Bhabha first-order wave equations. V. Indefinite metric and Foldy-Wouthuysen transformations*

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We prove the existence of a Foldy-Wouthuysen (FW) transformation which decouples the mass states of all the Bhabha Poincaré generators and diagonalizes the Hamiltonian. Since the Bhabha fields operate in an indefinite-metric space, such an existence is not *a priori* guaranteed. The FW-transformed generators are given and satisfy the Poincaré Lie algebra. We observe that although the FW transformation expressed as a power series in c^{-1} more clearly exhibits the physics of the situation, it is only in the Dirac and Duffin-Kemmer-Petiau special cases of the Bhabha fields that the FW-transformation power series can easily be summed to yield a closed-form expression. The general closed-form expression surrenders more readily to another technique. Therefore, as a first calculation, in this paper we present a method of generating the FW transformation as a power series in c^{-1}. Our discussion concentrates on the indefinite metric, the physics which is evident in the power-series form (such as size and types of *Zitterbewegung*), and on a detailed examination of special cases up to spin 3/2. In all the above, a special handling of the built-in subsidiary components of the integer-spin fields is once again necessary. We also comment on what the indefinite metric may be implying about the possibility of finding a totally consistent high-spin field theory.

I. INTRODUCTION

In our series of papers[1-4] on the first-order fields for arbitrary spin discussed by Bhabha,[5-9] Lubânski,[10] and Madhavarao[11-15] (footnote 9 of paper IV in Ref. 4 describes the work of the above authors), we first directly confronted the indefinite metric[8,16,17] associated with these fields in IV.[4] There, in the q-number theory, the fields had to be quantized with an indefinite metric in order to properly obtain causality. Of course, the indefinite metric also exists in the c-number theory, as we first alluded to in III.

For a field described by the equation

$$(\partial\cdot\alpha+\chi)\psi=0\,, \tag{1.1}$$

where the α_μ are again the so(5) generators $J_{\mu 5}$, then for a given so(5) representation $(\mathcal{S}, S)$ of the α_μ, Eq. (1.1) describes fields of spin between 0 or $\frac{1}{2}$ and $\mathcal{S}$, and mass states $\pm\chi/\mathcal{S}$, $\pm\chi/(\mathcal{S}-1)$, to $\pm 2\chi$ or $\pm\chi$, as $\mathcal{S}$ is half-integer or integer.[2] [We are using the same notational conventions as in I–IV, such as $a_\mu=(\vec{a}, a_4)=(\vec{a}, ia_0)$, $\mu=1,2,3,4$; $a\cdot b=a_\mu b_\mu$.]

The metric of the fields implies that the ground state $+\chi/\mathcal{S}$ has a positive norm, the first excited state $+\chi/(\mathcal{S}-1)$ has a negative norm, the norm oscillating back and forth with each additional excitation. The antiparticles have the same or opposite norm as the respective particles for half-integer- or integer-spin fields. Taking for definiteness the representation where α_4 is diagonal, both the Hamiltonian and the boost generators $\vec{K}$ involve the matrices $\vec{\alpha}$, and the $\vec{\alpha}$ couple adjacent mass states. This means a scattering matrix or boost can in principle couple different normed states in a manner not conserving probability.

However, in the *free c-number theory* one can show that this difficulty is avoided by demonstrating that there exists a Foldy-Wouthuysen (FW) transformation[18-20] which decouples the mass states, not only in the free Hamiltonian, but in all the Poincaré generators. This would mean that although in the standard representation probability is leaking from the positive-normed states into the negative-normed states, it is leaking back just as fast. It is the purpose of this paper to show that such a FW transformation does exist for the Bhabha fields and it has the described properties.

Before continuing we emphasize that the term "FW transformation" is used in three senses, each more or less restrictive than the others. In the original FW terminology, an operator is "odd" if it couples the different mass states and it is "even" if it does not. Thus (i) an FW transformation is any metric-unitary (see below) transformation which decouples the mass states for all the Poincaré generators. Discussions in the literature often define an FW transformation in an even looser sense, demanding only that (ii) the Hamiltonian be in "even" form, altogether ignoring the other generators, in particular, the boost. Of those FW transformations which decouple mass states for all the Poincaré generators, there is one (iii) which makes the Hamiltonian diagonal. For the main part in this paper, we will choose this FW transformation [(iii)] as *the* FW transformation, although the reader should remember that there is a metric-unitary equivalent class of transformations which can leave the generators even, but not leave the Hamiltonian diagonal.

FW transformations have been discussed in numerous places. There is the work on[21–26] the spin-$\frac{1}{2}$ Dirac field[27] and on[28–30] the two-component spin-0 and six-component spin-1 Sakata-Taketani[31–36] versions of the Duffin-[37] Kemmer-[38] Petiau[39] fields. FW transformations of other fields have also been discussed.[40–45] Further, other "unitary" transformations related to the FW transformation have been investigated.[46–51]

It is the word "unitary" that is critical. In an indefinite-metric space, norm-preserving transformations are no longer unitary and observables are no longer Hermitian. As emphasized elsewhere[52,53] the condition that a transformation U preserve the norm is that it be "metric-unitary", i.e.,

$$U^{\dagger} M U = M \,, \tag{1.2}$$

where M is the metric matrix operator. For the Bhabha field (see Sec. II), the metric is

$$M = \eta_4 \alpha_4 = M^{\dagger} . \tag{1.3}$$

An observable Θ is "metric-Hermitian" if

$$(M\Theta)^{\dagger} = M\Theta \,. \tag{1.4}$$

(Commonly the terminology "pseudounitary" and "pseudo-Hermitian" is used, but we prefer our more descriptive names.) In Refs. 52 and 53 we proved theorems and lemmas about FW transformations in general indefinite-metric spaces which will be of particular value in this paper. We refer the reader elsewhere for further readings on indefinite metrics,[52–74] covering mathematical papers and general discussions[52–60] and applications to the original QED problem[61–67] and to other field theories.[68–74] (See Refs. 75–78 for discussions involving nonlocal Hamiltonians and Ref. 79 for calculations with an extended Poincaré algebra.)

In Sec. II we discuss the Bhabha indefinite metric and the proof that there exists an FW transformation which diagonalized the Hamiltonian and also decouples the different mass and normed states in all the FW-transformed Poincaré generators. We will do the above first for the half-integer-spin Bhabha fields, and then separately for the "particle components" of the integer-spin fields. We will comment on the importance of this separate treatment of the particle components and the connection of intrinsic parity with the metric.

Then, in Sec. III, we review the known FW results for the free Dirac field,[18–26] and extend previous results on DKP[28–30] to all the Poincaré generators. We verify that these special cases are included in the general results described in Sec. II. They must be, of course, since the Dirac and DKP fields are the simplest special cases of Bhabha fields.

Our presentation of the arbitrary-spin Bhabha FW transformations will proceed in two modes. The first mode, given in this paper, describes how to obtain the FW transformation order by order in $1/c$.[80] This is done by first (a) showing what form the FW generators must have and (b) demonstrating that in this form they satisfy the associated Lie algebra

$$[P_i, P_j] = 0 \,, \tag{1.5a}$$

$$[P_i, H] = 0 \,, \tag{1.5b}$$

$$[J_k, H] = 0 \,, \tag{1.5c}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \,, \tag{1.5d}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k \,, \tag{1.5e}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \,, \tag{1.5f}$$

$$[K_i, P_j] = i\delta_{ij} H/c^2 \,, \tag{1.5g}$$

$$[K_i, H] = iP_i \,, \tag{1.5h}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k / c^2 \,. \tag{1.5i}$$

(In this paper we use $\hbar = 1$, but at least for now keep $c \neq 1$, since in part we will be dealing with expansions in powers of c^{-1}. See Ref. 81 for a discussion on the dimensions of the Poincaré generators.) Then we (c) explicitly obtain the FW transformation to order c^{-3}, and (d) show that the original Poincaré generators agree with the FW-transformed generators to this order. The above program will be carried out in Sec. IV for both the half-integer-spin generators and the particle components of the integer-spin generators, with emphasis on the differences and similarities between them.

This emphasis will be generalized in Sec. V, where we will demonstrate how the order-by-order and exact integer-spin particle-components generators, FW generators, and FW transformations can be functionally related to their half-integer-spin counterparts.

In Sec. VI we will discuss the physical implications of our results. We will point out the relation of the FW transformation to the indefinite metric and the intrinsic parity of oppositely normed states. We also include an explanation, based on the structure of the so(5) matrices α_μ, of the different types of Zitterbewegung obtained from, for example, the minimally coupled Dirac and DKP special cases. We finally will exhibit the special $\mathcal{S}=\frac{3}{2}$ case as a high-spin example.

The second mode of our presentation of the arbitrary-spin Bhabha FW transformations, obtaining the exact, closed-form expressions, will involve the development of different techniques. We shall defer this to a later paper[82] (VI of our series on solutions and exact FW transformations), and will end our series of papers with the conclusions we have come to on Bhabha fields[83] (VII of our series). We simply note here that, contrary to the Dirac and DKP special case, the Bhabha arbitrary-spin FW transformations, when expressed as a power-series expansion in c^{-1}, *cannot* easily be summed to yield an exact, closed-form expression. The reason for this is the complicated nature of the higher-spin algebras. This is important, since the "physics" is more transparent in the power-series form. For example, the power-series expansion directly gives the relativistic corrections to the nonrelativistic generators.

Before continuing, we wish to mention work on FW transformations for nonminimal coupling,[84–91] and discussions[92–97] of other fields and/or interactions which have come to our attention.

II. INDEFINITE METRIC AND FW EXISTENCE PROOF

A. The Bhabha metric

The Bhabha-field metric

$$M=\eta_4\alpha_4 \tag{2.1}$$

was discussed by us in Sec. II of paper III. We further investigated it in Sec. IV of paper IV, since we had to understand its nature in detail to show causality for the second-quantized fields. There we pointed out that from the defining equations [(IA18)] for η_4 and the characteristic equation [(I2.31)] for α_4 with α_4 diagonal, η_4 and α_4 have the forms

$$\eta_4(\mathcal{S}=n+\tfrac{1}{2})=\text{block diagonal}\left[\mathcal{G}^+_{\mathcal{S}},-\mathcal{G}^+_{\mathcal{S}-1},+\mathcal{G}^+_{\mathcal{S}-2},\ldots,(1)^{\mathcal{S}-1/2}\mathcal{G}^+_{1/2},(-1)^{\mathcal{S}+1/2}\mathcal{G}^-_{1/2},\ldots,+\mathcal{G}^-_{\mathcal{S}-1},-\mathcal{G}^-_{\mathcal{S}}\right], \tag{2.2}$$

$$\eta_4(\mathcal{S}=n)=\text{block diagonal}\left[\mathcal{G}^+_{\mathcal{S}},-\mathcal{G}^+_{\mathcal{S}-1},+\mathcal{G}^+_{\mathcal{S}-2},\ldots,(1)^{\mathcal{S}+1}\mathcal{G}^+_{1},(-1)^{\mathcal{S}+2}\mathcal{G}_0,(-1)^{\mathcal{S}+1}\mathcal{G}^-_{1},\ldots,-\mathcal{G}^-_{\mathcal{S}-1},+\mathcal{G}^-_{\mathcal{S}}\right], \tag{2.3}$$

$$\alpha_4=\text{block diagonal}\left[\mathcal{S}\mathcal{G}^+_{\mathcal{S}},(\mathcal{S}-1)\mathcal{G}^+_{\mathcal{S}-1},\ldots,(\mathcal{S}-1)\mathcal{G}^-_{\mathcal{S}-1},-\mathcal{S}\mathcal{G}^-_{\mathcal{S}}\right], \tag{2.4}$$

meaning M has the form

$$M(\mathcal{S}=n+\tfrac{1}{2})=\text{block diagonal}\left[+\mathcal{S}\mathcal{G}^+_{\mathcal{S}},-(\mathcal{S}-1)\mathcal{G}^+_{\mathcal{S}-1},+(\mathcal{S}-2)\mathcal{G}^+_{\mathcal{S}-2},\ldots,(-1)^{\mathcal{S}-1/2}\tfrac{1}{2}\mathcal{G}^+_{1/2},(-1)^{\mathcal{S}-1/2}\tfrac{1}{2}\mathcal{G}^-_{1/2},\ldots,-(\mathcal{S}-1)\mathcal{G}^-_{\mathcal{S}-1},+\mathcal{S}\mathcal{G}^-_{\mathcal{S}}\right], \tag{2.5}$$

$$M(\mathcal{S}=n)=\text{block diagonal}\left[+\mathcal{S}\mathcal{G}^+_{\mathcal{S}},-(\mathcal{S}-1)\mathcal{G}^+_{\mathcal{S}-1},+(\mathcal{S}-2)\mathcal{G}^+_{\mathcal{S}-2},\ldots,(-1)^{\mathcal{S}+1}\mathcal{G}^+_{1},0\mathcal{G}_0,-(-1)^{\mathcal{S}+1}\mathcal{G}^-_{1},\ldots,+(\mathcal{S}-1)\mathcal{G}^-_{\mathcal{S}-1},-\mathcal{S}\mathcal{G}^-_{\mathcal{S}}\right]. \tag{2.6}$$

