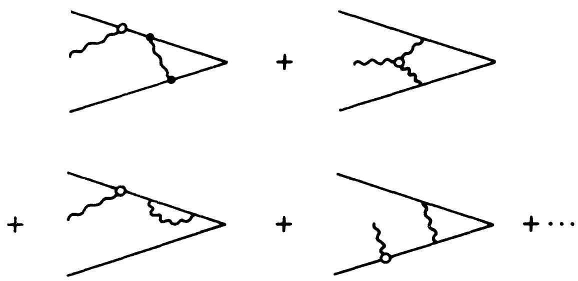

FIG. 4. Cancellation of on-shell IR divergences. The circled emission point occurs at the latest τ. The internal configurations have to be summed over all possibilities. Then the circled point is moved inside.

tions are performed. Combining the Coulomb and longitudinal vector exchange (which have the same source $-J_+^a$) leads to

$$\frac{1}{\eta^2}+\frac{K^2}{\eta^3(\nu-K^2/\eta)}=\frac{\nu}{\eta(\eta\nu-K^2)}. \qquad (63)$$

ν represents the energy difference due to the other lines. As in the previous argument, the integration over $\vec{K}$ and η is divided into two regions. The region $\eta\nu \ll K^2 < \infty$ now contributes to the wave-function renormalization graphs of Fig. 5(c) an expression of the form

$$\sim \nu \int_{0^+} \frac{d\eta}{\eta} \int_{K_1}^{K_2} \frac{d\vec{K}}{-K^2}. \qquad (64)$$

The divergence (64) is canceled, however, by the one-loop corrections to the near-"hard" gluon (namely $\eta' \gg \eta$) vertices attached to the line under consideration. [This cancellation is actually an instance of an identity between specific contributions to Z_1 and Z_2 (or Z_3).] The only subtlety in the verification involves the group factors where the simple identity $T^bT^aT^b = T^a(T^2-\frac{1}{2}t^2)$ is needed. The first term is used to cancel the divergence (64), while the second term cancels the contribution to the vertex correction due to the exchange of the soft gluon between the hard gluon and the original charged line. When the (off-shell) exchange takes place between two different internal lines, η takes both positive and negative values and we have

$$\text{Figs. 5(c)+5(d)} \underset{\substack{K^2\neq 0\\ \eta\to 0}}{\sim} \int_{-|\eta_-|}^{|\eta_+|} \frac{d\eta}{\eta} \int d\vec{K} J^a_{+f} \frac{\nu}{-K^2} J^a_{+i} < \infty. \qquad (65)$$

We are thus led back to the "IR corner" defined by Eq. (60). Now, however, the previous argument may be used to relate the IR singularity to the total incoming Q^a, which vanishes by assumption. Thus the off-shell ($\nu - \mathcal{H}_0 \neq 0$) part of the expansion is free of IR singularities provided the

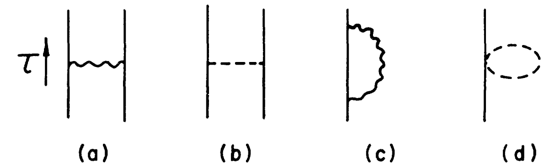

FIG. 5. Processes contributing to Coulomb (dashed lines) and adjacent-τ vector exchange.

external lines are neutral [in the sense of Eq. (62)]. Actually the above argument holds for one type of charged amplitude, namely the on-shell self-mass Δ. In the IR corner, the factor $\omega = K^2/\eta$ which multiplies J_+^a is equal [up to $O(K,\eta)$] to the Fourier frequency of J_+^a, since the rest of the momenta are changed only infinitesimally. We may thus use the equation $\dot{Q}^a=0$ to eliminate the off-shell IR divergence. Since Δ is irreducible and since there are no degenerate energy denominators we conclude that the self-mass is IR-finite and does not introduce singularities when substituted in the perturbation expansion through Eq. (50). The preceding discussion does not hold for the wave-function renormalization Z, which by definition includes on-shell contributions.

