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Kinematics of a Poincaré-covariant object having indecomposable internal structure*

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Ten generators for the Poincaré group are exhibited, defining the kinematics of an object with an internal space over two real variables. The Hamiltonian implies a general (Regge-band) discrete mass-spin relationship. The generators are given in two forms: in quasi-Newtonian coordinates and in Minkowski coordinates, the latter allowing one to introduce interactions.

I. INTRODUCTION AND SUMMARY

The essential problem posed by relativistic quantum mechanics, as emphasized by Dirac,[1] is to find ten operators describing the system in space-time whose commutation relations close on the Lie algebra of the Poincaré group. For elementary particles—which may be defined as effectively structureless—the problem is well posed and the complete solution has been given in the classic 1939 paper by Wigner[2] enumerating all irreducible unitary representations of the Poincaré group.

By contrast, for nonelementary "particles" and for "particles" having internal structure there is no consensus as to the proper basic concepts; it is clear, however, that such systems necessarily will involve reducible Poincaré representations having a finite or infinite number of irreducible components.

Rather than attempt an *a priori* categorization of possible composite structures, we believe it reasonable to employ empirical concepts. From particle physics, it is found empirically that the Chew-Frautschi hypothesis of roughly linear Regge bands [$(\text{mass})^2 = \text{linear function of spin}$] for related families of hadrons appears to be valid. If one idealizes this situation, by neglecting all decay widths (in the way familiar from dual resonance theories) one obtains the concept of an object having internal structure, characterized by a denumerably infinite number of discrete mass states, M, as a function of intrinsic spin, s: $M^2 = f(s) > 0$. The discreteness of the mass parameter implies that the object is indecomposable, that is, cannot fragment.[3]

It is the purpose of the present paper to demonstrate, by direct construction, that there exists a precise (mathematical) realization for the kinematics of such an object, for which we exhibit the required ten operators generating the Poincaré group.

Let us outline the plan of this paper. Since our work is based directly on the results of Wigner's 1939 paper, we summarize briefly in Sec. II the finite-mass Poincaré irreducible representations (irreps), and then give a uniform realization of the ten Poincaré generators in both momentum space and in "quasi-Newtonian"[4] form for the whole set of (M, s) systems. Much of the material in this section is well known, but we believe that our discussion of the generators (particularly for quasi-Newtonian coordinates uniformly) has not previously appeared in the literature.

Section III contains one of the two principal results of the present work. We first develop the necessary structures to be built on the internal space (ξ_1, ξ_2) of two variables over the real line, and then develop the necessary Poincaré generators for our object, first in momentum space and then in quasi-Newtonian coordinates. The weaving together of unlimitedly many Poincaré irreps is necessarily somewhat complicated, and the technical details as to precisely how this is done—although mandatory for precision—may not be of primary concern, especially on a first reading. *In this case we suggest that one consider Eqs. (3.32)–(3.35) directly, since these equations stand on their own and completely define the nature and properties of our structural entity.*

In Sec. IV we discuss possible group-theoretic views of this entity, and introduce the concept of a "dynamical stability group." It is this concept which defines the entity globally, replacing the unworkable (and according to O'Raifeartaigh's theorem[5] impossible) concept of a global Lie group (for our entity). By way of emphasizing this same idea in a different context, we construct, in Sec. V, a simple example of a model possessing relativistic SU(6) symmetry and having none of the hitherto expected unphysical difficulties.

In Sec. VI we give the second of our two main results, developing and exhibiting the ten Poincaré generators, realized now in Minkowski coordinates. These generators, Eqs. (6.9)–(6.12)—just as for the equivalent generators in Sec. III—do indeed stand on their own, but it is far from an easy task now to prove that the Poincaré commutation relations are verified. The existence of a Minkowski-space realization is essential in order to couple the object to interactions.

Concluding remarks are contained in Sec. VII.

II. REVIEW OF BASIC STRUCTURES

A. The Wigner representations $\{M,s\}$

All of the constructions of the present paper will be built upon the foundations laid by Wigner in his famous, and classic, work[2] on the unitary irreducible representations of the Poincaré group. Accordingly, it is helpful to briefly review Wigner's results, restricted, however, to the finite-mass case.

We use the name Poincaré group, denoted by $\mathcal{P}$, for the group of proper orthochronous inhomogeneous Lorentz transformations. The elements of the covering group of $\mathcal{P}$ are $(d;A)$, where d represents a space-time translation and A a complex 2×2 matrix with unit determinant. Invariance for the group $\mathcal{P}$ leads, for quantum mechanics, to a unitary representation of the covering group of $\mathcal{P}$, as first shown in Ref. 2. The unitary irreducible representation (unirrep) which corresponds to a particle of spin $\frac{1}{2}$ and mass M represents the Poincaré element (d,A) by the transformation

$$[U(d,A)\phi]_i(p')=e^{ip'd}\sum_{j=1}^{2}(B_{p'}{}^{-1}AB_p)_{ij}\phi_j(p),$$

where (2.1)

$$p'\equiv\Lambda(A)p,\quad p^2=p'^2=M^2,\quad p^0>0,\quad i=1,2.$$

Here B_p stands for the boost (pure Lorentz transformation in the plane of the time axis and the four-vector p) which transforms the (reference) four-vector $(0,0,0,M)$ into p. The hyperbolic angle ϕ of this boost is found from $p_0=M\cosh\phi$. The transformation B_p is then found to be

$$B_p=\exp(\tfrac{1}{2}\phi\vec{\sigma}\cdot\hat{p}),\tag{2.2}$$

where $\hat{p}$ is the unit three-vector in the direction of the three-vector $\vec{p}$ and $\vec{\sigma}$ stands for the Pauli matrices. Equation (2.2) can be written[14] as

$$B_p=(\{p/M\})^{1/2},\tag{2.3}$$

where

$$\{p/M\}=\frac{1}{M}\{p_0\underline{1}+\vec{p}\cdot\vec{\sigma}\}.\tag{2.4}$$

Note that

$$(\{p/M\})^{-1}=\frac{1}{M}\{p_0\underline{1}-\vec{p}\cdot\vec{\sigma}\}.\tag{2.5}$$

By expressing $\cosh\frac{1}{2}\phi$ and $\sinh\frac{1}{2}\phi$ in terms of $\cosh\phi$, one can rewrite (2.3) in the form

$$B_p=\frac{(M+p_0)\underline{1}+\vec{\sigma}\cdot\vec{p}}{[2M(M+p_0)]^{1/2}},\tag{2.6}$$

which is very close to the so-called Foldy-Wouthuysen transformation. From (2.2) and (2.6) it is clear that

$$B_p{}^{-1}=\frac{(M+p_0)\underline{1}-\vec{\sigma}\cdot\vec{p}}{[2M(M+p_0)]^{1/2}}.\tag{2.7}$$

Returning to Eq. (2.1), the functions $\phi_i(p)$ clearly satisfy

$$(p^2-M^2)\phi_i(p)=0.\tag{2.8}$$

Notice that the functions $\phi_i(p)$ are accordingly really functions only of the three-vector $\vec{p}$.

The Lorentz transformation corresponding to $B_{p'}{}^{-1}AB_p$ leaves the time axis invariant [since B_p changes $(0,0,0,M)$ into p, A changes p into $\Lambda(A)p=p'$, and $B_{p'}{}^{-1}$ brings p' back to $(0,0,0,M)$] and is therefore a rotation of the little group. Hence the 2×2 matrix, $B_{p'}{}^{-1}AB_p$, is unitary.