The $\mathcal{G}^{\pm}_j(\mathcal{S})$ are the mass state $\pm\chi/j$ projection operators and $\mathcal{G}_0(\mathcal{S})$ is the subsidary-components projection operator discussed in Secs. III B and III C of paper II. [The matrix M is the $\lambda=4$ case of matrices $\xi_\lambda=\eta_\lambda\alpha_\lambda$ (no sum) defined by Madhavarao, Thiruvenkatachar, and Venkatachaliengar.[14]]

For half-integer spin, Eq. (2.5) shows that the particle and antiparticle ground states $\pm\chi/\mathcal{S}$ have positive norms, the first excited states $\pm\chi/(\mathcal{S}-1)$ have negative norms, and so on. The simplest case $(\mathcal{S}=\frac{1}{2})$ is the Dirac equation which has only positive-normed states.

For integer spin, Eq. (2.6), there is an extra minus sign. From Eq. (IA18) this is due to the extra minus sign in η_4 caused by the extra subsidiary components block projected by $\mathcal{G}_0$. Thus for DKP, where $\mathcal{S}=1$, although the particle state has a positive norm, the antiparticle state has a negative norm. The standard resolution of this most simple negative norm is the Pauli-Weisskopf[98] device of saying that this is a charge probability density instead of the particle probability density of the Dirac case.

However, for $\mathcal{S}>1$, this resolution is not obviously applicable for either integer or half-integer spin, since the first excited state of a particle necessarily has the opposite norm, oscillating back and forth with each further excitation, with an added minus sign for the antiparticle normalization of integer-spin particles. This extra minus sign is important in the proper handling of the FW transformations of integer-spin fields.

We pointed out in Ref. 52 that it is not necessarily true that a metric-Hermitian operator in an indefinite-metric space can be diagonalized. Thus even the question of the existence of an FW transformation in an indefinite-metric space is open. The existence of such a transformation depends on the particular space and the operators under consideration. However, we will show that for the Bhabha generators such a transformation exists.

The crucial result needed to show this is the theorem[52] that there exists an FW transformation which will diagonalize a metric-Hermitian matrix Θ on a *nonsingular* indefinite-metric space *if and only if* all the eigenvalues of Θ are real and all the eigenvectors of Θ have nonzero norm. (The point is that these conditions are not always satisfied in an indefinite-metric space.) Under these conditions, the metric-unitary matrix U which will diagonalize Θ to D is given by

$$U\Theta U^{-1}=D, \quad U^{-1}=[\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_n], \qquad \hat{u}_j=|M_{jj}|^{1/2}u_j, \quad U=M^{-1}(U^{-1})^{\dagger}M, \tag{2.7}$$

where the u_j are the n independent eigenvectors of Θ, normed to ±1. This reduces to the standard answer in a Hilbert space.

Before continuing, note that the form (2.2) and (2.3) for η_4, which used the projection operators developed in III, clearly shows the intrinsic parities of the particles. (Recall from I that η_4 is the parity operator.) For half-integer-spin fields, the antiparticles have the opposite parity as the particles, and for integer-spin fields the antiparticles have the same parity as the particles. Further, observing that the ground state $+\chi/\mathcal{S}$ has its parity determined by η_4 only to a minus sign, one sees that the first excited state $+\chi/(\mathcal{S}-1)$ has the the opposite parity of the ground state $+\chi/\mathcal{S}$, and that each further excited state has the opposite parity to the previous one.

B. FW existence for half-integer-spin fields

First consider the Poincaré generators of half-integer-spin Bhabha fields given in Eqs. (III1.5)–(III1.9):

$$P_j=p_j=-i\partial_j, \tag{2.8}$$

$$J_k=-i\epsilon_{ijk}(x_i\partial_j+\alpha_i\alpha_j)\equiv L_k+S_k, \tag{2.9}$$

$$H(\mathcal{S}=n+\tfrac{1}{2})=\alpha_4^{-1}(c\vec{\partial}\cdot\vec{\alpha}+\chi c^2), \tag{2.10}$$

$$K_j=x_jHc^{-2}-tp_j+t_{4j}c^{-1}, \tag{2.11}$$

$$t_{4j}\equiv[\alpha_4,\alpha_j]. \tag{2.12}$$

Observe that in the rest system the eigenvalues of H are the rest masses, the eigenvectors $\hat{u}_j$ are just 1 in a particular row with zeros elsewhere, and the norm of $\hat{u}_j$ is never zero. A Lorentz transformation will take any one of these states to an arbitrary energy, and in particular the norm will never be zero. (From the discussion in Ref. 52 a zero norm would only ensue from some sort of binding which would take the total energy to zero.) Thus from the above theorem there does exist an FW transformation which will diagonalize the Hamiltonian H of Eq. (2.10).

The next question is whether the FW transformation which diagonalizes H will decouple the components of the different mass states (and hence different normed states) for all the Poincaré generators. That is, will this FW transformation leave all the generators in a form which has zeros outside of the mass blocks defined by the operators $\mathcal{G}_j^{\pm}(\mathcal{S})$? The answer is "yes," as can be seen by the following argument.

Since the Hamiltonian involves only $\vec{p}$, χ, c, and the α_μ, the transformation which will diagonalize it needs only to involve these quantities. They all commute with p_j, so the generators p_j trivially remain diagonal. With regard to the generators J_k, in the form (2.9) they already are mass-block diagonal. This follows since [see Eq. (III 5.17)]

$$[\alpha_4,[\alpha_i,\alpha_j]]=0 \quad (i,j)\neq 4. \tag{2.13}$$

That the J_k remain mass-block diagonal (in fact the same) would follow immediately from correctly *assuming* that the transformation is a space-scalar function of $\vec{p}$, χ, c, and the α_μ. However, it is useful to make the following, more general argument. First observe that the transformed generators still satisfy the Lie algebra defined by Eqs. (1.5). Also, the diagonal Hamiltonian will have diagonal matrix elements of the form (see Sec. IV A)

$$(H^{\mathrm{FW}})_{ij}=E_j(\vec{p})\delta_{ij} =\pm(c^2\vec{p}^{\,2}+\chi^2/q_j^{\,2})^{1/2}\delta_{ij}, \quad q_j\equiv(\alpha_4)_{jj}. \tag{2.14}$$

This means Eq. (1.5c) will have matrix elements

$$\begin{aligned}0&=[J_k^{\mathrm{FW}},H^{\mathrm{FW}}]_{ij}\\&=(J_k^{\mathrm{FW}})_{ij}E_j-E_i(J_k^{\mathrm{FW}})_{ij}\\&=(E_j-E_i)(J_k^{\mathrm{FW}})_{ij}+(W_k)_{ij},\end{aligned} \tag{2.15}$$

where

$$(W_k)_{ij}\equiv[(J_k^{\mathrm{FW}})_{ij},E_j]=\epsilon_{mnk}[(x_m^{\mathrm{FW}})_{ij}p_n,E_j]. \tag{2.16}$$

The last equality of (2.16) comes about because from the form (2.9) of $(J_k)_{ij}$, only $(x_m^{\mathrm{FW}})_{ij}$ will not commute with E_j. However, because of our previous comment that the FW transformation will only be a function of $\vec{p}$, α_μ, χ, and c, one has

$$(x_m^{\mathrm{FW}})_{ij}=x_m\delta_{ij}+\Delta x_m(\vec{p},\alpha_\mu,\chi,c)_{ij}, \tag{2.17a}$$

$$\Delta x_m\equiv U[x_m,U^{-1}], \tag{2.17b}$$

$$x_m=i\frac{\partial}{\partial p_m}. \tag{2.18}$$

Thus only the first term of (2.17a) contributes to (2.16), so using (2.18) yields

$$(W_k)_{ij}=\epsilon_{mnk}[x_m\delta_{ij}p_n,E_j]=0. \tag{2.19}$$

Putting (2.19) into (2.15) implies that $J_k^{\rm FW}$ only has nonzero matrix elements when $E_j = E_i$, i.e., within the diagonal mass blocks, so we have shown that $J_k^{\rm FW}$ is mass-block diagonal.

Finally, one can show that $K_i^{\rm FW}$ is mass-block diagonal. Starting from Eq. (1.5h), one has that

$$i(P_i^{\rm FW})_{jk} = ip_i\delta_{jk} = (K_i^{\rm FW})_{jk}(E_k - E_j) + (Z_i)_{jk}\,, \quad (2.20)$$

$$(Z_i)_{jk} \equiv [(K_i^{\rm FW})_{jk}, E_j]\,. \quad (2.21)$$

Then, as before, the only contribution to $(Z_i)_{jk}$ will come from the FW-transformed piece of K_i which is $(x_i)\delta_{jk}E_k$. Thus, using (2.18),

$$(Z_i)_{jk} = [x_i\delta_{jk}E_k, E_j] = ip_i\delta_{jk}\,, \quad (2.22)$$

so the first term in the second line of Eq. (2.20) is zero. This, as before, implies that the only pieces of $K_i^{\rm FW}$ which are nonzero are when $E_k = E_j$, i.e., within the diagonal mass blocks. Thus we have shown that $K_i^{\rm FW}$ is also mass-block diagonal, completing our demonstration that all FW Poincaré generators for half-integer-spin fields are mass-block diagonal and thus do not couple the different normed or mass states.