Having eliminated the off-shell IR singularities, we now turn to the on-shell divergences. Using the phase-space element (54), and the emission amplitude (58), the on-shell contribution of the intermediate state (Fig. 2) becomes

$$T_{\rm IR}^{\rm os} \sim J^a_{+f}\delta(\nu-\mu^2-\textstyle\sum\omega)d\Omega(\mu)\dfrac{\prod d\theta d\omega d\eta}{\eta}\,\omega J^a_{+i}. \qquad (66)$$

Clearly, energy conservation allows only finite ω's to contribute. Since the intermediate lines are on-shell, all τ orderings must be considered, and $J_+^a(\eta, K\to 0)$ cannot be set equal to zero. Rather, the Ward identity [Eq. (44)] must be used to eliminate ωJ_+^a. Since by assumption all external lines commute with $\lim_{\eta\to 0;\,K\to 0} J_+(\eta,K)$, the only nonvanishing contribution to the right-hand side of Eq. (44) comes from the equal-τ commutators of $J_+^a(\eta)$ and the on-shell annihilation operators in Eq. (52). If the latter is a fermion operator [say $b(\eta', \vec{K}')$]

$$\eta,\eta'>0:\ \ [J_+^a(\eta,\vec{K}), b(\eta',\vec{K}')] = -t^a b(\eta+\eta', \vec{K}+\vec{K}'). \qquad (67)$$

If the annihilation operator is $\vec{a}^b(\eta', \vec{K}')$

$$\eta,\eta'>0:\ \ [J_+^a(\eta,\vec{K}), \vec{a}^b(\eta',\vec{K}')] = -T^{abc}\left(1-\frac{\eta}{2(\eta+\eta')}\right)\vec{a}^c(\eta+\eta', \vec{K}+\vec{K}'). \qquad (68)$$

In Eq. (68) the a's are normalized covariantly

($[a, a^\dagger] = \eta\delta$) and the extra factor is due to the form of the expression (38) in momentum space. It is easy to verify that the mass-shell condition on $(k+k')^2$ becomes

$$\text{Fermi lines: } (\eta'\vec{K} - \eta\vec{K}')^2 + \eta^2 M^2 = 0\,, \qquad (69)$$

$$\text{Bose lines: } \eta'\vec{K} = \eta\vec{K}'\,. \qquad (70)$$

Equation (69) clearly has vanishing phase space, while Eq. (70) forces both k and k' into the IR corner:

$$\theta = \theta'\,, \quad (\eta, \eta') \to 0 \quad (\omega, \omega') \neq 0\,. \qquad (71)$$

In order to overcome the phase-space reduction due to (71) *both* vector mesons k and k' must be longitudinal, so that an extra ω'/η' will appear. Furthermore, Eq. (68) yields a wave function which is antisymmetric in group space (T^{abc}). Since the transverse momenta are parallel and the polarization equal (both $\hat{K}$-polarized), Bose symmetry forces an antisymmetry under either $(\omega \leftrightarrow \omega')$ or $(\eta \leftrightarrow \eta')$. Owing to Eqs. (70) and (71) this supplies (at least) one extra factor $(\eta - \eta') \to 0$, which eliminates the IR singularity

$$\frac{d\eta}{\eta} \to \frac{d\eta}{\eta}(\eta - \eta') = \frac{d\eta}{\eta}\eta\left(1 - \frac{\omega'}{\omega}\right). \qquad (72)$$

The above argument is clearly true for any number of IR intermediate vector gluons, and completes the elimination of all IR singularities from "gauge-invariant" amplitudes. I have no general proof that conventional gauge invariance is equivalent to the condition (62). However, special cases can be checked directly. In particular, it is straightforward to prove (62) for the fermion flavor bilinears

$$\bar{\psi}\lambda^A\gamma_\mu\psi\,, \quad \bar{\psi}\lambda^A\psi\,, \qquad (73)$$

where $[\lambda^A, t^a] = 0$. This is done in Appendix C. The situation regarding gauge-field objects such as $F^a_{\mu\nu}F^a_{\alpha\beta}$ is more complicated since the dependence of $F_{\mu\nu}$ on the canonical variables introduces explicit η^{-1} factors, and the algebra is much more complicated. Settling this issue one way or another might have some implication on the question of existence of "pure gluon" states in hadron physics.