The invariant inner product of two Wigner functions is given by

$$(\phi,\psi)\equiv\int\frac{d\vec{p}}{p_0}\sum_{i=1}^{2}\phi_i^*(\vec{p})\psi_i(\vec{p}).\tag{2.9}$$

For a unitary matrix A (which hence corresponds to a rotation), one finds from the definition of B_p, or via calculation using (2.6), that

$$B_{p'}{}^{-1}AB_p=A\ \text{ for } A\in SU(2).\tag{2.10}$$

The case of particles of spin s is entirely analogous to the spin-$\frac{1}{2}$ case. The only change is that the 2×2 unitary matrix $B_{p'}{}^{-1}AB_p$ is replaced by its $(2s+1)$-dimensional unitary representation $D^{(s)}_{mm'}(B_{p'}{}^{-1}AB_p)$, and that ϕ now has $2s+1$ components. For the case of spin s, Eq. (2.1) reads

$$[U(d,A)\phi]_m(p')=e^{ip'd}\sum_{m'=-s}^{s}D^{(s)}_{mm'}(B_{p'}{}^{-1}AB_p)\phi_{m'}(p).\tag{2.1$'$}$$

The functions $\phi_m(p)$ satisfy Eq. (2.8), but (2.9) has to be replaced by

$$(\phi,\psi)=\int\frac{d\vec{p}}{p^0}\sum_{m=-s}^{s}\phi_m^*(\vec{p})\psi_m(\vec{p}).\tag{2.9$'$}$$

Once again for $A\in SU(2)$ one has the special result

$$D^{(s)}_{mm'}(B_{p'}{}^{-1}AB_p)=D^{(s)}_{mm'}(A),\quad m=-s,-s+1,\ldots,s.\tag{2.10$'$}$$

B. The Poincaré group generators in the Wigner form

Symmetry considerations play a dual role in quantum physics, and lead not only to the structure of allowed states but also imply the proper observables (operators) of the symmetry. Thus having constructed the set of $\{M,s\}$ $\mathcal{P}$ irreps, it is important next to construct the explicit operators (observables) generating the symmetry structure.

It is clear from Eq. (2.1) that the displacement generators, $\{P_\alpha\}=(P_0, \vec{P})$, take the eigenvalues

$$\vec{P} \to \vec{p}, \tag{2.11}$$

$$P_0 = (\vec{P}^2 + M^2)^{1/2} \to +(\vec{p}^2 + M^2)^{1/2}. \tag{2.12}$$

Noting that the three-space rotations obey the simplifying result, Eq. (2.10), it is easily verified that the rotation generators are $[(ijk)=$ positive permutation of (123)$]$

$$M_{ij} = ip_i \frac{\partial}{\partial p_j} - ip_j \frac{\partial}{\partial p_i} + \sigma_k/2. \tag{2.13}$$

The really interesting generators are the boosts, M_{0i}, and it is somewhat more difficult to verify [from Eq. (2.1)] that these have the form

$$M_{0i} = (\vec{p}^2 + M^2)^{1/2} \frac{\partial}{\partial p_i} + \frac{p_j \sigma_k - p_k \sigma_j}{2[(\vec{p}^2 + M^2)^{1/2} + M]}. \tag{2.14}$$

To generalize from spin $\frac{1}{2}$ to spin j one need only replace the 2×2 matrices $\sigma_i/2$ by the corresponding $(2j+1)\times(2j+1)$-spin matrices $\vec{S}$ in Eqs. (2.13) and (2.14).

Note that—for the inner product given by Eq. (2.9)—the operator M_{0i} in Eq. (2.14) is indeed Hermitian.

It is readily verified, directly, that the ten generators given in Eqs. (2.11)–(2.14) close upon the commutation relations of the Poincaré group. [This uses the commutation rules $\vec{S}\times\vec{S} = i\vec{S}$, where $\{\vec{S}\}$ are the $(2j+1\times 2j+1)$ matrix realizations of the generators of SU(2).] Note that time displacements are generated by the Hamiltonian $H = P_0 = +(\vec{P}^2 + M^2)^{1/2}$, $P_0 > 0$; hence the time, t, cannot correspond to an operator but functions, correctly, as a c number.

It is quite obvious that the Wigner irreps, $\{M,s\}$, directly imply the momentum-space operator realizations given above; nonetheless these operators are of importance on their own. To our knowledge these operators have first been given explicitly by Bacry.[15] (Newton and Wigner[6] gave the results only for spin zero.)

C. The introduction of quasi-Newtonian coordinates

The Wigner irreps $\{M,s\}$ and the Wigner form of the generators are realized in momentum space, and it is an interesting question as to how to obtain configuration-space realizations. This problem was actually considered prior to the Wigner construction (1939) by Schrödinger, in his studies on the Dirac equation. As we shall show here, a comprehensive view can best be obtained directly from the Wigner construction of the set $\{M,s\}$; from this point of view the existence of the Dirac equation (for spin $\frac{1}{2}$) is irrelevant, and the historical accident that this equation came first has greatly confused the initial, and subsequent, discussions of the problem, based as they are on the particularities of the Dirac equation itself.

The problem is this: How shall we introduce configuration-space variables in place of the momentum-space variables of the Wigner irreps? At first glance, the question appears trivial; one should simply use a Fourier transformation. The difficulty is that the Wigner irreps are defined only on the mass hyperboloid in four-dimensional momentum space, and this constraint implies that the concept "Fourier transformation" is ill defined.

There are two distinct ways to proceed, leading to two very different results. We will designate the coordinates defined by these procedures as follows:

(a) Minkowski coordinates $\{x_\mu\}$ (which turn out to be the coordinates appropriate for coupling to the electromagnetic field), and

(b) Quasi-Newtonian coordinates $(\vec{X}, t)$, which have been discussed in the literature by a great many authors.[6-12]

To introduce the quasi-Newtonian coordinates one *postulates* the Fourier transform to be

$$\phi_i(\vec{X},t) = (2\pi)^{-3/2} \int d\vec{p}\,(p_0)^{-1/2} e^{i\vec{p}\cdot\vec{X}} e^{ip_0 t} \phi_i(\vec{p}). \tag{2.15}$$

From Eqs. (2.9), or (2.9′), one finds that the norm (ϕ,ψ) may be written in terms of these new functions as

$$(\phi,\psi) = \int d\vec{X} \sum_i \phi_i^*(\vec{X},t)\psi_i(\vec{X},t). \tag{2.16}$$

Once having this definition, the form taken by the Poincaré generators can be found directly from the Wigner realization, Eqs. (2.12)–(2.14):

$$P_i = -i\partial/\partial X_i, \tag{2.17}$$

$$\underline{M} = \vec{X}\times\vec{P} + \vec{S}. \tag{2.18}$$

From Eq. (2.12) one finds the Hamiltonian to be

$$P_0 = +(\vec{P}^2 + M^2)^{1/2}. \tag{2.19}$$

Finally one finds for the generators of the Lorentz boosts [using Eq. (2.14), and by partial integration in Eq. (2.15)]

$$\{M_{0i}\} \equiv \vec{K} = \tfrac{1}{2}(\vec{X}P_0 + P_0\vec{X}) + tP_0^{-1}\vec{P} - (P_0 + m)^{-1}\vec{S}\times\vec{P}. \tag{2.20}$$

Several important remarks are to be made at this point.

(1) The generators (2.17), (2.18), (2.19), and (2.20) are Hermitian for the inner product (2.16). That they satisfy the commutation rules for the Poincaré group is clear by the way they were derived from the Wigner generators; one can also verify this fact directly. Note that t is once again a c number.

(2) Keeping (2.17), (2.18), and (2.19) one can modify (2.20) so that it remains Hermitian and keeps the correct Poincaré commutation relations. The most general form is[6]

$$\vec{K}' = \vec{K} + f(p_0)\vec{P}\,, \tag{2.20'}$$

with $f(p_0)$ real. This freedom corresponds to the fact that instead of (2.15) one might have chosen

$$\phi_i(\vec{X}, t) = (2\pi)^{-3/2}\int d\vec{p}\, e^{ig(p^0)}(p_0)^{-1/2} \times e^{i\vec{p}\cdot\vec{X}} e^{ip_0 t}\phi_i(\vec{p})\,. \tag{2.15'}$$

In other words, these representations, corresponding to different choices $f(p^0)$, are all unitarily equivalent.