C. FW existence for integer-spin fields

For integer-spin fields we must first decide what to do about the built-in subsidiary components [those projected out by the $\mathcal{G}_0(\mathcal{S})$]. Since these components have zero norm, their remaining in the theory would rule out a metric-unitary transformation. However, in III we showed how to project out the particle components of the Poincaré generators, these then algebraically satisfying the Lie algebra (1.5). (The reader will recall that as long as the subsidiary components remain, the entire Lie algebra is satisfied only as an operator algebra on the fields themselves.) The integer-spin particle-components Poincaré generators were shown in III to be metric-Hermitian and, as the half-integer-spin generators, they have a complete set of nonzero-normed eigenstates. Thus they are the appropriate generators upon which to perform FW transformations. These particle-components generators were found in Eqs. (III5.28)–(III5.31) to be

$$P_j^{(P)} = P_j(1 - \mathcal{G}_0)\,, \quad (2.23)$$

$$J_j^{(P)} = J_j(1 - \mathcal{G}_0)\,, \quad (2.24)$$

$$\begin{aligned} H^{(P)} &= [Q(c\vec{\partial}\cdot\vec{\alpha} + \chi c^2) - Q(c\vec{\partial}\cdot\vec{\alpha})\mathcal{G}_0(\chi c^2)^{-1}(c\vec{\partial}\cdot\vec{\alpha})](1 - \mathcal{G}_0) \\ &= Q(c\vec{\partial}\cdot\vec{\alpha} + \chi c^2) - Q(c\vec{\partial}\cdot\vec{\alpha})\mathcal{G}_0[1 + (\chi c^2)^{-1}(c\vec{\partial}\cdot\vec{\alpha})]\,, \end{aligned} \quad (2.25)$$

$$\begin{aligned} K_j^{(P)} &= x_j H^{(P)} - tP_j^{(P)} \\ &\quad + c^{-1}(1 - \mathcal{G}_0)[\alpha_4, \alpha_j](1 - \mathcal{G}_0) \\ &\quad - c^{-1}(\alpha_4\alpha_j)\mathcal{G}_0(\chi c^2)^{-1}(c\vec{\partial}\cdot\vec{\alpha})(1 - \mathcal{G}_0)\,, \end{aligned} \quad (2.26)$$

where Q is defined in Eq. (II3.22) and has the numerical values of $(\alpha_4^{-1})_{ij}$ in the particle components.

For the above generators the same arguments hold which showed that there exists an FW transformation which decouples the mass states in the half-integer-spin generators. In particular, there exists a complete set of eigenstates of nonzero norm for the particle-components Hamiltonian, and so there exists a metric-unitary transformation which diagonalizes the particle-components Hamiltonian. Thus from the form of the Lie algebra, the metric-unitary transformation which diagonalizes the Hamiltonian also makes the rest of the Poincaré generators mass-block diagonal, meaning a complete FW transformation is possible here, too.

The device of considering only the particle components was used in the DKP investigations[28-30] we referred to, either by directly using[28,30] what we called in II the particle components of the Sakata-Taketani DKP equations, or by what amounted to such a projection.[29] Thus, even if our general indefinite-metric rationale was not the basis, the proper method was followed completely in Refs. 28 and 30, and essentially in Ref. 29.

In the succeeding sections, when we deal with integer spin, we sometimes use the superscript "P" for particle components, and sometimes use a caret. The two notations are for convenience. Also, when listing the metric, it will be understood to be the "surrounded" metric which algebraically still has the same form $\eta_4\alpha_4$, since

$$\begin{aligned} M^{(P)} &= (1 - \mathcal{G}_0)\eta_4\alpha_4[1 - \mathcal{G}_0(\chi c^2)^{-1}(c\vec{\partial}\cdot\vec{\alpha})](1 - \mathcal{G}_0) \\ &= \eta_4\alpha_4 \\ &= M\,. \end{aligned} \quad (2.27)$$

Further, when discussing the operators, the fact that the subsidiary components have been removed will sometimes be implicit. For example, the metric (2.6) is already zero in the subsidiary components. From (2.27) it is to be understood that the rows and columns of the subsidiary components of these operators are deleted from the matrix representation. In particular, the particle-components operators are no longer singular and so inverses exist. (Remember, there is a metric-unitary FW transformation for such nonsingular operators.)

III. SPECIAL CASE RESULTS

A. The Dirac case $(\mathcal{S},S)=(\frac{1}{2},\frac{1}{2})$

The FW transformation was originally developed[18-20] for the Dirac system. It is useful to give a review of those results for comparison with the general case. To obtain the Dirac Poincaré generators in the usual FW notation, first substitute into Eqs. (2.8)–(2.12)

$$\alpha_\mu=\gamma_\mu/2\,,\quad \chi=m/2\,, \tag{3.1}$$

which, from Eq. (I2.32), corresponds to the Dirac algebra

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\,. \tag{3.2}$$

Next use the matrices β^D and $\vec{\alpha}^D$ defined by

$$\beta^D=\gamma_4=\gamma_4^{-1}\,,\quad \vec{\alpha}^D=i\gamma_4\vec{\gamma} \tag{3.3}$$

to obtain the Dirac Poincaré generators

$$P_j=p_j\,, \tag{3.4}$$

$$J_k=\epsilon_{ijk}(x_ip_j-\tfrac{1}{4}\alpha_i^D\alpha_j^D)\equiv L_k+S_k$$

$$\equiv L_k+\tfrac{1}{2}\sigma_k\,, \tag{3.5}$$

$$H=c\vec{p}\cdot\vec{\alpha}^D+\beta^D mc^2\,, \tag{3.6}$$

$$K_j=x_jH/c^2-tp_j+t_{4j}/c\,, \tag{3.7a}$$

$$K_j=\frac{1}{2c^2}\{x_j,H\}-tp_j\,, \tag{3.7b}$$

$$t_{4j}=-i\beta^D[\beta^D,\alpha_j^D]\,. \tag{3.8}$$

Then, as described in the literature,[99,100] the transformation which diagonalizes the Hamiltonian (3.4) is

$$U^{-1}=e^{-iS}\,, \tag{3.9a}$$

$$\begin{aligned}U&=e^{iS}\\&=\exp[\beta^D\vec{p}\cdot\vec{\alpha}^D\theta(\vec{p})]\\&=\cos(p\theta)+\frac{\beta^D\vec{p}\cdot\vec{\alpha}^D}{p}\sin(p\theta)\,,\end{aligned} \tag{3.9b}$$

where the last equality comes from a power-series expansion. Explicitly, one obtains ($p\equiv|\vec{p}|$)

$$H^{FW}=UHU^{-1} \tag{3.10}$$

$$\begin{aligned}&=\left(\cos p\theta+\frac{\beta^D\vec{p}\cdot\vec{\alpha}^D}{p}\sin p\theta\right)(c\vec{p}\cdot\vec{\alpha}^D+\beta^D mc^2)\left(\cos p\theta-\frac{\beta^D\vec{p}\cdot\vec{\alpha}^D}{p}\sin p\theta\right)\\&=(c\vec{p}\cdot\vec{\alpha}^D+\beta^D mc^2)\left(\cos p\theta-\frac{\beta^D\vec{p}\cdot\vec{\alpha}^D}{p}\sin p\theta\right)^2\\&=(c\vec{p}\cdot\vec{\alpha}^D+\beta^D mc^2)\exp(-2\beta^D\vec{p}\cdot\vec{\alpha}^D\theta)\end{aligned}$$

$$=c\vec{p}\cdot\vec{\alpha}^D\left(\cos 2p\theta-\frac{mc}{p}\sin 2p\theta\right)+\beta^D(mc^2\cos 2p\theta+pc\sin 2p\theta) \tag{3.11}$$

$$\equiv\beta^D(m^2c^4+p^2c^2)^{1/2} \tag{3.12}$$

$$\equiv\beta^D E_p\,, \tag{3.13}$$

where (3.12) follows from (3.11) by using standard trigonometry and defining $\theta(\vec{p})$ as

$$\tan[2p\theta(\vec{p})]\equiv\frac{p}{mc}\,. \tag{3.14}$$

With U defined by Eqs. (3.9) and (3.14), it is an algebraic exercise to verify that[101]

$$\begin{aligned}\vec{S}^{FW}&=U\vec{S}U^{-1}\\&=\vec{S}+\frac{i\beta^D(\vec{\alpha}^D\times\vec{p})c}{2E_p}+\frac{\vec{p}\times(\vec{S}\times\vec{p})c^2}{E_p(E_p+m)}\,,\end{aligned} \tag{3.15}$$

and, with the aid of (2.18),

$$\begin{aligned}\vec{x}^{FW}&=U\vec{x}U^{-1}\\&=\vec{x}-\frac{i\beta^D\vec{\alpha}^D c}{2E_p}-\frac{i\beta^D(\vec{p}\cdot\vec{\alpha}^D)\vec{p}c^3}{2E_p^2(E_p+mc^2)}\\&\quad+\frac{(\vec{S}\times\vec{p})c^2}{E_p(E_p+mc^2)}\,.\end{aligned} \tag{3.16}$$

Given (3.15) and (3.16) one can calculate the rest of the FW-transformed generators as[102]

$$P_j^{FW}=P_j\,, \tag{3.17}$$

$$J_j^{FW}=UJ_jU^{-1}=J_j\,, \tag{3.18}$$

$$\begin{aligned}\vec{K}^{FW}&=U\vec{K}U^{-1}\\&=\frac{1}{2c^2}(\vec{x}^{FW}H^{FW}+H^{FW}\vec{x}^{FW})-t\vec{p}\\&=\frac{\beta^D}{2c^2}(\vec{x}E_p+E_p\vec{x})-t\vec{p}-\frac{\beta^D(\vec{S}\times\vec{p})}{(E_p+mc^2)}\end{aligned} \tag{3.19}$$

$$=\frac{\beta^D}{2c^2}(\vec{x}E_p)-t\vec{p}-\frac{i\beta^D\vec{p}}{2E_p}-\frac{\beta^D(\vec{S}\times\vec{p})}{(E_p+mc^2)}\,. \tag{3.20}$$

It is straightforward[102] to verify that the transformed generators (3.13) and (3.17)–(3.20) satisfy the Lie algebra (1.5).

B. The DKP case $(\mathcal{S},S)=(1,1)$ or $(1,0)$

As emphasized in the Introduction, because one wants metric-unitary FW transformations, when discussing integer-spin systems one should deal with the particle-components generators. For the special DKP case, the particle-components Poincaré generators are obtained by substituting

$$\alpha_\mu = \beta_\mu\,, \quad \chi = m \tag{3.21}$$

into Eqs. (2.23)–(2.26), to yield, as in Eqs. (III 4.11)–(III 4.14),

$$P_j^{(P)} = P_j\beta_4^{\,2}\,, \tag{3.22}$$

$$J_j^{(P)} = J_j\beta_4^{\,2}\,, \tag{3.23}$$

$$H^{(P)} = mc^2\beta_4 + \beta_4(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})m^{-1}\,, \tag{3.24}$$

$$H^{(P)} = mc^2\beta_4 + \beta_4\left(\frac{1+\eta}{2m}\right)\vec{p}\cdot\vec{p} - \beta_4\eta m^{-1}(\vec{S}\cdot\vec{p})^2\,, \tag{3.25}$$

$$K_i^{(P)} = c^{-2}x_iH^{(P)} - tP_i^{(P)} - i\beta_4\beta_i(\vec{p}\cdot\vec{\beta})(mc^2)^{-1}\,, \tag{3.26}$$

$$K_i^{(P)} = \frac{1}{2c^2}\{x_i, H^{(P)}\} - tP_i^{(P)} + \frac{i\beta_4}{2mc^2}[\vec{p}\cdot\vec{\beta},\beta_i]\,, \tag{3.27}$$

where the specific DKP algebra is

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\lambda} + \beta_\lambda\delta_{\nu\mu}\,, \tag{3.28}$$

$$\eta = \eta_1\eta_2\eta_3\,, \quad \eta_\lambda = 2\beta_\lambda^{\,2} - 1\,, \tag{3.29}$$

$$(1-\mathcal{G}_0) = \beta_4^{\,2}\,, \quad S_k = -i\epsilon_{ijk}\beta_i\beta_j\,. \tag{3.30}$$

As pointed out for the Hamiltonian by Sakata and Taketani,[31–36] the above operators can be put into "Pauli form." This is discussed in Refs. 31–36 and Sec. IV A of I.[1] Essentially it is the observation that with a Pauli (τ) space for particle-antiparticle and an orthogonal spin space, one has the required $2(2S+1)$ equals 2 or 6 components for the spin-0 and spin-1 DKP systems. The Pauli space is given by the association

$$\begin{gathered}(1-\mathcal{G}_0)\sim \mathrm{I}\,, \quad (1-\mathcal{G}_0)(-i\beta_4)(1-\mathcal{G}_0)^2\eta(1-\mathcal{G}_0)\sim\tau_2\,,\\ (1-\mathcal{G}_0)\eta(1-\mathcal{G}_0)\sim\tau_1\,, \quad (1-\mathcal{G}_0)\beta_4(1-\mathcal{G}_0)\sim\tau_3\,.\end{gathered} \tag{3.31}$$