We end this section with two remarks. First, the central condition [Eq. (62)] for IR finiteness is self-consistent. Once the tree-graph commutator is shown to be nonsingular at small momenta, higher orders of perturbation theory cannot invalidate the vanishing of expressions linear in $\vec{K}$ or η, since the latter will always overcome logarithmic divergences.[7] Second, the IR cutoff introduced in Sec. II is in fact redundant for IR-finite amplitudes provided Hamiltonian perturbation is used. However, if the more convenient Lagrangian Feynman rules are to be used, the IR cutoff is essential. The reason is that the divergences cancel only if they are summed in a τ-ordered fashion (Fig. 4). Conversely, in a skeleton-loop expansion the cancellations require the summation of whole classes of individually divergent graphs. I do not know whether other IR cutoffs (such as a principal-value prescription)[1] which violate some of the conditions listed in Sec. II can yield the correct result.

IV. ULTRAVIOLET STRUCTURE

In the Hamiltonian formalism, ultraviolet divergences are due to the behavior of the sums over intermediate states at high $\vec{K}$ (or small $\vec{R}$). In fact, as already mentioned, the η dependence is actually homogeneous due to boost invariance. Thus, in any given amplitude with a fixed set of external legs, all longitudinal momenta can be trivially rescaled to the interval $(-1, 1)$ by boosting the external (positive) total longitudinal momentum.

The validity of conventional power counting will now be demonstrated, and the divergences of perturbation theory shown to be logarithmic. Thus consider the passage to the next order of perturbation theory in the series (49) with a fixed set of external legs. This operation involves the insertion of a single four-point vertex and one extra $(\nu - \mathcal{H}_0)^{-1}$, or two three-point vertices and *two* factors of $(\nu - \mathcal{H}_0)^{-1}$, or one four-point and one three-point vertex and one $(\nu - \mathcal{H}_0)^{-1}$. Each additional loop generated by adding a vertex supplies a d^2K, while each energy denominator traversed by the $\vec{K}$ line behaves as $(\omega - K^2/\eta)^{-1}$. Bearing in mind that the three-point vertices are all linear in $\vec{K}$, and the four-point ones are $\vec{K}$-independent, and taking into consideration the reduction by two powers of $\vec{K}$ due to each $\delta(\sum\vec{K})$ in the vertices, it is easy to ascertain that in all the above cases the extra high-K behavior is at most $\sim d\vec{K}/K^2$, namely logarithmic (Fig. 3). There is one apparent violation of this rule, involving the second-order self-energy graph [Fig. 5(c)], which diverges quadratically:

$$\text{Fig. 5(c)} \sim \int \frac{d\vec{K}K^2}{\omega - K^2}\,. \qquad (74)$$

However, the second-order quadratic divergence is the only primitive one, while any higher-order quadratic self-energy divergences are due to overlaps of Fig. 5(c) and logarithmic vertex corrections. As is clear from Eq. (74), the divergence can be eliminated by $K^2 \to \omega - (\omega - K^2)$ and the addition of a counterterm $\propto \int d^2K$. Such a procedure had already been used in the treatment of IR divergences when the graph in Fig. 5(d) was added to 5(c). Graphs such as 5(d) correspond to a reordering of the operators in $\mathcal{H}$, and as may be verified by explicit one-loop calculations are actually generated automatically when the passage $\mathcal{L}_I \to -\mathcal{H}_I$ is performed.