(3) For the case of spin s the only change is that the Pauli matrices of the spin-$\frac{1}{2}$ case are replaced by the generators of the $(2s+1)$-dimensional unitary representation of SU(2). Hence we have obtained quasi-Newtonian coordinates for the general (M, s) representation uniformly.

(4) The coordinates $(\vec{X}, t)$ have a number of less desirable properties; this has been discussed in the literature extensively.[6,13]

(5) To our knowledge Eqs. (2.17)–(2.20) were first given in this generality by Thomas.[16]

The coordinates $(\vec{X}, t)$ have been designated here as "quasi-Newtonian" since $\vec{X}$ transforms under rotations as a three-vector, but $\vec{X}$ is *not* part of a (Minkowski) four-vector; as mentioned before, t is a c number. These coordinates are therefore nonrelativistic in appearance, i.e., "Newtonian"; but since $\vec{X}$ properly belongs to quantum mechanics as an operator (or q number) we accordingly call these coordinates quasi-Newtonian for short.

Such coordinates have been introduced many times,[6-12] but it is chiefly the discussions of Newton and Wigner (1949)—"Newton-Wigner position operator"—and of Foldy and Wouthuysen (1950)—"mean position operator"—that have been definitive. There are major problems posed by quasi-Newtonian coordinates (localization is not invariant to Lorentz transformations; moreover, in the next instant the system is completely dislocalized). Physically the difficulty is that no interaction couples to this coordinate.

We defer the discussion of the Minkowski position coordinates to Sec. VI.

III. POINCARÉ COVARIANCE FOR A DEFORMABLE COMPOSITE OBJECT

A. Dirac's new equation

Dirac's new relativistic wave equation[17] describes a composite particle, having intrinsic spin $s=0$ and mass M, whose internal structure is based upon two (degenerate) harmonic oscillators (π_1, ξ_1) and (π_2, ξ_2). Here π_i and ξ_i are the usual dimensionless momenta and coordinates $[\xi_i = (\omega\mu/\hbar)^{1/2} q_i$, $\pi_i = (\hbar\omega\mu)^{-1/2} p_i]$. It is convenient also to use the (non-Hermitian) operators $2^{1/2}\bar{a}_i \equiv i\pi_i + \xi_i$ and $2^{1/2} a_i \equiv -i\pi_i + \xi_i$ which obey the commutation rule $[\bar{a}_i, a_j] = \delta_{ij}$, all other commutators vanishing.

Dirac's construction—and the generalization to spin s in Ref. 18—are based upon the fact that, remarkably, the internal structure supports a realization of the symplectic group Sp(2, 2) whose ten generators $S_{\mu\nu}$ are given by[19]

$$\begin{aligned}
\{J\}:\ & J_1 = \tfrac{1}{2}(a_1\bar{a}_2 + a_2\bar{a}_1),\\
& J_2 = \tfrac{1}{2}(a_1\bar{a}_1 - a_2\bar{a}_2),\\
& J_3 = \tfrac{1}{2} i(a_1\bar{a}_2 - a_2\bar{a}_1),\\
\{K\}:\ & K_1 = \tfrac{1}{4}(a_1^2 - a_2^2 + \bar{a}_1^2 - \bar{a}_2^2),\\
& K_2 = -\tfrac{1}{2}(a_1 a_2 + \bar{a}_1\bar{a}_2),\\
& K_3 = \tfrac{1}{4} i(a_1^2 a_2^2 - \bar{a}_1^2 - \bar{a}_2^2),\\
\{V\}:\ & V_1 = \tfrac{1}{4} i(\bar{a}_1^2 - \bar{a}_2^2 + a_2^2 - a_1^2),\\
& V_2 = \tfrac{1}{2} i(a_1 a_2 - \bar{a}_1\bar{a}_2),\\
& V_3 = \tfrac{1}{4}(a_1^2 + a_2^2 + \bar{a}_2^2 + \bar{a}_1^2),\\
& V_0 = \tfrac{1}{2}(a_1\bar{a}_1 + a_2\bar{a}_2 + 1).
\end{aligned} \tag{3.1}$$

The Lorentz group generators $S_{\mu\nu}$ are the subset $S_{ij} = \epsilon_{ijk} J_k$, $S_{i0} = K_i$.

All ten generators operate on functions $\phi(\xi_1, \xi_2)$ and all are Hermitian within the usual norm

$$(\phi, \psi) = \int d\xi_1 d\xi_2 \phi^*(\xi_1, \xi_2)\psi(\xi_1, \xi_2)\,. \tag{3.2}$$

In this way the ten generators of (3.1) provide a unitary representation of the covering group of the de Sitter group [Sp(2, 2)], which we denote by $\mathfrak{U}(S)$. This representation induces a four-dimensional nonunitary representation in the following way[20]:

$$\mathfrak{U}(S)\begin{pmatrix} a_1 \\ a_2 \\ \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}\mathfrak{U}^{-1}(S) = \mathfrak{D}^{-1}(S)\begin{pmatrix} a_1 \\ a_2 \\ \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}, \tag{3.3}$$

where $\mathfrak{D}(S)$ is a 4×4 matrix representing S. One checks easily that (3.2) is true and that the matrices $\mathfrak{D}(S)$ form a representation of the de Sitter group S. The covering group of the homogeneous proper orthochronous Lorentz group is obtained from S by restricting oneself to the first six generators of Eq. (3.1). Writing for an element A of the restricted group

$$A=\exp\Big(\sum_{\mu,\nu}\alpha_{\mu\nu}M_{\mu\nu}\Big), \tag{3.4}$$

one has

$$\mathfrak{U}(A)=\exp\Big(\sum_{\mu,\nu}i\,\alpha_{\mu\nu}S_{\mu\nu}\Big). \tag{3.5}$$

For the representation $\mathfrak{D}(A)$ one finds

$$\mathfrak{D}(A)=\exp\Big(\sum_{\mu,\nu}\alpha_{\mu\nu}\tfrac{1}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\Big), \tag{3.6}$$

where the four matrices γ_μ, $\mu=1,2,3,0$, are an unusual representation of Dirac's matrices. Actually $\mathfrak{D}(A)$ is equivalent to a direct sum of A itself and of its complex conjugate.

The new Dirac equation is based on Eq. (3.3) and on the explicit form for $\mathfrak{D}$ given in Eq. (3.6). Dirac writes

$$(\gamma^\mu\partial_\mu-M)\begin{pmatrix}a_1\\a_2\\\bar{a}_1\\\bar{a}_2\end{pmatrix}\psi(x;\xi_1,\xi_2)=0, \tag{3.7}$$

where ∂_μ works on the Minkowski coordinate x and where the $\bar{a}$'s and $\bar{a}$'s work on ξ_1 and ξ_2. The functions $\psi(x;\xi_1,\xi_2)$ transform under (A,d) according to

$$\psi'(x;\xi_1,\xi_2)=\mathfrak{U}(A)\psi(\Lambda^{-1}(A)x-d,\xi_1,\xi_2), \tag{3.8}$$

where $\mathfrak{U}(A)$ is given by (3.5), with (3.1), and where $\mathfrak{U}(A)$ operates on ξ_1 and ξ_2.

To check the invariance of (3.7) for Poincaré transformations (d,A), one verifies that $\psi'(x;\xi_1,\xi_2)$ is a solution of (3.7) provided that $\psi(x;\xi_1,\xi_2)$ is a solution. Using (3.3), this verification becomes similar to the verification that the relativistic equation for the electron is invariant.