The operators in (3.31) satisfy the Pauli algebra

$$\tau_i\tau_j = \delta_{ij} + i\epsilon_{ijk}\tau_k\,. \tag{3.32}$$

Also, since $(1-\mathcal{G}_0)$ commutes with the S_k of (3.30) and because of the DKP algebra (3.28), the Pauli form S_k retains the commutation relations

$$S_iS_jS_k + S_kS_jS_i = S_i\delta_{jk} + S_k\delta_{ij}\,. \tag{3.33}$$

Putting (3.31)–(3.33) into the generators (3.22)–(3.27) gives the Pauli form of the DKP particle-components generators

$$P_j^{(\tau)} = P_j\,, \tag{3.34}$$

$$\vec{J}^{(\tau)} = (\vec{r}\times\vec{p}) + \vec{S}\,, \tag{3.35}$$

$$H^{(\tau)} = mc^2\tau_3 + \frac{(\tau_3+i\tau_2)}{2m}p^2 - \frac{i\tau_2}{m}(\vec{p}\cdot\vec{S})^2\,, \tag{3.36}$$

$$\vec{K}^{(\tau)} = \frac{\vec{x}H^{(\tau)}}{c^2} - t\vec{p} + \frac{i(\tau_3+i\tau_2)}{2mc^2}\vec{p} + \frac{\tau_2(\vec{p}\cdot\vec{S})\vec{S}}{2mc^2} - \frac{\tau_3(\vec{S}\times\vec{p})}{2mc^2} \tag{3.37}$$

$$= \frac{1}{2c^2}\{\vec{x}, H^{(\tau)}\} - t\vec{p} - \frac{\tau_3(\vec{S}\times\vec{p})}{2mc^2}\,. \tag{3.38}$$

To diagonalize the Hamiltonian,[28–30] we note that τ_3 is the even operator and τ_2 is the odd operator, meaning, as in the magnetic field analogy of Ref. 99, we expect the FW transformation to be of the form

$$\begin{aligned}\Theta^{\mathrm{FW}} &= U\Theta U^{-1}\\ &= \exp(-i\tau_1\phi/2)\Theta\exp(i\tau_1\phi/2)\,,\end{aligned} \tag{3.39a}$$

$$\Theta^{\mathrm{FW}} = \sum_{n=0}^{\infty}(-\tfrac{1}{2}i)^n\frac{\phi^n}{n!}([\tau_1,)^n\Theta(])^n\,, \tag{3.39b}$$

$$U^{-1} = \cos(\phi/2) + i\tau_1\sin(\phi/2)\,. \tag{3.40}$$

Equation (3.39b) is a special case of theorem I of Ref. 53, and we are using the notation that the quantities in the parentheses are to be written out the number of times indicated in the superscripts. For example,

$$([\tau_1,)^3\Theta(])^3 \equiv [\tau_1,[\tau_1,[\tau_1,\Theta]]]\,. \tag{3.41}$$

Using (3.32) and the derivative results

$$([\tau_1,)^{2n+1}\tau_k(])^{2n+1} = (2)^{2n}[\tau_1,\tau_k]\,, \quad n=0,1,2,\ldots \tag{3.42}$$

$$([\tau_1,)^{2n+2}\tau_k(])^{2n+2} = (2)^{2n}[\tau_1,[\tau_1,\tau_k]]\,, \quad n=0,1,2,\ldots \tag{3.43}$$

yields

$$\begin{aligned}\tau_2^{\mathrm{FW}} &= \tau_2 + \sum_{n=0}^{\infty}\frac{(-i)^{2n+1}\phi^{2n+1}}{2^{2n+1}(2n+1)!}(2)^{2n}[\tau_1,\tau_2] + \sum_{n=0}^{\infty}\frac{(-i)^{2n+1}\phi^{2n+2}}{2^{2n+2}(2n+2)!}(2)^{2n}[\tau_1,[\tau_1,\tau_2]]\\ &= \tau_2 - i\sum_{n=0}^{\infty}\frac{(i\phi)^{2n+1}}{(2n+1)!}\tau_3 + \sum_{n=0}^{\infty}\frac{(i\phi)^{2n+2}}{(2n+2)!}\tau_2\\ &= \tau_2\cos\phi + \tau_3\sin\phi\,.\end{aligned} \tag{3.44}$$

Similarly,

$$\tau_3^{\mathrm{FW}} = \tau_3 \cos\phi - \tau_2 \sin\phi. \quad (3.45)$$

Thus

$$H^{\mathrm{FW}} = \tau_3 \left[\left(mc^2 + \frac{p^2}{2m} \right) \cos\phi + i \left(\frac{p^2}{2m} - \frac{(\vec{p}\cdot\vec{S})^2}{m} \right) \sin\phi \right] + \tau_2 \left[i \left(\frac{p^2}{2m} - \frac{(\vec{p}\cdot\vec{S})^2}{m} \right) \cos\phi - \left(mc^2 + \frac{p^2}{2m} \right) \sin\phi \right]. \quad (3.46)$$

Letting

$$\tan\phi = \frac{i[p^2/2m - (\vec{p}\cdot\vec{S})^2/m]}{(mc^2 + p^2/2m)} \equiv \frac{\sin\phi}{\cos\phi} \quad (3.47)$$

means

$$\cos\phi = \frac{+(mc^2 + p^2/2m)}{E_p},$$

$$\sin\phi = \frac{+i[p^2/2m - (\vec{p}\cdot\vec{S})^2/m]}{E_p}. \quad (3.48)$$

With a little algebra, (3.46) and (3.47) imply

$$H^{\mathrm{FW}} = \tau_3 E_p \equiv \tau_3 (m^2c^4 + p^2c^2)^{1/2}. \quad (3.49)$$

In both the equalities of (3.48) there is an overall minus sign which is arbitrary, but once chosen must be consistently held to.[103] Given Eqs. (3.39) and (3.47), one can determine

$$\vec{x}^{\mathrm{FW}} = U\vec{x}U^{-1} = \vec{x} - \frac{(E_p - mc^2)(\vec{p}\times\vec{S})}{2mE_p(E_p + mc^2)} - \tau_1 \left[\frac{i\vec{p}c^2}{2E_p^2} + \frac{i\vec{p}(\vec{p}\cdot\vec{S})^2c^2}{mE_p^2(E_p + mc^2)} - \frac{i\{\vec{S}, \vec{p}\cdot\vec{S}\}}{mE_p} \right], \quad (3.50)$$

and, with the aid of the spin algebra (3.33), one also finds that

$$\vec{S}^{\mathrm{FW}} = \vec{S}\frac{(mc^2 + p^2/2m)}{E_p} - \frac{(E_p - mc^2)(\vec{p}\cdot\vec{S})\vec{p}}{2mE_p(E_p + mc^2)} + \frac{i\tau_1}{2mE_p}\{(\vec{p}\times\vec{S}), \vec{p}\cdot\vec{S}\}. \quad (3.51)$$

With the aid of these results one can calculate the Poincaré generators $\vec{J}^{\mathrm{FW}}$ and $\vec{K}^{\mathrm{FW}}$,

$$\vec{J}^{\mathrm{FW}} = \vec{J}^{(\tau)} = (\vec{x}\times\vec{p}) + \vec{S}, \quad (3.52)$$

$$\vec{K}^{\mathrm{FW}} = \frac{\tau_3}{2c^2}\{\vec{x}, E_p\} - t\vec{p} - \frac{\tau_3(\vec{S}\times\vec{p})}{(E_p + mc^2)}, \quad (3.53)$$

$$\vec{K}^{\mathrm{FW}} = \tau_3 \vec{x} E_p/c^2 - t\vec{p} - \tau_3 \left[\frac{i\vec{p}_k}{2E_p} + \frac{(\vec{S}\times\vec{p})}{(E_p + mc^2)} \right], \quad (3.54)$$

and also, of course,

$$P_j^{\mathrm{FW}} = P_j^{(\tau)} = p_j. \quad (3.55)$$

The FW generators (3.49) and (3.52)–(3.55) are all even and satisfy the Lie algebra (1.5). Thus the transformation of (3.49) and (3.47) has completely decoupled the positive- and negative-energy mass states of the DKP system. To see that it is a metric unitary transformation, note that in the Pauli form

$$M = \eta_4\beta_4 = \beta_4 \sim \tau_3. \quad (3.56)$$

This means that

$$U^\dagger M U = M, \quad (3.57)$$

implying U is metric-unitary, or equivalently

$$(MS)^\dagger = MS, \quad (3.58)$$

implying S is metric-Hermitian.

IV. ORDER-BY-ORDER FW TRANSFORMATIONS

A. Half-integer spin

The FW transformation (3.9) and (3.14) for the Dirac system was obtained in closed form with the aid of the specific Dirac algebra. In particular, because of the algebra (3.2), the FW transformation (3.9) could be expressed as two terms. For the arbitrary Bhabha field this is no longer true. The algebra is of order $(2\mathcal{S}+1)$, meaning more terms are involved before they begin to close with multiplicative coefficients. Although such a method is in principle doable, the exact transformation turns out to be elegantly calculable from a study of the eigenvectors, as we will show in VI.[82] However, it still is informative to obtain the transformation for arbitrary Bhabha fields as a power-series expansion to some order; the physics is more transparent in the power-series expansion.

From Sec. II the FW transformation which diagonalizes the Hamiltonian will also transform the other Poincaré generators to mass-block diagonal form. Further, one knows what the form of the diagonal Hamiltonian H^{FW} is.[104] For all the mass states, it will be the particular energy (in terms of the three-momentum and mass) times a diagonal matrix which will be unity, $\mathscr{I}_j^+(\mathcal{S})$, for that particular particle mass-block state and minus unity, $-\mathscr{I}_j^-(\mathcal{S})$, for the antiparticle state. Thus the idea will be to first find the transformation which diagonalizes H to H^{FW} to a particular order. Then, since we can also show[104] what the other FW generators are, we will show that the U which diagonalizes H will also transform all the generators (2.8)–(2.12) to the mass-block diagonal FW generators to order c^{-3}.