The degree of divergence of any amplitude is thus fixed in the usual manner by the physical dimension and Ward identities of the external lines. In the case of external gauge-invariant Fermi bilinears [Eq. (72)] the relevant identities are the flavor currents (partial) conservation laws and commutation rules; since the verification of these identities ($Z_1 = Z_2$, etc.) depends only on shifts of internal fermion momenta, the IR problem is irrelevant and they will hold true. In particular, if all external lines are conserved currents there is no overall divergence (except for the irrelevant real part of the two-point function). Since all internal divergences are logarithmic they can be controlled by dimensionally regularizing all $d\vec{K}$ integrations [Eq. (48)], thus keeping all important operator relations intact. The analysis of the UV divergences can now be carried out in the usual fashion. It is easy to see that only irreducible internal two-, three-, and four-point functions actually diverge. Moreover, the insertion of any $\mathcal{H}_I$ into the expansion adds an extra $(\omega - K^2/\eta)^{-1}$ even if the extra $\mathcal{H}_I$ does not couple to the loops through which $\vec{K}$ flows. Thus an irreducible internal subgraph will actually occur if and only if all the vertices which contribute to it are τ-adjacent. For example, in Fig. 3 the vertex subgraph (12) is UV-convergent due to the six energy denominators which contain $\vec{K}$-dependent energies, while the one associated with the points (3, 4, 5) is log-divergent. As a result, all UV divergences will be manifested as poles in ϵ, with coefficients which are polynomials in the transverse momenta of the "external" lines of the relevant subgraph. In particular, there is no dependence on the external energy ν and the momenta of other lines. Note that virtual propagator wave-function renormalization divergences which seem to depend on $(\nu - \mathcal{H}_0)$ linearly [the factor ω in Eq. (74)] actually lose that dependence due to the adjacent $(\nu - \mathcal{H}_0)^{-1}$'s, and combine with the vertex divergences in the usual manner (Fig. 6). Thus, all UV divergences are local in τ and can be canceled in each order by subtracting from $\mathcal{H}$ a finite number of bilinear (mass renormalization), trilinear, and quadrilinear terms in the canonical fields.

All the symmetries (gauge invariance, Galilei invariance, and full Lorentz invariance) of $\mathcal{H}$ are respected in $2-\epsilon$ transverse dimensions provided the amplitude is IR-finite. The latter condition is

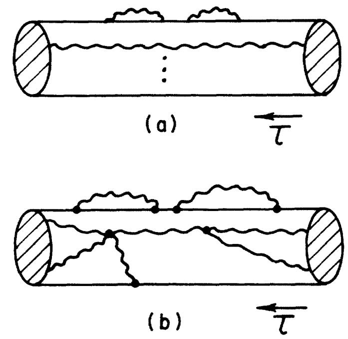

FIG. 6. The factor $(\nu - H_0)^2$ which multiplies the off-shell wave-function renormalizations cancels in (a). Diagram (b) is convergent.

needed to ensure full Lorentz invariance, which mixes $\vec{K}$ and η integrations (see Appendix B). Therefore, the contact terms which multiply the ϵ-pole terms must generate a Lorentz-invariant gauge-invariant infrared-finite tree graph. It is clear from the Lagrangian formulation of the tree-graph theory that $\mathcal{H}$ [Eq. (36)] is the most general function of the canonical variables $(\vec{\phi}, \chi, \chi^\dagger)$ which is consistent with the requirements of dimensionality (momentum dependence of the vertices) and symmetry. Thus all UV divergences will be eliminated by coupling constant and fermion mass renormalization:

$$g \to g_0(g, \epsilon)\,, \tag{75}$$

$$M \to M_0(g, \epsilon, M)\,. \tag{76}$$

The fact that no gluon mass term can be generated follows from the gauge invariance of the tree-graph theory (actually, a gluon mass would violate Lorentz invariance).

