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Magnetic charge in general relativity*

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We consider a dual-charged test particle with electric charge e_0 and magnetic charge g_0 moving in the field of a dual-charged massive central body with charges e and g. A simple method used by Schwinger to study charge quantization in flat space can be generalized to this situation to show that $e_0g - g_0e$ is an integer multiple of $\hbar c$. It is remarkable that the curvature of the space plays no role and that this is the same result that is obtained for flat space. We then show that the test particle can remain at rest in the field only if $ee_0 + gg_0 = \kappa M\mu$ and $e^2 + g^2 = \kappa M^2$, where κ is the gravitational constant and M and μ are the masses of the central and test body, respectively.

I. INTRODUCTION

Dirac's classic discussion of magnetic monopoles[1] provides such an elegant basis for understanding the otherwise mysterious quantization of charge that many physicists continue to have faith in the existence of such particles,[2] independent of the results of experimental searches with either negative or positive results.[3] In particular, Schwinger has based his dyon theory of matter on the existence of particles with both electric and magnetic charge.[2] In this context he has given perhaps the simplest derivation of the classic relation $eg = n\hbar c$ (n an integer) between the fundamental electric charge e and the fundamental magnetic monopole charge g.

In this work we study the effect of the nonlinear nature of the Einstein-Maxwell equations on the above electric-magnetic charge relation and related properties of magnetically charged particles. In Sec. II we review for completeness the elegant and elementary discussion of Schwinger on the charge-quantization relation for two interacting dual-charged particles; this yields a generalization of the classic relation $ge_0 - eg_0 = n\hbar c$, where (e,g) and (e_0,g_0) are the (electric, magnetic) charges of the two particles, and n is an integer. In Sec. III we obtain the electric and magnetic fields and the metric of a dual-charged massive point particle, which we refer to as a dual-charged Reissner-Nordström (RN) field.[4] In Sec. IV we study the motion of a dual-charged test particle in the dual-charged RN field, and demonstrate the remarkable fact that despite the nonlinearity of the Einstein-Maxwell equations and the curved nature of the RN space the above charge-quantization condition still holds. In Sec. V we show that the conditions $ee_0 + gg_0 = \kappa M\mu$ and $e^2 + g^2 = \kappa M^2$ are necessary to allow the test particle to remain at rest in the RN field; the first relation expresses an obvious balance between electromagnetic and gravitational forces identical to the relation necessary in flat space, and the second is a more subtle geometric condition which we refer to as optimally charged.

It must be emphasized that our study treats only the central massive body nonlinear effects. The test particle is assumed to have negligible effect on the fields, e.g., $e_0 \ll e$ and is thus implicitly treated in a linear manner. Hence our results should not be assumed necessarily to hold for two arbitrary dual-charged particles, although one may feel intuitively that they should. Indeed, a study by Parker indicates that for at least some exact two-body solutions contrary results occur.[5] Since the physical nature of the singularities involved in these solutions is not totally clear, further work should be of interest.

II. FLAT-SPACE ANALYSIS

Schwinger's derivation of the charge-quantization condition on dual-charged particles is extraordinarily simple, and we include it here for completeness.[2] Consider a dual-charged particle with mass μ and charges (e_0, g_0) moving at nonrelativistic velocity $\vec{v}$ in the electric field $\vec{E}$ and magnetic field $\vec{H}$ of a stationary dual-charged particle with charges (e,g). The equation of motion is a modified Lorentz equation[2,4]

$$\mu \frac{d\vec{v}}{dt} = e_0\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right) + g_0\left(\vec{H} - \frac{\vec{v}}{c} \times \vec{E}\right). \tag{2.1}$$

The second term is obtained from the first by the dual replacements $\vec{E} \to \vec{H}$, $\vec{H} \to -\vec{E}$, $e_0 \to g_0$. With the fields given by

$$\vec{E} = e\hat{r}/r^2, \quad \vec{H} = g\hat{r}/r^2, \tag{2.2}$$

where $\hat{r}$ is a radial unit vector, we obtain

$$\mu \frac{d\vec{v}}{dt} = (ee_0 + gg_0)\frac{\hat{r}}{r^2} + (ge_0 - eg_0)\frac{\vec{v}}{c} \times \frac{\hat{r}}{r^2}. \tag{2.3}$$

The torque acting on the mechanical angular momentum is therefore

$$\vec{r}\times\mu\frac{d\vec{v}}{dt}=(ge_0-eg_0)\frac{d}{dt}\left(\frac{\hat{r}}{c}\right). \tag{2.4}$$

Thus the mechanical angular momentum is not conserved but the following combination is:

$$\vec{J}=\vec{r}\times\mu\vec{v}-(ge_0-eg_0)\hat{r}/c. \tag{2.5}$$

It is clear that the second term in this expression represents angular momentum contained in the electromagnetic field; what is remarkable is that it is independent of the separation of the dual-charged particles. Quantization of this component of the angular momentum leads to a generalization of the classical relation of Dirac

$$ge_0-eg_0=n\hbar c, \tag{2.6}$$

where n is an integer.