B. An alternative formulation

Let us return to the mapping defined by Eq. (3.3). Consider an arbitrary transformation A, belonging to the covering group of the proper homogeneous Lorentz group [SL(2, C)], i.e., $\pm A$ leads to the same Lorentz transformation $\Lambda(A)$. Let us define for an arbitrary (A)

$$\begin{pmatrix}a_1(A)\\a_2(A)\\\bar{a}_1(A)\\\bar{a}_2(A)\end{pmatrix}\equiv\mathfrak{U}(A)\begin{pmatrix}a_1\\a_2\\\bar{a}_1\\\bar{a}_2\end{pmatrix}\mathfrak{U}^{-1}(A). \tag{3.9}$$

The quantities on the left-hand side have the same commutation relations as the quantities sandwiched on the right-hand side. If the original ground state is $u_e^0(\xi_1,\xi_2)$ then the operators on the left-hand side of (3.9) define a new ground state

$$u_A^0(\xi_1,\xi_2)=\mathfrak{U}(A)u_e^0(\xi_1,\xi_2). \tag{3.10}$$

The ground state $u_e^0(\xi_1,\xi_2)$ is defined by

$$\bar{a}_i u_{e_0}^0(\xi_1,\xi_2)=0,\quad i=1,2. \tag{3.11}$$

The first three generators in (3.1) make it clear that $u_e^0(\xi_1,\xi_2)$ is invariant for rotations; the next three show that it is not invariant for boosts. In this way the ground state $u_e^0(\xi_1,\xi_2)$ may be labeled by the unit four-vector in the time direction: $e=(0,0,0,1)$. The ground state obtained from $u_e^0(\xi_1,\xi_2)$ by a boost will be denoted $u_{e_B}^0(\xi_1,\xi_2)$; it is characterized by the unit four-vector e_B into which $(0,0,0,1)$ is changed by the boost. Again this is a good characterization because the ground state $u_{e_B}^0(\xi_1,\xi_2)$ is invariant for those transformations A which leave the four-vector e_B invariant (little group of e_B). We found, for a boost in a spatial direction specified by θ and ϕ, and over a hyperbolic angle χ,

$$u_{e_B}^0(\xi_1,\xi_2)=\pi(\cosh\chi+\sinh\chi\sin\theta\sin\phi)^{-1/2}$$
$$\times\exp\{-(2\cosh\chi+2\sinh\chi\sin\theta\sin\phi)^{-1}[(\xi_1{}^2+\xi_2{}^2)+2i\sinh\chi\cos\theta\,\xi_1\xi_2+i\sinh\chi\sin\theta\cos\phi(\xi_2{}^2-\xi_1{}^2)]\}. \tag{3.12}$$

A useful alternative parametrization uses the unit four-vector p/M:

$$u_{p/M}^0(\xi_1,\xi_2)=\left[\frac{M}{\pi(p_0+p_3)}\right]^{1/2}\exp\left\{-\frac{M}{2(p_0+p_3)}\left[(\xi_1{}^2+\xi_2{}^2)-\frac{2ip_2}{M}\,\xi_1\xi_2+\frac{ip_1}{M}(\xi_2{}^2-\xi_1{}^2)\right]\right\}. \tag{3.13}$$

The plane-wave solutions to Dirac's new equation are given by

$$\exp(ip\cdot x)\mathfrak{u}_{p/M}(\xi_1,\xi_2), \tag{3.14}$$

where $p^2 - M^2 = 0$. This should be no surprise, because for a plane wave of four-momentum p Dirac's new equation may be written using (3.9) as

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(B_p) \\ a_2(B_p) \\ \bar{a}_1(B_p) \\ \bar{a}_2(B_p) \end{bmatrix} \phi(p;\xi_1,\xi_2)=0,$$

where B_p is the boost which changes $(0,0,0,M)$ into the four-vector p.

C. A uniform presentation of the Wigner basis functions

Once we have obtained this view of the structure underlying Dirac's result, the generalization to nonzero spin is immediate; the result will define the Wigner basis functions for general spin. First one defines the basis states for four-momentum $(0,0,0,M)$ as

$$\mathfrak{u}_e^{s,m}(\xi_1,\xi_2)=\frac{(a_1)^{s+m}(a_2)^{s-m}}{[(s+m)!\,(s-m)!]^{1/2}}\,\mathfrak{u}_e^0(\xi_1,\xi_2), \tag{3.15}$$

which characterizes spin s and projection m for a particle with four-momentum $(0,0,0,M)$.

The spin states for a particle with four-momentum p are then given by

$$\mathfrak{u}_{p/M}^{s,m}(\xi_1,\xi_2)=\frac{[a_1(B_p)]^{s+m}[a_2(B_p)]^{s-m}}{[(s+m)!\,(s-m)!]^{1/2}}\,\mathfrak{u}^0_{p/M}(\xi_1,\xi_2), \tag{3.16}$$

where B_p boosts $(0,0,0,1)$ to p/M, and where $\mathfrak{u}^0_{p/M}(\xi_1,\xi_2)$ is given by (3.13).

The corresponding Wigner basis function, having mass M, sharp four-momentum p, sign s, and index m, is then

$$\mathfrak{v}^{s,m}(p)=e^{ip\cdot x}\mathfrak{u}_{p/M}^{s/m}(\xi_1,\xi_2). \tag{3.17}$$

[To make the relation to Eq. (2.1′) clearer we have suppressed the parameters M, ξ_1, ξ_2 in the Wigner basis functions on the left.]

Let us verify the transformations of these basis functions under Poincaré transformations (d,A). The transformation of the functions $\mathfrak{v}(p)$ is given by (3.8), and the translation part checks immediately; we are left with considering A.

Next, we write $\mathfrak{U}(A)$, of (3.8), as

$$\mathfrak{U}(A)=\mathfrak{U}(B_{\Lambda p})\mathfrak{U}(B_{\Lambda p}{}^{-1}AB_p)\mathfrak{U}(B_p{}^{-1}), \tag{3.18}$$

which is an obvious identity. [Here B denotes a boost, p denotes the initial four-momentum, and $\Lambda p=\Lambda(A)p$ denotes the final four-momentum just as for Eq. (2.1).] As we remarked in connection with (2.1), $B_{\Lambda p}{}^{-1}AB_p$ leaves $(0,0,0,1)$ invariant, and, therefore, is a rotation, $R=B_p{}^{-1}AB_p$. In this way

$$\begin{aligned}\mathfrak{U}(A)\mathfrak{u}_{p/M}^{s,m} &= \mathfrak{U}(B_{\Lambda p})\mathfrak{U}(R)\mathfrak{U}^{-1}(B_p)\mathfrak{u}_{p/M}^{s,m}\\ &= \mathfrak{U}(B_{\Lambda p})\mathfrak{U}(R)\mathfrak{u}_e^{s,m}\\ &= \mathfrak{U}(B_{\Lambda p})\sum_{m'}\mathfrak{D}^s_{m'm}(R)\mathfrak{u}_e^{s,m'}\\ &= \sum_{m'}\mathfrak{D}^s_{m'm}(R)\mathfrak{U}(B_{\Lambda p})\mathfrak{u}_e^{s,m'}\\ &= \sum_{m'}\mathfrak{D}^s_{m'm}(R)\mathfrak{u}^{s,m'}_{\Lambda p/M}.\end{aligned} \tag{3.19}$$