The constants g and M can be fixed by normalizing any convenient set of gauge-invariant objects at an arbitrary external momentum, say the flavor currents two-point function $\langle \mathcal{F}_\mu \mathcal{F}_\nu \rangle_0$ and scalar density $(\bar{\psi}\psi)$ two-point function $\langle SS \rangle_0$. In fact the usual Euclidean-momentum renormalization can be carried out by continuing the external ν in Eq. (49) to negative values, or dispersing Eq. (50). Note incidentally that in order to define the perturbative series M has to be shifted from its arbitrary renormalization point value to the fermion mass shell. We have already seen that the fermion self-mass is IR-finite in perturbation theory, so that this is a finite renormalization.

It should perhaps be reiterated that the renormalized Hamiltonian is meaningful only when applied to the IR-finite (gauge-invariant?) sector of the Hilbert space. Apart from the divergences inherent in gauge-variant amplitudes, Lorentz invariance can no longer be used to restrict the counterterms, which will have to respect only the Galilei invariance. Thus, for example, different Coulomb and vector emission coupling constants could be generated. In the light of this comment, the fermion mass-shell renormalization should *not* be interpreted as a perturbative definition of single fermion states. Rather, it is a process which fixes the continuum threshold of the gauge-invariant sector at $2M$. The generation of gauge-invariant bound states may of course change this threshold and cause perturbation theory to diverge, but this is (in principle) a controllable dynamical process.

V. GAUGE INVARIANCE

Having proved the perturbative existence of a renormalizable, unitary, Lorentz-invariant infrared-finite gauge-field theory on the null plane, it remains to show that in the gauge-invariant sector it yields the same amplitudes as do the conventional covariant gauges (say the Feynman gauge). This will be done by a brute-force construction of the (actually simple) gauge transformation which connects the NP gauge and the covariant gauges (CG). Since gauge invariance is best handled by using the action and the associated Feynman rules, the Hamiltonian formulation will be abandoned at this point. As noted in Sec. III, this necessitated the introduction of an IR cutoff which is compatible with gauge invariance. A simple discretization of the η axis and a corresponding finite ζ box with periodic boundary conditions suffice as was noted in Sec. II. We remark in this connection that the decoupling of the 0 mode imposed in Sec. II arbitrarily is clearly consistent with the IR finiteness of the gauge-invariant sector. In order to pass continuously from one gauge to another, the UV and IR regularization schemes have to be applicable at both ends. The applicability of the NPG discrete η-$(2-\epsilon)$ transverse-dimension procedure to the CG's will now be demonstrated by proving a simple theorem. All CG gauge-invariant amplitudes are expressions of the general form[6]

$$T_\epsilon(p)=\int\prod d^{4-\epsilon}l\,\frac{P(q,p)}{\Pi(q^2-\mu^2+i0)}\,, \qquad (77)$$

where l, q, and p are loop, propagator, and external momenta, and $\mu=0$ or M. By representing the denominators as

$$(q^2-\mu^2+i0)^{-1}=\int_0^\infty \frac{d\alpha}{i}\,e^{i\alpha(q^2-\mu^2+i0)}\,, \qquad (78)$$

and performing the Gaussian dl integrations, $T_\epsilon(p)$ becomes

$$T_\epsilon(p)=\int(d\alpha)\frac{e^{iS(p,\mu^2,\alpha)}}{[D(\alpha)]^{(4-\epsilon)/2}}K(p,\alpha)\,. \qquad (79)$$

The functions K, S, and D depend only on the external lines, the topology of the graph, and the structure of the Lagrangian. The only dependence on ϵ occurs in the exponent of D, and it is clear that as long as Eq. (77) is infrared-finite, the integrations leading to Eq. (79) can be performed by using

$$d^{4-\epsilon}l\to\tfrac{1}{2}d\omega\, d\eta\, d^{2-\epsilon}K\,. \qquad (80)$$