The general half-integer-spin FW generators should reduce to the Dirac case and have the same properties. Specifically looking at the Dirac FW generators (3.13) and (3.17)–(3.20), one first wants a Hamiltonian which is diagonal and has the properties enunciated in the last paragraph. Further, the momentum and angular momentum should remain in the ordinary mass-block diagonal form, and the boost generators should have the type of relation to the spin and momentum as in (3.19) and (3.20). First defining

$$\mathcal{E}_p \equiv (c^2p^2\alpha_4^{\,2} + \chi^2c^4)^{1/2}, \tag{4.1}$$

these properties are maintained by the following set of generators[104,105]:

$$H^{\rm FW} = \alpha_4^{-1}(c^2p^2\alpha_4^{\,2} + \chi^2c^4)^{1/2} \equiv \alpha_4^{-1}\mathcal{E}_p, \tag{4.2}$$

$$\vec{\rm P}^{\rm FW} = \vec{\rm p}, \tag{4.3}$$

$$\vec{\rm J}^{\rm FW} = \vec{\rm x}\times\vec{\rm p} + \vec{\rm S}, \quad S_k \equiv -i\epsilon_{ijk}\,\alpha_j\,\alpha_k\ , \tag{4.4}$$

$$\vec{\rm K}^{\rm FW} = \vec{\rm x}H^{\rm FW}/c^2 - t\vec{\rm p} + \left(\frac{-i\vec{\rm p}\alpha_4}{2\mathcal{E}_p}\right) + \left(\frac{-(\vec{\rm S}\times\vec{\rm p})\alpha_4}{\mathcal{E}_p + \chi c^2}\right) \tag{4.5}$$

$$= \frac{1}{2c^2}\{\vec{\rm x}, H^{\rm FW}\} - t\vec{\rm p} - \frac{(\vec{\rm S}\times\vec{\rm p})\alpha_4}{\mathcal{E}_p + \chi c^2}. \tag{4.6}$$

The generators (4.2)–(4.6) do reduce to the Dirac FW generators (3.13) and (3.17)–(3.20) in the $(\frac{1}{2}, \frac{1}{2})$ representation and they satisfy the Lie algebra (1.5). To help see that they are the FW-transformed generators, we now perform an expansion in c^{-1} to third order. Taking the metric-unitary transformation U as

$$U = e^{iS}, \tag{4.7}$$

meaning S (not to be confused with the spin) is metric-Hermitian, we have from lemma I of Ref. 53 that

$$U\mathcal{O}U^{-1} = \sum_{n=0}^{\infty}\frac{i^n}{n!}([S,\ )^n\mathcal{O}(])^n = \mathcal{O}^{\rm FW}. \tag{4.8}$$

Thus if

$$S \equiv \sum_{n=0}^{\infty} S_n c^{-n}, \quad S_0 \equiv 0 \tag{4.9}$$

where the S_n should not be confused with the components of the spin, then from the form of H,

$$H = (\alpha_4^{-1}\chi)c^2 + (\alpha_4^{-1}i\vec{\rm p}\cdot\vec{\alpha})c \equiv ec^2 + oc\ , \tag{4.10}$$

where "e" and "o" signify the generalization of "even" and "odd," one obtains to third order in c^{-1} that

$$(\alpha_4^{-1}\chi)c^2 = H^{\rm FW}(c^2) = (\alpha_4^{-1}\chi)c^2\ , \tag{4.11}$$

$$0 = H^{\rm FW}(c)$$
$$= (\alpha_4^{-1}i\vec{\rm p}\cdot\vec{\alpha})c + i[S_1, \alpha_4^{-1}\chi]c\ , \tag{4.12}$$

$$\frac{p^2\alpha_4}{2\chi} = H^{\rm FW}(c^0)$$
$$= i[S_2, \alpha_4^{-1}\chi] + i[S_1, \alpha_4^{-1}i\vec{\rm p}\cdot\vec{\alpha}] + \frac{i^2}{2!}[S_1,[S_1, \alpha_4^{-1}\chi]], \tag{4.13}$$

$$0 = H^{\rm FW}(c^{-1})$$
$$= i[S_3, \alpha_4^{-1}\chi]c^{-1} + i[S_2, \alpha_4^{-1}i\vec{\rm p}\cdot\vec{\alpha}]c^{-1} + \frac{i^2}{2!}[S_2,[S_1, \alpha_4^{-1}\chi]]c^{-1} + \frac{i^2}{2!}[S_1,[S_2, \alpha_4^{-1}\chi]]c^{-1} + \frac{i^2}{2!}[S_1,[S_1, \alpha_4^{-1}i\vec{\rm p}\cdot\vec{\alpha}]]c^{-1} + \frac{i^3}{3!}[S_1,[S_1,[S_1, \alpha_4^{-1}\chi]]]c^{-1}. \tag{4.14}$$

The metric-Hermitian solutions to the above equations are

$$S_1 = \frac{[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4]\alpha_4}{\chi}, \tag{4.15}$$

$$S_2 = \frac{-i}{2\chi^2}\vec{\rm p}\cdot\vec{\alpha}[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4]\alpha_4 + \frac{ip^2\alpha_4^{\,2}}{4\chi^2}, \tag{4.16}$$

$$S_3 = -\frac{1}{24\chi^3}\{8(\vec{\rm p}\cdot\vec{\alpha})^2[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4] + 2[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4]^3 + 5\alpha_4^{\,2}[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4]p^2 - 3[\vec{\rm p}\cdot\vec{\alpha}, \alpha_4]\alpha_4^{\,2}p^2\}\alpha_4. \tag{4.17}$$

We emphasize metric-Hermitian solutions because of the following: The S_1 of Eq. (4.15) is the (already metric-Hermitian) solution of Eq. (4.12). But given that S_1, then just the first term of (4.16) would be a solution to (4.13). However, that term by itself would not give a metric-Hermitian S_2. The second term is needed for that, and then S_2 also reduces to the Dirac special case solution. The above S_1, S_2, and S_3 all, as they should, reduce to the power-series expansion of the Dirac transformation (3.9),

$$S_1^D = \frac{-i\beta^D\vec{\rm p}\cdot\vec{\alpha}^D}{2m}, \tag{4.18}$$

$$S_2^D = 0, \tag{4.19}$$

$$S_3^D = \frac{i\beta^D(\vec{\alpha}^D\cdot\vec{\rm p})p^2}{6m^3}. \tag{4.20}$$

Given the above transformation to order c^{-3}, one can check to see how it transforms the other Poincaŕe generators. First, since from (4.2), (4.8), and (4.10) U will only be a function of p^2, $\vec{\rm p}\cdot\vec{\alpha}$, α_4, χ, and c, one can say to all orders in c^{-1} that

$$\vec{P}^{FW} = U\vec{P}U^{-1} = \vec{P} = \vec{p} \; . \quad (4.21)$$

In general, if $\mathcal{O}$ is of a definite order of c^{-1},

$$U\mathcal{O}U^{-1} \equiv \mathcal{O} + \frac{\Delta\mathcal{O}_1}{c} + \frac{\Delta\mathcal{O}_2}{c^2} + \frac{\Delta\mathcal{O}_3}{c^3} + \cdots, \quad (4.22)$$

$$\Delta\mathcal{O}_1 = i[S_1, \mathcal{O}], \quad (4.23)$$

$$\Delta\mathcal{O}_2 = i[S_2, \mathcal{O}] + \frac{i^2}{2!}[S_1,[S_1,\mathcal{O}]], \quad (4.24)$$

$$\Delta\mathcal{O}_3 = i[S_3, \mathcal{O}] + \tfrac{1}{2}i[S_2, \Delta\mathcal{O}_1] + \tfrac{1}{2}i[S_1, \Delta\mathcal{O}_2] + \tfrac{1}{12}[S_1,[S_1,\Delta\mathcal{O}_1]]. \quad (4.25)$$

From the above,

$$\Delta\vec{x}_1 = \frac{[\vec{\alpha}, \alpha_4]\alpha_4}{\chi}, \quad (4.26)$$

$$\Delta\vec{x}_2 = \frac{-i}{2\chi^2}\{\vec{\alpha}, [\vec{p}\cdot\vec{\alpha}, \alpha_4]\}\alpha_4 + \frac{\vec{S}\times\vec{p}}{2\chi^2}\alpha_4^{\,2}, \quad (4.27)$$

$$\Delta\vec{x}_3 = -\frac{1}{2\chi^3}\{[\vec{p}\cdot\vec{\alpha}, \alpha_4]^2[\vec{\alpha}, \alpha_4]\alpha_4 + [(\vec{p}\cdot\vec{\alpha})^2, \alpha_4]\vec{\alpha}\alpha_4 + [\vec{p}\cdot\vec{\alpha}, \alpha_4]\alpha_4^{\,3}\vec{p}\} - \frac{p^2}{4\chi^3}[\vec{\alpha}, \alpha_4]\alpha_4, \quad (4.28)$$

which again agrees with the Dirac results

$$\Delta\vec{x}_1^D = \frac{-i\beta^D\vec{\alpha}^D}{2m}, \quad (4.29)$$

$$\Delta\vec{x}_2^D = \frac{-\vec{S}\times\vec{p}}{2m^2}, \quad (4.30)$$

$$\Delta\vec{x}_3^D = \frac{i\beta^D(\vec{\alpha}^D\cdot\vec{p})\vec{p}}{4m^3} + \frac{i\beta^D(\vec{\alpha}^D)p^2}{4m^3}\; . \quad (4.31)$$

By explicit calculation one can find that the transformation of the spin operator is given by

$$\Delta\vec{S}_n = -\Delta\vec{x}_n \times \vec{p}, \quad n = 1, 2, 3. \quad (4.32)$$

In fact, (4.32) is true in general, as can be seen from the following argument. Consider the transformation of the angular momentum generator. We have with (4.21) that

$$\vec{J}^{FW} = \vec{x}^{FW} \times \vec{p} + \vec{S}^{FW}. \quad (4.33)$$

However, as stated above, from the set of equations (4.2) and (4.7)-(4.10) which define the solutions to the transformations S_n, the S_n will only be functions of p^2, $\vec{\alpha}\cdot\vec{p}$, α_4, and χ. That is, the S_n are scalar functions in three-space. Thus

$$[\vec{J}, S_n] = 0 \quad (4.34)$$

to all orders so that

$$\vec{J}^{FW} = \vec{J}, \quad (4.35)$$

which also means that

$$\Delta\vec{S}_n = -\Delta\vec{x}_n \times \vec{p}, \quad \text{all } n. \quad (4.36)$$

Finally, for the transformation of the boost generators, we have in terms of $(\vec{t}_4)_j \equiv t_{4j}$

$$\vec{K}^{FW} = \vec{x}^{FW}H^{FW}/c^2 - t\vec{p} + \vec{t}_4^{\,FW}/c, \quad (4.37)$$

meaning that we need $\Delta(\vec{t}_4)_n$ for $n = 1, 2$. Direct calculation yields

$$\Delta(\vec{t}_4)_1 = \frac{-i}{\chi}\left\{-[\vec{p}\cdot\vec{\alpha}, \alpha_4]\vec{\alpha} + [\vec{\alpha}, \vec{p}\cdot\vec{\alpha}]\alpha_4\right\}, \quad (4.38)$$

$$\Delta(\vec{t}_4)_2 = \frac{1}{2\chi^2}\left\{[\vec{p}\cdot\vec{\alpha}, \alpha_4]^2[\vec{\alpha}, \alpha_4] + [(\vec{p}\cdot\vec{\alpha})^2, \alpha_4]\vec{\alpha} + [\vec{p}\cdot\vec{\alpha}, \alpha_4]\alpha_4^{\,2}\vec{p} - [\vec{\alpha}, \alpha_4]\alpha_4^{\,2}p^2 + \tfrac{1}{2}p^2[\vec{\alpha}, \alpha_4]\right\}. \quad (4.39)$$

Combining Eqs. (4.11)–(4.14), (4.26)–(4.28), and (4.37)–(4.39) gives to order c^{-3}

$$\vec{K}^{FW} = \frac{1}{c^2}\vec{x}\left(\alpha_4^{-1}\chi c^2 + \frac{p^2}{2\chi}\alpha_4\right) - t\vec{p} + \frac{1}{c}\delta\vec{k}_1 + \frac{1}{c^2}\delta\vec{k}_2 + \frac{1}{c^3}\delta\vec{k}_3, \quad (4.40)$$

$$\delta\vec{k}_1 = \Delta\vec{x}_1\alpha_4^{-1}\chi + \vec{t}_4 = 0, \quad (4.41)$$

$$\delta\vec{k}_2 = \Delta\vec{x}_2\alpha_4^{-1}\chi + \Delta(\vec{t}_4)_1 = -\left[\frac{(\vec{S}\times\vec{p}) + i\vec{p}}{2\chi}\right]\alpha_4, \quad (4.42)$$

$$\delta\vec{k}_3 = \frac{\Delta\vec{x}_1 p^2\alpha_4}{2\chi} + \Delta\vec{x}_3\alpha_4^{-1}\chi + \Delta(\vec{t}_4)_2 = 0. \quad (4.43)$$

The above agrees with the exact $\vec{K}^{FW}$ up to third order in c^{-1}.