Hence we find from (3.8) that the basis vectors $\mathfrak{u}^{s,m}(p)$ transform according to

$$(d,A):\ \mathfrak{u}^{s,m}(p)\to e^{i\Lambda p\cdot d}\mathfrak{D}_{m'm}(R)\mathfrak{u}^{s,m'}(\Lambda p),$$

where (3.20)

$$R\equiv B_p{}^{-1}AB_p.$$

The transformation (3.20) for the basis functions (unit vectors in Hilbert space) implies the transformation (2.1′) for the *components* $\phi^s_m(p)$ of a state

$$\phi=\int\frac{d\vec{p}}{p^0}\sum_m \phi^s_m(p)\mathfrak{u}^{s,m}(p).$$

It is of interest to note that there is an alternative way to proceed.[20] To see this, let us use a different identity for decomposing $\mathfrak{U}(A)$:

$$\mathfrak{U}(A)=\mathfrak{U}(AB_pB_{\Lambda p}{}^{-1})\mathfrak{U}(B_{\Lambda p})\mathfrak{U}(B_p{}^{-1}). \tag{3.21}$$

The transformation $AB_pB_{\Lambda p}{}^{-1}$ belongs to the little group of Λp, since $B_{\Lambda p}{}^{-1}$ takes Λp into $(0,0,0,M)$ and AB_p takes it to $\Lambda(A)p$. Thus we may equivalently write the transformation induced by (d,A) in the form

$$(d,A):\ \mathfrak{u}^{s,m}(p)\to e^{i\Lambda p\cdot d}\mathfrak{D}^s_{m'm}(AB_pB_{\Lambda p}{}^{-1})\,\mathfrak{u}^{s,m'}(\Lambda p). \tag{3.22}$$

The significance of this alternative form of the transformation, Eq. (3.22), is that it enables one to give an explicit operator form of the generators of the stability group (little group) of Λp. This stability group is generated by three operators $S_{ij}(B_{\Lambda p})$, $i,j=1,2,3$, which are obtained by substituting the $a_i(B_p)$ and the $\bar{a}_j(B_p)$ for the a_i and the $\bar{a}_j$ in the first three lines of (3.1).

D. Poincaré generators

1. Momentum-space coordinates (Wigner form)

Once one has obtained a uniform presentation of the set of all Wigner-Poincaré wave functions (on the harmonic-oscillator basis above) the question occurs as to whether there exists a unified set of generators yielding precisely this set of functions. It is clear, by analogy with the derivation of the Wigner generators given in Sec. II, that generators (for each mass separately) must exist, but the real question is: Can one incorporate a *nonconstant* mass parameter m? It is here that we go beyond the concepts used in the Wigner-Poincaré construction.

It is the merit of the Galilean subdynamics approach[18,20] that it suggests an answer to these questions. By ascribing a literal meaning to the composite structure it was demonstrated that this imposed, consistently, a mass-spin constraint on the set of Wigner wave functions. This constraint takes the form

$$M^2 = f(s), \tag{3.23}$$

where s is the spin such that $2s = n$, the number of quanta.

The form of the function f is arbitrary (but of course M^2 must be non-negative). The special, linear, form $M^2 = \alpha + \beta s$ is suggested by the empirical data on hadronic Regge bands. (We discuss this case further in Sec. V.)

Since the number of quanta is the boson operator $2V_0$ belonging to the generators of the de Sitter group, we may give an operator formulation of this constraint:

$$M^2 = f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1). \tag{3.24}$$

[Note, for preciseness, that these boson operators are the original ones defined in connection with Eq. (3.1).]

Let us now demonstrate that there exist generators explicitly validating these ideas. Following the Wigner construction, we first define an invariant inner product

$$(\phi,\psi) \equiv \int d^3p \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2\, \phi^*(\xi_1\xi_2;p)\frac{1}{P_0} \times \psi(\xi_1\xi_2;p). \tag{3.25}$$

Space-time translations are now realized by the operators

$$\vec{P} \to \vec{p}, \qquad P^0 \equiv [\vec{P}^2 + f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)]^{1/2} \to E \equiv [\vec{p}^{\,2} + f(s)]^{1/2}, \tag{3.26}$$

where

$$[\vec{P}, P^0] = 0. \tag{3.27}$$

Just as earlier the rotation generators follow directly from the realization of the e_{0001} little group by the rotation matrices. This now implies that the rotations are generated by the operators

$$M_{ij} = i\left(P_i \frac{\partial}{\partial p_j} - P_j \frac{\partial}{\partial p_i}\right) + S_{ij} \tag{3.28}$$

Here the operators S_{ij} are the boson operators $\vec{J}$ given in Eq. (3.1).

Finally the boosts are found to be generated by the operators

$$M_{0i} \equiv iP_0 \frac{\partial}{\partial p_i} + \frac{(\vec{P}\times\vec{S})_i}{P_0 + M(s)} . \tag{3.29}$$

The existence of this operator realization depends critically on the fact that the "mass operator," $M^2 = f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)$, commutes with $\vec{P}$, $\vec{S}$, and hence with $M_{\mu\nu}$.

One may verify directly that these generators (designated the "Wigner form") satisfy the Poincaré group algebra.

2. *Quasi-Newtonian coordinates for the generators (Thomas form)*

It is straightforward now to introduce quasi-Newtonian coordinates, $(\vec{X}, t)$, in complete analogy to the earlier example in Sec. II. Let us define the wave function by

$$\phi(\vec{X}, t; \xi_1\xi_2) \equiv (2\pi)^{-3/2} \int \frac{d^3p}{(p_0)^{1/2}} e^{i(\vec{p}\cdot\vec{X} + p_0 t)} \times \phi(\vec{p}; \xi_1\xi_2) . \tag{3.30}$$

The norm now takes the form

$$(\phi, \psi) = \int d^3X\, d\xi_1\, d\xi_2\, \phi^*(\vec{X}, t; \xi_1\xi_2)\, \psi(\vec{X}, t; \xi_1\xi_2) . \tag{3.31}$$

Introducing the Wigner form of the generators, Eqs. (3.26)–(3.29), into Eq. (3.30), it follows that the quasi-Newtonian form of the generators is given by

$$P_i = -i\,\partial/\partial X_i , \tag{3.32}$$

$$P_0 = +[\vec{P}^2 + f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)]^{1/2} , \tag{3.33}$$

$$M_{ij} = \epsilon_{ijk}(\vec{X}\times\vec{P})_k + S_{ij} , \tag{3.34}$$

$$M_{0i} = \tfrac{1}{2}(X_i P_0 + P_0 X_i) + tP_i/P_0 + (P_0 + M)^{-1}(\epsilon_{ijk} P_j S_k) , \tag{3.35}$$

where $S_k \equiv \epsilon_{ijk} S_{ij}$ and $M \equiv [f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)]^{1/2}$; cf. Eq. (3.1).

It is verified directly that these operators close on the commutation relations of the Poincaré algebra.

We designate these generators, in quasi-Newtonian coordinates, Eqs. (3.32)–(3.35), as the "Thomas form" since the equations have (for a fixed numerical mass M-number) precisely the form given by Thomas in his discussion of an isotropic spinning particle.[16] [There is a slight difference in that Thomas sets $t=0$, which is an allowed simplification; compare remark (2) after (2.20).]

It is remarkable, we believe, that such a simple formulation exists for an object realizing all the Wigner representations $[M(s), s]$ each s once and only once.

Having once obtained these ten generators, we can forget where they came from and consider the generators on their own merits. (Thus, for example, our previous considerations of Galilean subdynamics can be considered as heuristic only, and, if desired, can be discarded.) *The generators given by Eqs. (3.32)–(3.35) define completely the free motion of a relativistic composite object whose mass-spin quantum numbers lie on a discrete Regge band, $M^2=f(2s)$; the Hamiltonian is given by Eq. (3.33).*

It will be observed that these generators are actually in Hamiltonian form, and the operator P_0 is indeed the Hamiltonian. The motion generated is, however, the free motion of the object and there is no essential dynamics involved. Thus the real content of these generators is kinematical in nature as emphasized in the title.