Equation (80) holds even in the Euclidean region, since boost invariance allows the total incoming p_+ to be rescaled to 1, and the Euclidean continuation to be achieved by

$$\omega\to-\omega\,,\quad (\eta,\vec{\mathrm{P}})\to(\eta,\vec{\mathrm{P}})\,. \qquad (81)$$

Finally, since the integral (77) is IR-finite, the $d\eta$ integration can be approximated by a Riemann sum over a discrete η axis. Consider now the NPG path-integral average of a gauge-invariant object,

$$\langle G\rangle_{\mathrm{NPG}}=\int d\phi\, e^{iI[\phi_+=0]}G[\phi,\psi,\bar\psi]d\psi\, d\bar\psi\,. \qquad (82)$$

Define a new variable $\phi_+'^a$, and perform on the NPG fields $(\vec\phi,\phi_-,\psi,\bar\psi)$ the gauge tranformation defined by

$$t^a\phi_\mu^a(x)=g^{-1}\mathcal{U}^\dagger(x)D_\mu'\mathcal{U}(x)\,,$$

$$\psi(x)=\mathcal{U}(x)\psi'(x)\,,\quad \bar\psi(x)=\bar\psi'(x)\mathcal{U}^\dagger(x)\,, \qquad (83)$$

$$\mathcal{U}(\zeta,\vec{\mathrm{R}},\tau)=\left[\exp\left(i\tfrac{1}{2}g\int^\zeta d\zeta' t^a\phi_+'^a(\zeta',\vec{\mathrm{R}},\tau)\right)\right]_+\,, \qquad (84)$$

where $[\ ]_+$ is a ζ-ordering symbol. In terms of the primed variables we have

$$I[\phi_+=0,\phi]=I[\phi_+',\phi']\,. \qquad (85)$$

Also, the measure $d\phi$ is invariant as is the function G. We thus have, up to an infinite normalization constant,

$$\langle G\rangle_{\mathrm{NPG}}=\int(d\phi')e^{iI[\phi']}G[\phi',\psi',\bar\psi']d\psi'd\bar\psi'\,. \qquad (86)$$

At this stage the usual gauge-breaking term and FFP ghost may be introduced in the conventional manner.[8]

In order to implement the above formal transformation in perturbation theory the function $\mathcal{U}$ should be expanded in a power series of g, and (83) inserted into the CG Lagrangian. New vertices[9] are thus generated (Fig. 7) which depend explicitly on the Fourier transform of the exponent of $\mathcal{U}$, namely $\eta^{-1}\phi_+'(\eta)$. The formal manipulations leading to Eq. (86) are safe so long as the cutoffs are kept (recall that $\eta=0$ is decoupled by

FIG. 7. Terms generated by Eq. (83). Dashed lines are η^{-1}, dots are products of gT^a's, and the "springs" are ϕ_+'.

definition). Since both the NPG and CG expressions are separately IR- and UV-finite (after g and M renormalization) the cutoffs may be removed. Moreover, since Eq. (86) holds in 4-ϵ dimensions, and since the UV counterterms are IR-finite, $g_0(g,\epsilon)$ and $M_0(g,\epsilon,M)$ will be the same power series in both gauges provided g and M are fixed by the same gauge-invariant objects at the same external momenta. It should be remarked that the order of cutoff removal is irrelevant, since the Lorentz-noninvariant counterterms which may be generated in the presence of the IR cutoff vanish order by order when $N\to\infty$ [Eq. (47)]. Again, it is important to remember that the gauge transformation (83) may be implemented only if the NPG amplitude is IR-nonsingular. In particular, the situation with respect to pure-glue gauge-invariant objects needs clarification. When the transformation (83) is applied to (finite) covariant gauge-variant Green's functions the resultant NPG amplitude will diverge. Since unitarity in our approach is defined by the NPG Hamiltonian theory the corresponding CG amplitudes will violate unitarity.