Thus we have explicitly verified up to third order in c^{-1} that the FW transformation which diagonalizes the Hamiltonian (2.10) to the form (4.2) also mass-block diagonalizes the rest of the Poincaré generators (2.8), (2.9), and (2.11) to the forms (4.3)–(4.5).

B. Integer spin

For the particle components of the integer-spin case, the discussion proceeds as for the half-integer-spin case, modulo the complications of the removal of the subsidiary components with the aid of the particle-components projection operators $\mathcal{G}^{(P)} \equiv (1 - \mathcal{G}_0(8))$ and the fact that in place of α_4^{-1} one deals with the operators $Q = Q\mathcal{G}^{(P)}$. The FW-transformed generators must then have the same set of properties as for half-integer spin, and this leads to the set of FW generators[104]

$$H^{(P)FW} = Q(p^2c^2\alpha_4^{\,2} + \chi^2c^4)^{1/2}(1 - \mathcal{G}_0) \equiv Q\mathcal{E}_p\mathcal{G}^{(P)}, \quad (4.44)$$

$$\vec{P}^{(P)\mathrm{FW}} = \vec{P}(1-\mathcal{G}_0) = \vec{p}(1-\mathcal{G}_0) = \vec{p}\mathcal{G}^{(P)}, \quad (4.45)$$

$$\vec{J}^{(P)\mathrm{FW}} = \vec{J}(1-\mathcal{G}_0) = (\vec{r}\times\vec{p}+\vec{S})(1-\mathcal{G}_0), \quad (4.46)$$

$$\vec{K}^{(P)\mathrm{FW}} = \left\{\frac{1}{c^2}\vec{x}H^{(P)\mathrm{FW}} - t\vec{p} + \left(-\frac{i\vec{p}\alpha_4}{2\mathcal{E}_p}\right) + \left[\frac{-(\vec{S}\times\vec{p})\alpha_4}{\mathcal{E}_p+\chi c^2}\right]\right\}(1-\mathcal{G}_0). \quad (4.47)$$

As they should, the generators (4.44)–(4.47) reduce to the DKP FW generators (3.49) and (3.52)–(3.55) in the $(\mathcal{S},S)=(1,1)$ and $(1,0)$ representations $(\beta_4 \to \tau_3, \beta_4{}^2 \to (1-\mathcal{G}_0) \to I)$, and they also satisfy the Lie algebra (1.5). The projection operator $(1-\mathcal{G}_0)$ is the continual reminder that we are working with the particle components, the subsidiary components having now been removed. With the exception of the operators $(1-\mathcal{G}_0)$ and Q, which is the particle-components analog of $\alpha_4{}^{-1}$, the above generators are functionally the same as the FW half-integer-spin generators. This functional similarity will continue below.

We will now derive to order c^{-3} the transformation from the particle-components generators (2.23)–(2.26) to the FW generators (4.44)–(4.47). For now we simply note that this transformation will have a close functional resemblance to the half-integer-spin transformation. In the next section we will show that this functional dependence is true to all orders, and in fact one can use this dependence to derive the exact particle-components integer-spin transformation in terms of the exact half-integer-spin transformation results.

Defining the particle-components FW transformation as

$$H^{(P)\mathrm{FW}} = \hat{U}H^{(P)}\hat{U}^{-1}, \quad (4.48)$$

$$\hat{U} = \mathcal{G}^{(P)}\exp(i\hat{S})\mathcal{G}^{(P)} = \mathcal{G}^{(P)}\exp\left(i\sum_{n=1}^{\infty}\hat{S}_n c^{-n}\right)\mathcal{G}^{(P)}, \quad (4.49)$$

then with the aid of (4.8) and the $H^{(P)}$ in (2.25) we obtain, similarly to the half-integer-spin case, the equations

$$(1-\mathcal{G}_0)(Q\chi)c^2 = H^{(P)\mathrm{FW}}(c^2) = (Q\chi)c^2(1-\mathcal{G}_0), \quad (4.50)$$

$$0 = H^{(P)\mathrm{FW}}(c) = \mathcal{G}^{(P)}\{(Qi\vec{p}\cdot\vec{\alpha})(1-\mathcal{G}_0)c + i[\hat{S}_1, Q\chi(1-\mathcal{G}_0)]c\}\mathcal{G}^{(P)}, \quad (4.51)$$

$$\begin{aligned}\frac{p^2\alpha_4(1-\mathcal{G}_0)}{2\chi} &= H^{(P)\mathrm{FW}}(c^0)\\ &= \mathcal{G}^{(P)}\Big\{i[\hat{S}_2, Q(1-\mathcal{G}_0)\chi] + i[\hat{S}_1, Q(1-\mathcal{G}_0)i\vec{p}\cdot\vec{\alpha}(1-\mathcal{G}_0)] + \frac{i^2}{2!}[\hat{S}_1,[\hat{S}_1, Q(1-\mathcal{G}_0)\chi]]\\ &\quad + (1-\mathcal{G}_0)Q(\vec{p}\cdot\vec{\alpha})\mathcal{G}_0(\vec{p}\cdot\vec{\alpha})(1-\mathcal{G}_0)\chi^{-1}\Big\}\mathcal{G}^{(P)}, \end{aligned} \quad (4.52)$$

$$\begin{aligned} 0 &= H^{(P)\mathrm{FW}}(c^{-1})\\ &= \mathcal{G}^{(P)}\Big\{i[\hat{S}_3, Q(1-\mathcal{G}_0)\chi]c^{-1} + i[\hat{S}_2, Q(1-\mathcal{G}_0)i\vec{p}\cdot\vec{\alpha}(1-\mathcal{G}_0)]c^{-1}\\ &\quad + \frac{i^2}{2!}[\hat{S}_2,[\hat{S}_1, Q(1-\mathcal{G}_0)\chi]]c^{-1} + \frac{i^2}{2!}[\hat{S}_1,[\hat{S}_2, Q(1-\mathcal{G}_0)\chi]]c^{-1} + \frac{i^2}{2!}[\hat{S}_1,[\hat{S}_1, Q(1-\mathcal{G}_0)i\vec{p}\cdot\vec{\alpha}(1-\mathcal{G}_0)]]c^{-1}\\ &\quad + \frac{i^3}{3!}[\hat{S}_1,[\hat{S}_1,[\hat{S}_1, Q(1-\mathcal{G}_0)\chi]]]c^{-1} + i[\hat{S}_1, (1-\mathcal{G}_0)Q(\vec{p}\cdot\vec{\alpha})\mathcal{G}_0(\vec{p}\cdot\vec{\alpha})(1-\mathcal{G}_0)\chi]\chi^{-1}\Big\}\mathcal{G}^{(P)}. \end{aligned} \quad (4.53)$$

The apparent differences between these equations and Eqs. (4.11)–(4.14) for half-integer spin are the last terms in Eqs. (4.52) and (4.53). These extra terms are due to the elimination of the subsidiary components, but can be accounted for by the projection operators $\mathcal{G}^{(P)} \equiv (1-\mathcal{G}_0)$. In fact, when one explicitly solves the equations, one finds that the metric-Hermitian solutions to (4.50)–(4.53) are, up to a phase,

$$\hat{S}_n(\vec{p},\vec{\alpha},\alpha_4,\chi) = (1-\mathcal{G}_0)S_n(\vec{p},\vec{\alpha},\alpha_4,\chi)(1-\mathcal{G}_0), \quad (4.54)$$

$$n = 1, 2, 3$$

where the S_n are functionals with the exact form as the half-integer-spin transformations S_n of Eqs. (4.15)–(4.16). In the next section we will show that (4.54) holds for all n. As is necessary, the results (4.54) are equal to the first three terms in the power-series expansion of the DKP FW transformation (3.39) and (3.47),

$$\hat{S}_1^{\mathrm{DKP}} = 0, \quad (4.55)$$

$$\hat{S}_2^{\mathrm{DKP}} = \frac{i}{m}\left(\frac{p^2}{2m} - \frac{(\vec{p}\cdot\vec{\beta})^2}{m}\right)\beta_4{}^2 - \frac{i\eta}{2m}\left(\frac{p^2}{2m} - \frac{(\vec{p}\cdot\vec{S})^2}{m}\right) = \frac{-\eta\phi_2}{2}, \quad (4.56)$$

$$\phi \equiv \sum_{n=1}^{\infty} \phi_n c^{-n}, \tag{4.57}$$

$$\hat{S}_3^{\mathrm{DKP}} = 0. \tag{4.58}$$

It will be shown in the next section that if the particle components of an operator in an integer-spin representation are designated by $\hat{\Theta} = \Theta^{(P)}$ and in a half-integer-spin representation by Θ, and if

$$[\Theta^{(P)}, \alpha_4] \equiv [\hat{\Theta}, \alpha_4] = 0, \quad [\Theta, \alpha_4] = 0, \tag{4.59}$$

then the power-series expansion of $\hat{\Theta}$,

$$\hat{U}\hat{\Theta}\hat{U}^{-1} = \hat{\Theta} + \frac{\Delta\hat{\Theta}_1}{c} + \frac{\Delta\hat{\Theta}_2}{c^2} + \cdots, \tag{4.60}$$

can be functionally related to the power-series expansion (4.22)–(4.25) of Θ as

$$\Delta\hat{\Theta}_n = (1 - \mathcal{I}_0)\Delta\Theta_n(1 - \mathcal{I}_0) = \Delta\Theta_n(1 - \mathcal{I}_0), \quad \text{all } n. \tag{4.61}$$

This implies that

$$\Delta\vec{x}_n^{(P)} = \Delta\vec{x}_n(1 - \mathcal{I}_0), \quad \text{all } n \tag{4.62}$$

and in particular for the $n = 1, 2, 3$ of Eqs. (4.26)–(4.28). Similarly, following the discussion leading up to Eq. (4.36), one can exactly say that

$$\Delta\vec{S}_n^{(P)} = -\Delta\vec{x}_n^{(P)} \times \vec{p}, \quad \text{all } n \tag{4.63}$$

which yields the exact results

$$\Delta\vec{S}^{(P)} = -\Delta\vec{x}^{(P)} \times \vec{p}, \tag{4.64}$$

$$\vec{J}^{(P)\mathrm{FW}} = \vec{J}^{(P)} = \vec{J}(1 - \mathcal{I}_0), \tag{4.65}$$

along with the trivial result

$$\vec{p}^{(P)\mathrm{FW}} = \vec{p}^{(P)} = \vec{p}(1 - \mathcal{I}_0). \tag{4.66}$$

Now it just remains to obtain $\vec{K}^{(P)\mathrm{FW}}$. From $\vec{K}^{(P)}$ of (2.26) we have in terms of $(\vec{t}_4)_j \equiv t_{4j}$

$$\vec{K}^{(P)\mathrm{FW}} = \frac{1}{c^2}(\vec{x}^{(P)\mathrm{FW}} H^{(P)\mathrm{FW}}) - t\vec{p}^{(P)\mathrm{FW}} + c^{-1}(\vec{t}_4)^{(P)\mathrm{FW}}, \tag{4.67}$$

where $(\vec{t}_4)^{(P)}$ can be written as

$$(\vec{t}_4)^{(P)} = [(1 - \mathcal{I}_0)[\alpha_4, \vec{\alpha}](1 - \mathcal{I}_0)] + \left[(1 - \mathcal{I}_0)\alpha_4\vec{\alpha}(-\mathcal{I}_0)\frac{ip \cdot \alpha}{c\chi}(1 - \mathcal{I}_0)\right] \equiv (\vec{t}_4(c^0))^{(P)} + c^{-1}(\vec{t}_4(c^{-1}))^{(P)}. \tag{4.68}$$