Let us remark, for completeness, that in their comprehensive discussion of the theory of a relativistic spherical top, Regge and Hanson[21] arrived at a canonical formalism for a constrained Hamiltonian system ("Dirac brackets") which is formally quite similar in structure to the results above. (Cf. Ref. 21, Sec. E, p. 535.) However, the Hanson-Regge realization introduced a set of internal coordinates R^{ij} which served as the carrier space for the usual rotation matrices; this results not only in a $(2j+1)^2$ multiplicity but also introduces nontrivial ordering problems in realizing the commutators. Hence, though similar in spirit, the work of Ref. 21 is in actuality rather different from the results given above.

[It might be of interest to note that a suitable limit for the structure given in Eqs. (3.32)–(3.35) yields the Wigner massless infinite spin irreps of $\mathcal{P}$.]

IV. DYNAMICAL STABILITY GROUPS

The existence of the ten Poincaré generators $(\vec{\mathrm{P}}, \underline{M})$, given by Eqs. (3.26), (3.27), and (3.29), or by (3.32), (3.33), (3.34), and (3.35) whose associated wave functions are the set of $[M(s), s]$ Wigner irreps, serves to define an entity which, for want of a better name, we may call a (relativistic) "composite object." We seek, in the present section, to understand better the structure of this object.

The phrase "composite object" is meant in contradistinction to the phrase "elementary particle" (presumably structureless), since this latter must surely mean, at the least, an *irreducible* Poincaré representation. It would seem reasonable to consider a composite particle to be one that belongs to a Regge band; if the band has discrete masses only then the particle presumably does not fragment, but remains a single entity.[3] The object described by the generators $(\vec{\mathrm{P}}, \underline{M})$ possesses these characteristics of compositeness with no possibility of fragmenting.

Let us now demonstrate that we can consider this structure from a group-theoretic viewpoint which unites the set of $[M(s), s]$ irreps into an entity. Let us take the object to be in its rest frame. In this frame, the operators $S_{\mu\nu} \equiv (\vec{\mathrm{J}}, \vec{\mathrm{K}})$ generate two irreps of the Lorentz group ($\mathfrak{L}$):

(a) the half-integer band $s=\frac{1}{2}, \frac{3}{2}, \ldots,$ and
(b) the integer band $s=0, 1, 2, \ldots.$

These two irreps are easily recognized as rotational bands belonging to a (noncompact) version of the rotator symmetry group SU(2)×SU(2).

If we now use the Hamiltonian of Eq. (3.26), $H=[f(a_1\bar{a}_1+a_2\bar{a}_2+1)]^{1/2}$, we see that the two bands each comprise states of different energies, and that the bands exhaust the set of all eigenstates of the energy. These properties identify the group generated by $S_{\mu\nu}$ as a "dynamical symmetry group" of the Hamiltonian H.

These considerations show that one may consider the structure in two distinct ways:

(a) The rest-frame system is that of a harmonic oscillator, with $n=2s$ or

(b) the rest-frame Hamiltonian has a dynamical symmetry group $\mathfrak{L}$.

In both views the states split (superselection rule) into two subsystems, the two bands above.

These results, which for $p=(0,0,0,M)$ are almost too obvious, become significant when it is realized that they are *Poincaré-covariant*. That is to say, the structure is in view (a) that of a *relativistic harmonic oscillator*, or in view (b) that of a *Poincaré-covariant dynamical symmetry*. From this viewpoint the results are possibly surprising.

We have based these results on considerations for $p=(0,0,0,M)$. But if one recalls the proofs in Sec. III C it is clear that, by using boson operators adapted to general p/M, precisely the same group-theoretic properties obtain. In other words, one has, in view (a) a relativistic harmonic oscillator of the stability group for general p/M, or in view (b) a dynamical stability group for p/M. (The relevant generators are the $S_{\mu\nu}$ using bosons adapted to p/M.)

There is a third view of this structure which is, to us, the most physically appealing. It is well known that, because of parity, the Regge-band structure (for integer spin) splits into bands having steps of two units of spin. To incorporate this feature, Dothan, Gell-Mann, and Ne'eman[22] proposed that the appropriate dynamical symmetry, for both hadronic and nuclear physics, was not the rotator ($\mathcal{L}$) but rather the group SL$(3,R)$—the group of rotations and volume-preserving deformations (shears) of three-space. To incorporate half-integer spin the (twofold) covering group cov[SL$(3,R)$] must be used.

Let us demonstrate that the cov[SL$(3,R)$] group is indeed a dynamical stability group for our composite object. It is convenient to realize this group by choosing the representation (3.32)–(3.35) [or alternatively, (3.26)–(3.29)] as the structure into which to introduce cov[SL$(3,R)$]. The representation (3.32)–(3.35) involves the original boson operators, defined in connection with Eq. (3.1). Using these same boson operators, the group cov[SL$(3,R)$] is generated by the vector operator J_i of Eq. (3.1) and by the quadrupole operator $\underline{T}_2$, defined by

$$T_2^M=\left[\frac{3!}{(2+m)!(2-m)!}\right]^{1/2}(V_0)^{-1/2}[(a_1)^{2+M}(a_2)^{2-M}+(-\bar{a}_2)^{2+M}(\bar{a}_1)^{2-M}](V_0)^{-1/2}, \tag{4.1}$$

where V_0 is the number operator given by the last line of Eq. (3.1). It is easily verified that T_2 is Hermitian and transforms as a quadrupole operator under $\vec{J}$. Hence we get the desired commutation rule

$$[J,T_2^M]=C_{Mi}^{212}\,T_2^{M+i}\ . \tag{4.2}$$

All remaining commutation rules of cov[SL$(3,R)$] are obtainable (by commutation with J) from a single relation, the cov[SL$(3,R)$] condition:

$$\begin{aligned}[T_2^2,T_2^{-2}]\,\mathrm{u}^{s,m}&=-4J_3\mathrm{u}^{s,m}\\&=-4m\mathrm{u}^{s,m}\,,\end{aligned} \tag{4.3}$$

where $\mathrm{u}^{s,m}$ is defined by Eq. (3.15).

One can verify, after a bit of algebra, that the quadrupole operator T_2^M, Eq. (4.3), does indeed satisfy[23] Eq. (4.3). We have thereby proved that our realization does indeed generate cov[SL$(3,R)$] *on the given basis*.[24]

It can be shown[23] that the two invariant operators I_2 and I_3 of cov[SL$(3,R)$] assume the *unique* eigenvalues on our basis,

$$I_2=\tfrac{1}{12}\,(J^2-\tfrac{1}{2}T^2)\to\tfrac{1}{4},\quad I_3\to 0 \tag{4.4}$$

corresponding to the generalized Young pattern labels $[p,q,0]$ $p=\frac{3}{2}$, $q=0$. [Here T^2 is just $\sum_M(-1)^M T_2^M T_2^{-M}$, the usual invariant.] Note in particular that these eigenvalues belong to the *discrete* spectrum.

One recognizes at once that since T transfers four-quanta, the space $\{M,s\}$ splits into four disjoint pieces, namely $2s\equiv 2k$ (mod 4), $k=0,\frac{1}{2},1,\frac{3}{2}$. Thus our construction yields three distinct primitive unirreps[25]

(a) $s=0,2,4,\ldots$ $(k=0)$,
(b) $s=1,3,5,\ldots$ $(k=1)$,
(c) $s=\frac{1}{2},\frac{5}{2},\frac{9}{2},\ldots$ $(k=\frac{1}{2})$,

and a fourth "quasirep"

(d) $s=\frac{3}{2},\frac{7}{2},\frac{11}{2},\ldots$ $(k=\frac{3}{2})$.

Since these four bands have precisely the content of four known Regge sequences[26] one may label them as the π, ρ, N, and Δ Regge trajectories if desired.