VI. CONCLUDING REMARKS

The null-plane Hamiltonian formulation of gauge theory was shown to yield a consistent IR-finite Lorentz-invariant theory which is equivalent to the covariant gauge formulation. All of these properties hold in the gauge-invariant sector of the null-plane Hilbert space (possibly not even for *all* gauge-invariant states). When the system is quantized in a finite longitudinal box the Hamiltonian is defined on all states, but perturbation theory yields finite wave functions only in the gauge-invariant sector. Gauge-variant states are dominated by modes whose wavelength is comparable to the box size and thus yield divergent (actually vanishing) overlaps with locally produced states. The actual form of the IR divergences was related in Sec. III to the behavior of the gauge-group null-plane charge density at low momenta. We remark in this connection that in QED the charge is carried by a massive fermion, so that its behavior at $(\eta,\vec{K})\to 0$ is controlled by the initial-state wave packet. The analysis of Sec. II then shows that the IR-divergent terms are independent of the τ development and can be factored out and eliminated by redefining the initial (electron) state.[7] This procedure is impossible in the non-Abelian case, since in perturbation theory the (non-Abelian) gauge charge spreads throughout the box owing to the radiation of massless gluons. In particular, in each order a large finite fraction of the null-plane energy (ω) will be found at infinite transverse and longitudinal distances. Whether this perturbative result has anything to do with confinement is unclear.

For gauge-invariant initial states the infrared singularities cancel, and the null-plane Hamiltonian is a finite operator in this subspace. The momentum scale which governs the cancellation in perturbation theory is essentially (up to logs) the external momentum or fermion mass (whichever is smaller). Since gauge systems are infrared-unstable, the external momenta should be replaced by the renormalization scale in a nonperturbative situation.

One issue remains unresolved, namely, the behavior when the fermion masses vanish. The discussion of Sec. III breaks down in that case [see Eq. (69)] since Bose symmetry is not available any more. We recall that QED also breaks down for massless fermions.[10]

Although not explicitly discussed, it is clear that the short-distance operator-product expansion can be used in the NPG, since the Wilson coefficients and operators are manifestly gauge-invariant. The only complication is with regard to the singlet twist-four expansion[1] which contains an FFP ghost contribution. The problem lies in the interpretation of this operator in the NPG. The direct, brute-force approach is to perform the gauge transformation (83) order by order. It is hoped that there exists a cleverer way.

Clearly, null-plane-gauge perturbation theory is far less convenient than (say) the Feynman-gauge version. However, the null-plane Hamiltonian formalism, owing to the trivial vacuum and the manifest appearance of infrared singular terms, provides a chance of nonperturbative approaches to hadron physics.

APPENDIX

1. Phase space

The invariant phase-space element of the process depicted in Fig. 2 is

$$d\Omega(\Lambda\,|\{p\}\{k\})$$

$$=\delta\left(\Lambda^2-\sum\frac{p^2+m^2}{x}-\sum\frac{K^2}{\eta}\right)\delta(\Sigma\vec{P}+\Sigma\vec{K})$$

$$\times\,\delta(1-\Sigma x-\Sigma\eta)\prod\frac{dx}{x}\,d\vec{P}\prod\frac{d\eta}{\eta}\,d\vec{K}\,. \quad (A1)$$

Introducing the mass μ of the p subsystem and using Galilei and boost invariance

$$d\Omega(\Lambda|\{p\}\{k\}) = d\mu^2 d\Omega(\mu|\{p\}) \times \delta((\Lambda^2-\Sigma\omega)(1-\Sigma\eta)-\mu^2-(\Sigma\vec{K})^2)\prod\frac{d\eta}{\eta}\,d\vec{K}\,. \tag{A2}$$

Defining

$$\eta\omega = K^2,\quad \hat{K}=(\cos\theta,\sin\theta)\,, \tag{A3}$$

we find

$$d\Omega(\Lambda|\{p\},\{k\}) \underset{\eta\to 0}{\simeq} d\mu^2 d\Omega(\mu|\{p\}) \times \delta((\Lambda^2-\mu^2-\Sigma\omega)-(\Sigma\eta)(\mu^2+\lambda)) \times \prod\frac{d\theta}{2}\,d\eta\, d\omega\,, \tag{A4}$$

$$\lambda=(\Sigma\eta)^{-1}\sum_{i,j}(\omega_i\omega_j\eta_i\eta_j)^{1/2}\cos\theta_{ij}>0\,. \tag{A5}$$