Already having $\vec{x}^{(P)\mathrm{FW}}$ and $H^{(P)\mathrm{FW}}$ to order c^{-3}, to finish obtaining all the terms of (4.67) to order c^{-3} we need to calculate $\Delta(\vec{t}_4)_1^{(P)}$ and $\Delta(\vec{t}_4)_2^{(P)}$. This calculation is more complicated than for the half-integer-spin case because, from the second line of (4.68), $(\vec{t}_4)^{(P)}$ has terms of order c^0 and c^{-1}. Thus, the Δ expansion similar to Eqs. (4.22)–(4.25) has two sets of terms. Keeping track of them, one finds that

$$(\vec{t}_4)^{(P)\mathrm{FW}} = (\vec{t}_4(c^0))^{(P)} + \frac{\Delta(\vec{t}_4)_1^{(P)}}{c} + \frac{\Delta(\vec{t}_4)_2^{(P)}}{c^2}, \tag{4.69}$$

$$\Delta(\vec{t}_4)_1^{(P)} = (\vec{t}_4(c^{-1}))^{(P)} + i[\hat{S}_1, (\vec{t}_4(c^0))^{(P)}], \tag{4.70}$$

$$\Delta(\vec{t}_4)_2^{(P)} = i[\hat{S}_1, (\vec{t}_4(c^{-1}))^{(P)}] + i[\hat{S}_2, (\vec{t}_4(c^0))^{(P)}] + \frac{i^2}{2!}[\hat{S}_1, [\hat{S}_1, (\vec{t}_4(c^0))^{(P)}]]. \tag{4.71}$$

Calculating these quantities in terms of the half-integer-spin functionals, one obtains after some algebra that

$$(\vec{t}_4(c^0))^{(P)} = (1 - \mathcal{I}_0)\vec{t}_4(1 - \mathcal{I}_0), \tag{4.72}$$

$$\Delta(\vec{t}_4)_1^{(P)} = (1 - \mathcal{I}_0)\Delta(\vec{t}_4)_1(1 - \mathcal{I}_0), \tag{4.73}$$

$$\Delta(\vec{t}_4)_2^{(P)} = (1 - \mathcal{I}_0)\Delta(\vec{t}_4)_2(1 - \mathcal{I}_0). \tag{4.74}$$

Combining all these results, we finally have to order c^{-3} that

$$\vec{K}^{(P)\mathrm{FW}} = \frac{1}{c^2}\vec{x}\left(Q\chi c^2 + \frac{p^2}{2\chi}\alpha_4\right)(1 - \mathcal{I}_0) - t\vec{p}(1 - \mathcal{I}_0) + \frac{1}{c}\delta\vec{k}_1^{(P)} + \frac{1}{c^2}\delta\vec{k}_2^{(P)} + \frac{1}{c^3}\delta\vec{k}_3^{(P)}, \tag{4.75}$$

$$\delta\vec{k}_1^{(P)} = (1 - \mathcal{I}_0)(\Delta\vec{x}_2 Q\chi + \vec{t}_4)(1 - \mathcal{I}_0) = 0, \tag{4.76}$$

$$\delta\vec{k}_2^{(P)} = (1 - \mathcal{I}_0)[\Delta\vec{x}_2 Q\chi + \Delta(\vec{t}_4)_1](1 - \mathcal{I}_0) = -(1 - \mathcal{I}_0)\left(\frac{\vec{S} \times \vec{p} + i\vec{p}}{2\chi}\right)\alpha_4(1 - \mathcal{I}_0), \tag{4.77}$$

$$\delta\vec{k}_3^{(P)} = (1 - \mathcal{I}_0)\left[\frac{\Delta\vec{x}_1 p^2\alpha_4}{2\chi} + \Delta\vec{x}_3 Q\chi + \Delta(\vec{t}_4)_2\right](1 - \mathcal{I}_0) = 0, \tag{4.78}$$

which agrees with (4.47) to order c^{-3}.

Thus we have again explicitly verified up to third order in c^{-1} that the FW transformation which diagonalizes the particle-components Hamiltonian (2.25) to the form (4.44) also mass-block diagonalizes the rest of the particle-components Poincaré generators (2.23), (2.24), and (2.26) to the forms (4.45)–(4.47).

V. FUNCTIONAL RELATIONSHIP OF PARTICLE-COMPONENTS INTEGER-SPIN OPERATORS TO HALF-INTEGER-SPIN OPERATORS

In Sec. IV we gave emphasis to the functional relationships encountered between the order-by-order particle-components integer-spin FW generators and the half-integer-spin operators. In fact, this relationship is general. If we take a half-integer-spin operator

$$\mathcal{O} = \mathcal{O}(\alpha_4^{-1}, \alpha_\mu), \tag{5.1}$$

then the analogous particle-components integer-spin operator is directly functionally obtained as

$$\mathcal{O}^{(P)} = (1 - \mathcal{g}_0)\mathcal{O}(Q, \alpha_\mu)\left(1 - \frac{i\mathcal{g}_0\vec{p}\cdot\vec{\alpha}}{\chi c}\right)(1 - \mathcal{g}_0). \tag{5.2}$$

The operation (5.2) is a combination of two simultaneous things, changing α_4^{-1} to Q in $\mathcal{O}$, and "surrounding" the operator $\mathcal{O}$ with the generalized Sakata-Taketani operators [as described in Eqs. (II6.12) and (III5.25)] to yield the integer-spin particle components.

Previously in this series, we obtained the particle components in two steps. First the entire integer-spin quantities were obtained, which include the integer-spin subsidiary components. Then the generalized Sakata-Taketani (ST) reduction was done on the entire integer-spin operators to obtain the particle-components integer-spin operators. Recall that the first step of obtaining the entire integer-spin quantities was nontrivial since α_4 does not have an inverse in integer-spin representations, owing to its zero eigenvalues in the subsidiary components. However, as discussed in II, by using the "decoupling equations" one could obtain the entire integer-spin quantities in terms of Q, which is the integer-spin analog of α_4^{-1} in the particle components but remains zero in the subsidiary components. These entire integer-spin quantities then had the subsidiary components removed by the generalized ST reduction, yielding the particle components.

The single operation (5.2) is equivalent to the above two-step procedure. In II and III, where we were interested in the properties of both the entire integer-spin operators and the particle-components integer-spin operators, it was advantageous to use the two-step procedure. However, for FW transformations we need to work in a nonsingular metric space,[52,53] meaning with the particle components, and so the procedure of (5.2) is more direct and transparent.

The procedure (5.2) is intuitively understandable by remembering that once one has disregarded the subsidiary components, the two sets of operators should be functionally related since, after all, the difference between the half-integer-spin and the integer-spin fields is algebraically just that they are different representations of the so(5) algebra.

One will note that all our results are in agreement with (5.2). For example: (i) The particle-components integer-spin Poincaré generators (2.23)–(2.26) are functionally related to the half-integer-spin Poincaré generators (2.8)–(2.12) by Eq. (5.2); (ii) the particle-components integer-spin FW Poincaré generators (4.44)–(4.47) are functionally related to the half-integer-spin FW Poincaré generators (4.2)–(4.6) by Eq. (5.2); and (iii) of course, the same result holds for the order-by-order FW Poincaré generators. Finally, (iv) the *exact* integer-spin FW transformation is functionally related to the half-integer-spin FW transformation by (5.2). In fact, the second term in the large parentheses of (5.2) vanishes in this case, so that

$$\hat{U}(\alpha_\mu, p_\mu, \chi, c) = (1 - \mathcal{g}_0)U(\alpha_\mu, p_\mu, \chi, c)(1 - \mathcal{g}_0), \tag{5.3}$$

as we will now show.

First observe that (5.3) is to be intuitively expected. The FW transformations U^{-1} can be expressed either as the matrix whose columns are all of the independent eigenvectors of H or as a particular functional of the α_μ. However, since for integer spin the particle-components eigenvectors are given by

$$\psi_j^{(P)} = (1 - \mathcal{g}_0)\psi_j, \tag{5.4}$$

the matrix of the particle-components eigenvectors $\hat{U}$ will be related to the matrix U' of the entire integer-spin eigenvectors by [see Eq. (2.7)]

$$\hat{U}^{-1} = (1 - \mathcal{g}_0)[\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n](1 - \mathcal{g}_0) = (1 - \mathcal{g}_0)(U')^{-1}(1 - \mathcal{g}_0). \tag{5.5}$$

Since from (2.1), (2.7), and (5.8) below U^{-1}, like U, is not a functional of α_4^{-1}, this means one would expect that $(U')^{-1}$ has the same functional form as U^{-1} so that the functional relation of (5.5) should hold with $(U')^{-1} \to U^{-1}$, and hence with $U' \to U$.

To see that this is true, consider the expansions of $\hat{U}$ and U given in Sec. IV. We know that

$$\hat{U} \equiv \mathcal{g}^{(P)}[\exp(i\hat{S})]\mathcal{g}^{(P)} \equiv \mathcal{g}^{(P)}\left[\exp\left(\sum_{n=1}^{\infty}\hat{S}_n c^{-n}\right)\right]\mathcal{g}^{(P)} = (1 - \mathcal{g}_0)U\left(1 - \frac{i\mathcal{g}_0\vec{p}\cdot\vec{\alpha}}{\chi c}\right)(1 - \mathcal{g}_0), \tag{5.6}$$

$$U \equiv \exp(iS) \equiv \exp\left(\sum_{n=1}^{\infty} S_n c^{-n}\right). \tag{5.7}$$

Now, from Eqs. (4.15)–(4.17) S_1, S_2, and S_3 have solutions of the form

$$S_n = A_n\alpha_4, \quad n = 1, 2, 3. \tag{5.8}$$

In fact, by looking at (4.11)–(4.14) one sees that the set of equations an arbitrary S_n must satisfy implies all the S_n must be of the form

$$S_n = A_n\alpha_4, \quad \text{all } n. \tag{5.9}$$

(This is because the commutators in these equations always involve some combination of commutators of the S_n commuting with an object that has an α_4^{-1} on the left.) Thus if one makes an expansion in powers of the exponent of U in Eq. (5.7), then since

$$\alpha_4 \mathcal{G}_0 = 0, \quad (1-\mathcal{G}_0)\mathcal{G}_0 = 0, \tag{5.10}$$

one has that the second term in the large parentheses on the right of (5.6) is zero, leaving one with

$$\hat{U} = (1-\mathcal{G}_0)U(1-\mathcal{G}_0). \tag{5.11}$$

Further, since

$$\alpha_4(1-\mathcal{G}_0) = \alpha_4, \quad (1-\mathcal{G}_0)^2 = (1-\mathcal{G}_0), \tag{5.12}$$

one can make the algebraic relation

$$\begin{aligned}(1-\mathcal{G}_0)U(1-\mathcal{G}_0) &= (1-\mathcal{G}_0)\left\{1 + (1-\mathcal{G}_0)\sum_{n=1}^{\infty} A_n\alpha_4(1-\mathcal{G}_0) + \frac{1}{2}\left[(1-\mathcal{G}_0)\sum_{n=1}^{\infty} A_n\alpha_4(1-\mathcal{G}_0)\right]^2 + \cdots\right\}(1-\mathcal{G}_0)\\ &= (1-\mathcal{G}_0)\left[1 + \sum_{n=1}^{\infty}\hat{S}_n c^{-n} + \frac{1}{2}\left(\sum_{n=1}^{\infty}\hat{S}_n c^{-n}\right)^2 + \cdots\right](1-\mathcal{G}_0),\end{aligned} \tag{5.13}$$

where the exterior $(1-\mathcal{G}_0)$'s are the built-in unity operator of the particle-components space. Thus

$$\hat{S}_n = (1-\mathcal{G}_0)S_n(1-\mathcal{G}_0), \quad \text{all } n \tag{5.14}$$

or

$$\hat{S} = (1-\mathcal{G}_0)S(1-\mathcal{G}_0). \tag{5.15}$$

Finally, the result (4.61) that if

$$[\Theta, \alpha_4] = 0 \ \text{ and } [\hat{\Theta}, \alpha_4] = 0, \tag{5.16}$$

then the powers-series expansion

$$\hat{U}\hat{\Theta}\hat{U}^{-1} = \hat{\Theta} + \frac{\Delta\hat{\Theta}_1}{c} + \frac{\Delta\hat{\Theta}_2}{c^2} + \cdots \tag{5.17}$$

can be functionally related to the analogous expansion for Θ,

$$\hat{\Theta}_n = (1-\mathcal{G}_0)\Delta\Theta_n(1-\mathcal{G}_0), \quad \text{all } n \tag{5.18}$$

now follows trivially. Since $\mathcal{G}_0$ is a function of α_4, it will commute with Θ and $\hat{\Theta}$, meaning the second term in the large parentheses of (5.2) is zero by (5.10). This directly implies (5.18).