To summarize: We have shown that the most interesting way to view this "relativistic composite object" is to view the structure as realizing four distinct Regge bands and incorporating the *dynamical stability group* cov[SL$(3,R)$], the group of rotations and volume-preserving deformations of three-space.[27] We suggest that the concept of a dynamical stability group is the proper concept to replace the unworkable concept of a global Lie group symmetry in relativistic quantum mechanics. [It may not be amiss to remark here that in the "bag" models of hadrons it is an essential structural feature that the (energy/volume) be constant. This feature implies that SL$(3,R)$ is an inherent symmetry of the overall "bag-model" structures.]

V. AN OVERLY SIMPLE, BUT CORRECT, RELATIVISTIC SU(6) MODEL

The present section is in the nature of an application of the model-building concepts exemplified in the preceding sections. We wish to apply these ideas in a straightforward way to construct a logically precise relativistic model displaying a discrete Regge band incorporating SU(6) symmetry in a nontrivial way. The model is intended as a cautionary example, and since it experiences, clear defects (e.g., an observable 6-plet) we do not wish it to be taken overly seriously.

Shortly after the impressive success of SU(6) symmetry in accounting for the 56-plet of baryons, and the 35-plet of mesons, hasty attempts were made to construct relativistic symmetry structures by the abuse of group theory. The defects in these approaches led to a number of theorems (MacGlinn's theorem, O'Raifeartaigh's theorem) sharply delimiting the scope of such model building. The contrary view then became prevalent—as for example in Dalitz's summary at the Berkeley conference[28]—that relativistic SU(6) was not possible.[29] Further attempts avoiding earlier errors and using infinite-component wave functions[30] seemingly ran into equally severe problems[31] [e.g., violation of the spectral conditions (spacelike and lightlike solutions)].

To demonstrate explicitly the feasibility of a relativistic SU(6) model, let us give an improved version of the model of Ref. 32. The improved model is invariant for all Poincaré transformations and has all the properties which one would like a simple relativistic SU(6) model to have. The model is obtained by extending the construction of Sec. III to three pairs of oscillators: The generators act on wave functions of the form

$$\phi(p;\{\xi_i^A\}),\quad i=1,2,\quad A=1,2,3, \tag{5.1a}$$

with inner product

$$(\phi,\psi)=\int d\vec{p}\prod_{i,A}d\xi_i^A\,\phi^*(p;\{\xi_i^A\})\frac{1}{P_0}\psi(p;\{\xi_i^A\}). \tag{5.1b}$$

In momentum space the Poincaré generators take the form

$$\vec{P}=\vec{p}\,, \tag{5.2}$$

$$P_0=+\left[\vec{p}^2+f\left(\sum_{A=1}^{3}a_1^A\bar{a}_1^A+\bar{a}_2^Aa_2^A\right)\right]^{1/2}, \tag{5.3}$$

$$M_{ij}=ip_i\frac{\partial}{\partial p_j}-ip_j\frac{\partial}{\partial p_i}+\sum_{A=1}^{3}J_{ij}^{(A)}, \tag{5.4}$$

$$M_{0i}=P_0\frac{\partial}{\partial p_i}+\left[P_0+f^{1/2}\left(\sum_{A=1}^{3}a_1^A\bar{a}_1^A+\bar{a}_2^Aa_2^A\right)\right]^{-1}\times\tfrac{1}{2}\epsilon_{ijk}\left(p_j\sum_{A=1}^{3}J_k^{(A)}-p_k\sum_{A=1}^{3}J_j^A\right). \tag{5.5}$$

These generators can be written in the Thomas form by introducing quasi-Newtonian coordinates.

The masses described by this model clearly fall into SU(6) multiplets, since they belong to eigenstates of a 6-fold degenerate harmonic oscillator. Denoting the total number of quanta by n, one finds that the mass eigenstates are characterized by the Regge band

$$M^2=f(n+3),\quad n=0,1,\ldots\,. \tag{5.6}$$

For each n the associated mass multiplet belongs to the (totally symmetric) SU(6) irrep $[n0]$. The standard reduction[33] shows that, for example, the zero-quanta state has spin 0; the $n=1$ multiplet is an SU(3) triplet of spin-$\frac{1}{2}$ states; $n=2$ has a 6-plet with spin 1 and an antitriplet with spin 0; $n=3$ is the familiar SU(6) 56-plet.

Breaking of SU(6) symmetry is achieved easily by giving the three oscillators instead of a common f three different f's.

This simple model suffices to show that a fully relativistic SU(6) model does indeed exist having the following properties:

(a) Poincaré covariance,

(b) distinct, discrete SU(6) mass multiplets,

(c) no spurious (spacelike, timelike) states,

(d) the symmetry is not trivial [not the direct product SU(6) × $\mathcal{P}$],

(e) the dynamical stability group is realized by unitary operators acting in physical Hilbert space.

The model does not contradict O'Raifeartaigh's theorem since this theorem contains the extremely stringent hypothesis that the entire structure of the system be comprised in a finite-rank Lie group.[5] There is no such overall Lie group for our model, and the theorem is accordingly not applicable.

Actually this situation is already clear from Sec. IV. There we demonstrated that the set of states belonging to each of the three bands (0, $\frac{1}{2}$, and 1) formed the carrier space of a dynamical stability group and although fully relativistic, the structure was certainly not a direct product with $\mathcal{P}$. These results emphasize once again that the relevant structure for physics is the Hamiltonian and *its* symmetries (dynamical or otherwise); the problem posed by relativity is to complete the set of ten Poincaré generators. It is an unnecessary and overly restrictive view to require that the $\mathcal{P}$ generators plus the symmetry generators must themselves fit into a single overall Lie group.[5]

VI. INTRODUCTION OF MINKOWSKI POSITION COORDINATES

Let us now complete the program begun in Sec. II and introduce Minkowski position coordinates in place of the quasi-Newtonian coordinates discussed there (and in succeeding sections). The use of Minkowski coordinates is essential in constructing electromagnetic interactions.

The first step is to change from the Wigner momentum-space wave functions to wave functions more appropriate to describe interactions. For the spin-$\frac{1}{2}$ case there are two possibilities leading to the introduction of either dotted or undotted spinors. Let us sketch the development, for undotted spinors, and then proceed to a similar construction for the composite structure described in Sec. III.

From Sec. II we found the Wigner representation to be

$$[U(d,A)\phi]_i(p') = e^{ip'\cdot d}\sum_{j=1}^{2}(B_{p'}{}^{-1}AB_p)_{ij}\phi_j(p); \tag{6.1a}$$

observe that the spin matrix transformation depends on the value of p. This is often inconvenient, and the inconvenience can be removed simply by defining the undotted spinor $\chi_i(p)$ as

$$\chi_i(p) = \sum_{j=1}^{2}(B_p)_{ij}\phi_j(p). \tag{6.1b}$$

Clearly then for the Poincaré element (d,A), we obtain the representation

$$\chi_i'(p') = \sum_{j=1}^{2}(B_{p'})_{ij}\phi_j'(p') = e^{ip'\cdot d}\sum_{j=1}^{2}A_{ij}\chi_j(p). \tag{6.2}$$