B. Lorentz invariance

The null-plane rotating Lorentz transformations are

$$\delta\vec{R}=-2\vec{U}\xi,\quad \delta\tau=-4\vec{U}\cdot\vec{R},\quad \delta\xi=0\,, \tag{B1}$$

$$\delta\vec{P}=\vec{U}\omega,\quad \delta\eta=2\vec{U}\cdot\vec{P},\quad \delta\omega=0\,. \tag{B2}$$

Since ϕ_+ transforms like η, the NP gauge condition (13) is violated and a compensating gauge transformation is needed. We find

$$\delta\vec{\phi}=\vec{U}\phi_- - 2\vec{D}\partial_\xi^{-1}(\vec{\phi}\cdot\vec{U})\,. \tag{B3}$$

The Fermi fields transform in the usual way and (χ,ξ) get mixed.

Equation (B3) is a canonical transformation [when Eqs. (29) and (30) are substituted for ϕ_- and ξ], and an infinitesimal generator can be derived from the action (18). The generator $\vec{M}$ has two parts, a linear (free) part M_0 and a nonlinear g-dependent part M_I. Operating on the on-shell perturbative wave function we get (all τ dependence is due to $\mathcal{H}_0$)

$$(M_0+M_I)\left[\exp\left(\frac{1}{i}\int_{-\infty}^0 d\tau\,\mathcal{H}_I(\tau)\right)\right]_+ = \left[\exp\left(\frac{1}{i}\int_{-\infty}^0 d\tau\,\mathcal{H}_I(\tau)\right)\right]_+(M_0+M_I) + \frac{1}{i}\int_{-\infty}^0 U(0,\tau)d\tau[(M_0+M_I)(\tau),(\mathcal{H}_0+\mathcal{H}_I)(\tau)]\,U(\tau,-\infty)\,. \tag{B4}$$

Since $[M,\mathcal{H}]=0$ the second term drops. Also, the term M_I which is nonlinear in the fields does not contribute on-shell (no single-particle pole). Thus order by order in perturbation theory the transformation of the external lines of Eq. (51) is compensated by performing the transformation (B2) on the intermediate on-shell lines [the $\delta(\nu-\mathcal{H}_0)$ term]. All the internal-energy denominators and phase-space elements are invariant. Although $\vec{K}$ transforms quadratically, all transverse integrations depend on K^2 which transforms linearly in $\vec{K}$, so that the UV degree of divergence in unchanged. However, it is clear that the transformation (B3) cannot be implemented on an IR-singular integrand.

C. Fermion bilinears

Equation (62) is trivial for the operators J_+^A since the current-algebra-like relation

$$[J_+^a(\vec{R},\xi),J_+^A(\vec{R}',\xi')]=0 \tag{C1}$$

holds due to the commutation rules. No Schwinger terms appear because $\mathrm{tr}\,t^a\lambda^A=0$.

Equation (C1) does not hold for $\vec{J}^A$ because the latter explicitly depends on the gauge fields

$$\vec{J}^A=\xi^\dagger\vec{\sigma}\lambda^A\chi+\text{H.c.}\,, \tag{C2}$$

where ξ is given by Eq. (30). However, using Eqs. (67) and (68) it is straightforward to verify that

$$(\vec{K},\eta)\to 0:\quad [J_+^a(\vec{K},\eta),\vec{J}^A(\vec{P},\eta')]\simeq O(\eta,K)\,. \tag{C3}$$

Phase-space considerations "protect" Eq. (C3) when the momenta of the fermions or vector boson contained in $\vec{J}^A$ become small.

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