VI. DISCUSSION

In this paper we have shown the existence of an FW transformation for arbitrary-spin Bhabha fields, discussed its properties, and have demonstrated how to obtain the power-series expansion in c^{-1}.

The question of the existence of the FW transformation was due to the Bhabha indefinite metric, as was discussed in Sec. II. It is to be observed that the Bhabha indefinite metric tells us that it is the second excited state which has the same norm as the ground state. Also, since the "intrinsic parity" operator is η_4, the intrinsic parity operator has a direct relation to the norm of a state.

For half-integer spin, given that the ground state has positive norm and positive intrinsic parity, the first excited state will have negative norm and intrinsic parity, the second excited state will have positive norm and intrinsic parity, and so forth. Thus, in the $(\frac{5}{2}, \frac{1}{2})$ representation, one sees from Table I of paper II that the ground state with spin $\frac{1}{2}$ has positive norm and parity, the first excited states, since they are spin $\frac{1}{2}$ and $\frac{3}{2}$ particles, have negative norm and parity, and the second excited states, since they are spin-$\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ particles, have positive norm and parity. For half-integer spin, the antiparticles have the same norm but opposite intrinsic parity as the particles.

In the same way, for integer spin, if the ground state has positive norm with, for definiteness, positive parity (take negative parity for the usual pseudoscalar mesons if one will), then with each further excited state, the norm and intrinsic parity change by a minus sign. However, this time, given the norm and intrinsic parity of a particle, the antiparticle has *opposite norm* but the *same intrinsic parity*. This extra change of sign compared to the half-integer-spin case is due to the zero eigenvalue block of α_4, and hence the extra minus sign in that block for η_4. η_4, which is itself the intrinsic parity operator, is contained in the metric $M = \eta_4\alpha_4$.

It is this same extra zero eigenvalue block of

α_4 which allows an algebraic understanding of why the *Zitterbewegung* term is of the order of p/mc for Dirac, but of the order of p^2/m^2c^2 for the DKP (or KG) case of minimal electromagnetic coupling. Recall that the FW transformation diagonalizes the Hamiltonian with respect to mass states. For the $(\frac{1}{2}, \frac{1}{2})$ Dirac case with α_4 diagonal, the term $(\vec{p}^-)\cdot(\vec{\alpha})c$ is nonzero only in the upper right and lower left (2×2) blocks of the (4×4) $\vec{\alpha}$ matrices. Thus the particle fields are coupled directly to the antiparticle fields. On the other hand, because of the built-in subsidiary components for integer spin, the particle fields are connected to the antiparticle fields by first being coupled to the subsidiary components, and then from the subsidiary components to the antiparticle fields. Thus a two-step coupling is involved, and technically that is why the *Zitterbewegung* term, which comes from the decoupling of the particle fields from the antiparticle fields, is of different order in (p/mc) for the Dirac and DKP cases. This also means that for high-spin fields there will be many kinds of *Zitterbewegung* terms, owing to the off-diagonal couplings among the different mass states of the multimass fields and/or the corresponding antiparticle states. Further, this different-strength coupling will be evident in the nonzero components of high-spin eigenvectors. There the different components will be of the order of (p/mc) to some power with respect to the nonzero component of the rest state eigenvector, as we will see exactly in VI[82] and approximately in our power series $(\mathcal{S}, S)=(\frac{3}{2}, \frac{1}{2})$ special case example below. [This same argument explains why the Iachello first-order equation[76] for spin-0 mesons has a *Zitterbewegung* term of order (p/mc). His algebra matrices contain *no* subsidiary components, and so the particle and antiparticle states are coupled directly.]

Given the fact that the FW transformation exists, then it follows by the same arguments that it also exists as a matter of principle with minimal electromagnetic coupling. Then the question arises if one could not simply take the FW representation, meaning the Poincaré generators are mass-state diagonal, and have a perfectly acceptable q-number field theory by removing the negative-normed states from the space. After all, the FW generators tell us that the original representation had the probability coming back into the positive-normed states as fast as it was leaking out, and so the positive probability was preserved.

The trouble is that although we can solve the "indefinite-metric problem" by using the FW representation, in doing so another problem has arisen. The Poincaré generators, and in particular the Hamiltonian, can no longer be expressed as a simple first-order polynomial in $\vec{p}$. (In fact, even in the free case the generators are an infinite power series in $\vec{p}$.) This destroys all the nice causality features demonstrated in the ordinary representation in IV.

Thus the Bhabha fields appear of necessity to have the indefinite metric, with whatever implications this has for high-spin fields, which have not as yet been properly understood in any formalism. Note that this built-in or "kinematic" indefinite metric appears to be necessary. This is contrary to the case of "dynamic" indefinite metrics where the conclusions of certain studies[60,107] argue that those can be removed in a quantized formalism.

We know that high-spin particles exist in nature which are stable under electromagnetic interactions. The Ω^- is such a particle. Why, then, can we not find a single-mass, single-spin, high-spin field theory which is devoid of problems? One can answer this fundamental question by declaring as a matter of faith that the explanation is either that (a) the answer lies in a unified field theory of weak and electromagnetic interactions, meaning the weak decay of the Ω^- is the solution, or that (b) the fundamental fields we must deal with are quarks and gluons, so that no fundamental high-spin field is required. But if one chooses not to believe either of the above two answers, then the question remains open.

It is the implication of this series of papers that the answer *may lie* in accepting a multimass multispin field theory, *with* an indefinite metric. Perhaps we need a new interpretation of physics with respect to metrics and probability. We are not arguing that the Bhabha system is the answer, but rather that it may be an indication of the direction where the answer could be found. For example, certain uses of the exceptional groups in physics[108] would change the present probability interpretation.

As a final point, we wish to exhibit an example of a high-spin power-series FW transformation to third order in (c^{-1}). Specifically this is the $(\mathcal{S}, S)=(\frac{3}{2}, \frac{1}{2})$ representation. From Table I of II,[2] this is a 16-dimensional representation with a spin-$\frac{1}{2}$ ground state of mass $2\chi/3$, and two excited states of mass 2χ, one of spin $\frac{1}{2}$, the other of spin $\frac{3}{2}$.

A calculational aid is to rotate the coordinate system until the momentum is only in the z direction. Then we can conveniently write the α_4 and α_3 matrices in what amounts to an 8×8 form which is reminiscent of the ρ matrix representation of the γ matrices which Dirac has described.[109] Specifically, one can write

$$\alpha_4 = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \otimes I(2\times 2)\,, \tag{6.1}$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix} \otimes I(2\times 2)\,, \tag{6.2}$$

where $I(2\times 2)$ is a two-dimensional unit matrix which multiplies each element of the 8×8 matrix to yield the full (16×16)-dimensional representation. If we were to have $\vec{p}$ not parallel to $\hat{z}$, then the use of the α_1 and α_2 matrices in our discussion would prevent us from using the convenient 8×8 form.

Continuing, from (4.7) and (4.9) we have that

$$U^{-1}(c^{-3}) = I + (-iS_1)c^{-1} + (-iS_2 - \tfrac{1}{2}S_1^{\,2})c^{-2} + (-iS_3 - \tfrac{1}{2}S_1S_2 - \tfrac{1}{2}S_2S_1 + \tfrac{1}{6}iS_1^{\,3})c^{-3}\,. \tag{6.3}$$

Then cmbining (6.1)–(6.3) with the definitions (4.15)–(4.17) of S_1, S_2, and S_3 one obtains after matrix multiplication that, in terms of

$$R \equiv \left(\frac{p}{\chi c}\right), \tag{6.4}$$

$$U^{-1}(c^{-3}) = \begin{bmatrix} 1+\frac{9}{32}R^2 & 0 & i\sqrt{3}\left(\frac{R}{4}-\frac{R^3}{128}\right) & 0 & 0 & -\frac{\sqrt{3}}{16}R^2 & 0 & \frac{i27}{64}R^3 \\ 0 & 1-\frac{1}{32}R^2 & 0 & 0 & -i\left(\frac{R}{4}-\frac{3R^3}{128}\right) & 0 & 0 & 0 \\ -i3\sqrt{3}\left(\frac{R}{4}-\frac{9R^3}{128}\right) & 0 & 1+\frac{5}{32}R^2 & 0 & 0 & -\frac{1}{2}iR & 0 & -\frac{9\sqrt{3}}{16}R^2 \\ 0 & 0 & 0 & 1-\frac{1}{32}R^2 & 0 & 0 & -i\left(\frac{R}{4}-\frac{3R^3}{128}\right) & 0 \\ 0 & -i\left(\frac{R}{4}-\frac{3R^3}{128}\right) & 0 & 0 & 1-\frac{1}{32}R^2 & 0 & 0 & 0 \\ -\frac{9\sqrt{3}}{16}R^2 & 0 & -\frac{1}{2}iR & 0 & 0 & 1+\frac{5}{32}R^2 & 0 & -i\sqrt{3}\left(\frac{3R}{4}-\frac{27R^3}{128}\right) \\ 0 & 0 & 0 & -i\left(\frac{R}{4}-\frac{3R^3}{128}\right) & 0 & 0 & 1-\frac{1}{32}R^2 & 0 \\ \frac{i27}{64}R^3 & 0 & -\frac{\sqrt{3}}{16}R^2 & 0 & 0 & i\sqrt{3}\left(\frac{R}{4}-\frac{R^3}{128}\right) & 0 & 1+\frac{9}{32}R^2 \end{bmatrix} \otimes I(2\times 2)\,. \tag{6.5}$$

When we describe the closed-form FW transformation in VI,[82] the reader will see that (6.5) is indeed the expansion to order $(c^{-1})^3$ of the exact expression. However, already in (6.5) the physics of the situation is clear. Remember that the columns of (6.5) must be properly normalized eigenvectors $\hat{u}_j$, expanded to order c^{-3}. The first column is the ground state of mass $2\chi/3$, spin $\frac{1}{2}$. The second and fourth columns correspond to the excited state of mass 2χ, spin $\frac{3}{2}$. The third column corresponds to the excited state of mass 2χ, spin $\frac{1}{2}$. Similarly, one has the antiparticle states in columns five to eight.

Observe that all the spin-$\frac{1}{2}$ states are coupled in the FW matrix. The spin-$\frac{3}{2}$ states are coupled in two groups of two. Also, the diagonal elements of the matrix correspond to the nonzero elements in the rest system ($\vec{p}=0$). As one goes away from the diagonal element in a particular column (or row for that matter), each additional nonzero matrix element is of order R, i.e., $(p/\chi c)$, with respect to the previous one. These are properties which we said would occur.

Now that we understand the physics of the situation, we will proceed in VI[82] to describe how, starting from the eigensolutions of the Hamiltonian, the use of certain theorems will allow an analytic, closed form, to be given for the exact FW transformation for arbitrary-spin Bhabha fields. We will also discuss the relationship to other work.[47-49, 110, 111]

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