The transformation matrix A_{ij} now depends only on the Lorentz group element A, as desired. The functions $\chi(p)$ are much easier to use in constructing invariant interactions and Lagrangians. One can also, with their aid, easily construct Minkowski wave functions by using the Fourier transformation

$$\chi(x) = \int\frac{d\vec{p}}{p_0}e^{ip\cdot x}\chi(p). \tag{6.3}$$

Clearly then one finds

$$\chi'(x') = A\chi(\Lambda^{-1}(A)x - d). \tag{6.4}$$

Using this procedure, which is standard, as the proper approach let us now apply it to the momentum-space wave functions of Secs. III C and III D. From Eq. (3.20) we find these functions transform as

$$[U(d,A)\phi](p', \xi_1, \xi_2) = e^{ip'd}\mathfrak{U}(B_{p'}{}^{-1}AB_p) \times \phi(p, \xi_1, \xi_2), \tag{6.5}$$

where $B_{p'}{}^{-1}AB_p$ is a rotation and where U is defined in Sec. III A. This form suggests that one define, in analogy with Eq. (6.1),

$$\chi(p; \xi_1, \xi_2) \equiv \mathfrak{U}(B_p)\phi(p; \xi_1, \xi_2). \tag{6.6}$$

Then for a Poincaré transformation (d, A) one has

$$\chi'(p'; \xi_1, \xi_2) = e^{ip'\cdot d}\mathfrak{U}(A)\chi(p; \xi_1, \xi_2). \tag{6.7}$$

The transformation (6.7) can indeed be carried out, but needs some elaboration as the boost $U(B_p)$ contains p_0 defined as an operator by (3.27). To proceed, one first changes the coordinates of $\phi(p; \xi_1, \xi_2)$ from $\vec{p}$ [P^0 determined via (3.27)] to a unit four-vector l_p parallel to p [the length, M, of the four-vector p is given by $M^2 = f(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)$]. Then, the transformation $\mathfrak{U}(B_p)$ depends, via (2.2), only on l_p and not on the length, M, and the transformation (6.6) is well defined. The unitary similarity transformation $\mathfrak{U}(B_p)$ changes the operator $a_1\bar{a}_1 + a_2\bar{a}_2 + 1$ into

$$\mathfrak{U}(B_p)(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)\mathfrak{U}^{-1}(B_p) = a_1\bar{a}_1(B_p) + a_2\bar{a}_2(B_p) + 1,$$

using Eq. (3.9). Hence, in the new coordinates Eq. (3.24) reads

$$M^2 = f(a_1\bar{a}_1(B_p) + a_2\bar{a}_2(B_p) + 1). \tag{6.8}$$

The infinitesimal generators are given most easily by considering the p in $\chi(p; \xi_1, \xi_2)$ to have four independent components,

$$P \to p, \tag{6.9}$$

and by introducing the mass spectrum via the supplementary condition

$$p^2 = f(a_1\bar{a}_1(B_p) + a_2\bar{a}_2(B_p) + 1), \tag{6.10}$$

the remaining generators being given by

$$M_{ij} = i\left(p_i\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial p_i}\right) + S_{ij}, \tag{6.11}$$

$$M_{0i} = i\left(p_0\frac{\partial}{\partial p_i} + p^i\frac{\partial}{\partial p_0}\right) + S_{0i}, \tag{6.12}$$

with $S_{\mu\nu}$ given by Eq. (3.1).[34] One can verify[35] that the argument $a_1\bar{a}_1(B_p) + a_2\bar{a}_2(B_p) + 1$ of the right-hand side of (6.8) commutes with the generators (6.9), (6.11), and (6.12).

Having the functions $\chi(p; \xi_1, \xi_2)$, which transform according to (6.7), it is now possible to define Minkowski coordinates x by a Fourier transformation, just as in (6.4):

$$\chi(x; \xi_1, \xi_2) \equiv \int\frac{d\vec{p}}{p_0}e^{ip\cdot x}\chi(p; \xi_1, \xi_2). \tag{6.13}$$

It follows from (6.7) that $\chi(x; \xi_1, \xi_2)$ transforms under the Poincaré element (d,A) as

$$\chi'(x; \xi_1, \xi_2) = \mathfrak{U}(A)\chi(\Lambda(A)^{-1}x - d; \xi_1, \xi_2). \tag{6.14}$$

The generators on this structure are $P_\mu = -i\partial/\partial x^\mu$, $M_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}x_\alpha P_\beta + S_{\mu\nu}$, where the $S_{\mu\nu}$ are listed in Eq. (3.1), and are taken in the fixed frame a_1, etc. The functions satisfy, by construction, the wave equations

$$\{\Box - f[a_1\bar{a}_1(B_p) + a_2\bar{a}_2(B_p) + 1]\}\chi(x; \xi_1, \xi_2) = 0. \tag{6.15}$$

Vector and tensor fields can be constructed with the aid of the operators of (3.1), where a Minkowski-space vector field is given by

$$V_\mu(x)=\int d\xi_1 d\xi_2 \chi^*(x;\xi_1,\xi_2)V_\mu\chi(x;\xi_1,\xi_2). \quad (6.16)$$

A spinor field can be constructed as

$$\begin{pmatrix} s_1(x)\\ s_2(x)\\ s_3(x)\\ s_4(x)\end{pmatrix}=\int d\xi_1 d\xi_2\chi^*(x;\xi_1,\xi_2)\begin{pmatrix} a_1\\ a_2\\ \bar{a}_2\\ -a_1\end{pmatrix}\chi(x;\xi_1,\xi_2). \quad (6.17)$$

These Minkowski-space fields, really wave functions in the sense of quantum mechanics (i.e., not yet field operators), should be useful in constructing interactions, in particular with the electromagnetic field and for our earlier example of a dual resonance model.

The Minkowski-space wave functions χ have several remarkable features that distinguish them from the quasi-Newtonian form given in Sec. III. The Hamiltonian, $H\equiv P_0$, has the unusual form, as one sees from Eq. (6.15),

$$H\equiv P_0=+\{\vec{P}^2+f[a_1\bar{a}_1(B_p)+a_2\bar{a}_2(B_p)+1]\}^{1/2}. \quad (6.18)$$

This result was found earlier in studying Galilean subdynamics and posed difficult questions about the consistency of that approach. The problem, as one can see from Eq. (6.10), is that the oscillators (defining the "mass operator") *are to be adapted to a frame determined by the four-vector momentum operators themselves*. The consistency, even the lack of circularity, of such a structure was not evident; here the consistency is guaranteed by the properties of the ten Poincaré generators in the quasi-Newtonian coordinates of Sec. III D.

It was argued after Eq. (6.12) that the operator $a_1\bar{a}_1(B_p)+a_2\bar{a}_2(B_p)+1$ commutes with the generators (6.9), (6.11), and (6.12) therefore the Hamiltonian, P_0, commutes with the rotation generators (6.11). However, the orbital ($\vec{X}\times\vec{P}$) and spin (S_{ij}) angular momenta do not separately commute with the Hamiltonian P_0. This is a characteristic feature distinguishing Minkowski vs quasi-Newtonian variables (recall the similar behavior of the "mean spin" and "mean orbital angular momentum" as discussed by Foldy and Wouthuysen). That this behavior occurs here, uniformly for all the many spins, we regard as an encouraging feature of the model.

VII. CONCLUDING REMARKS

We have, throughout the previous discussion, kept the form of the mass-spin relationship essentially arbitrary, but empirically the linear form $M^2=\alpha s+\beta$ is strongly indicated, and there is some reason to believe the slope is characteristic for hadronic physics.

The results presented above indicate quite a different reason to prefer the linear form. From Eq. (3.27) one sees that the use of a linear Regge band *implies that the Hamiltonian belongs to a quadratic form in which a single dimensional constant sets the relative scale between the internal and external spaces*. Equally suggestive is the fact that such an eight-dimensional quadratic form has a unique factorization over the Cayley numbers which does not increase the space-time-spin variables (beyond adjoining negative mass). The resultant structure (*if* it can be implemented consistently) suggests the possibility of new intrinsic superselection spaces.[35]

Interesting as such possibilities may be, they remain at present speculations which are quite independent of the validity and usefulness of the present entity. *By virtue of the explicit realizations given for the Poincaré generators in Secs. III and VI, this deformable object is established in its own right, and merits consideration on its own.*

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