III. DUALLY CHARGED REISSNER-NORDSTRÖM FIELD

We need the electric, magnetic, and metric field of a massive dual-charged point particle. Owing to the symmetry of the Maxwell equations under duality transformations it is clear that the correct metric field can be obtained from the classic RN metric by simply replacing the square of the electric charge, e^2, by the sum of the squares of the electric and magnetic charges, e^2+g^2. Thus in the usual notation, for spherical coordinates,

$$\begin{aligned} &ds^2=g_{00}c^2dt^2-g_{00}{}^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2),\\ &g_{00}=1-\frac{2m}{r}+\frac{\Lambda}{r^2},\quad m\equiv\frac{\kappa M}{c^2},\quad \Lambda\equiv\frac{\kappa}{c^4}(e^2+g^2), \end{aligned} \tag{3.1}$$

where κ is the gravitational constant and M is the mass of the central body. (We use the notation of Ref. 4, but with cgs electromagnetic units.)

Similarly we can obtain the Minkowski tensor for a dual-charged source by a duality rotation,

$$F^{\alpha\beta}=\cos\gamma\, F^{\alpha\beta}_{(e)}+\sin\gamma\, {}^*F^{\alpha\beta}_{(e)}, \tag{3.2}$$

where $F^{\alpha\beta}_{(e)}$ is the Minkowski tensor for the RN field of an electrically charged source and ${}^*F^{\alpha\beta}_{(e)}$ is its dual tensor; these tensors have nonzero components

$$\begin{aligned} &F^{01}_{(e)}=-F^{10}_{(e)}=e/r^2,\\ &{}^*F^{23}_{(e)}=-{}^*F^{32}_{(e)}=e/(r^4\sin\theta). \end{aligned} \tag{3.3}$$

Thus the Minkowski tensor and its dual tensor for the dual-charged system have nonzero components

$$\begin{aligned} &F^{01}=-F^{10}=e/r^2,\\ &F^{23}=-F^{32}=-g/(r^4\sin\theta),\\ &{}^*F^{01}=-{}^*F^{10}=g/r^2,\\ &{}^*F^{23}=-{}^*F^{32}=e/(r^4\sin\theta), \end{aligned} \tag{3.4}$$

with the charges e and g given by $e\cos\gamma\to e$ and $-e\sin\gamma\to g$. In this the dual symmetric nature of the field is manifest, i.e., ${}^*F^{\alpha\beta}$ is obtained from $F^{\alpha\beta}$ by $e\to g$ and $g\to -e$.

IV. ANGULAR MOMENTUM AND CHARGE QUANTIZATION

In direct analogy with the discussion in Sec. I, we consider a dual-charged test particle with charges (e_0,g_0) and mass μ moving in the dual-charged RN field. The equations of motion are an obvious generalization of the Lorentz force law in covariant form:

$$\mu\left(\ddot{x}^\alpha+\left\{{}^{\;\alpha}_{\mu\;\nu}\right\}\dot{x}^\mu\dot{x}^\nu\right)=-e_0F^{\alpha\nu}\dot{x}_\nu-g_0{}^*F^{\alpha\nu}\dot{x}_\nu. \tag{4.1}$$

In this section we will deal only with the angular equations $\phi=x^3$ and $\theta=x^2$. It is straightforward to obtain these from the metric (3.1) and the fields (3.5).

The equation of motion for $x^3=\phi$ is

$$\frac{d}{ds}[\mu r^2\sin^2\theta\,\dot{\phi}+(ge_0-eg_0)\cos\theta]=0. \tag{4.2}$$

We can interpret this easily. The first term in the square brackets is the mechanical angular momentum of the test particle in the ϕ direction, that is the z component; it is not conserved. The second term in the square brackets is (e_0g-g_0e) times the z component of a unit vector in the radial direction. We must interpret this term as representing the angular momentum in the electromagnetic field associated with the existence of the two dual-charged particles but independent of their separation, precisely analogous to the flat-space situation discussed in Sec. II. Only the sum of mechanical plus field angular momentum is conserved. Note incidentally that motion is thus not in a plane.

For $x^2=\theta$ we obtain

$$\frac{d}{ds}(\mu r^2\dot{\theta})-\mu r^2\sin\theta\cos\theta\,\dot{\phi}^2+(ge_0-eg_0)\sin\theta\,\dot{\phi}=0. \tag{4.3}$$

This equation similarly expresses the conservation of total angular momentum, mechanical plus field, along an axis orthogonal to both the z axis and the radial direction.

It is quite remarkable that no gravitational effects of the central body are present in these conservation equations; they are the same as in flat space. As in Sec. II, we may thus impose quantization of angular momentum in the radial direction to obtain the Schwinger relation (2.6). It must be emphasized, however, that since we deal with a test particle we must have $(e_0, g_0)\ll(e,g)$ in order that the net field not be appreciably perturbed by its presence, and similarly the mass must be considered negligible. Thus the integer n in (2.6) should be supposed large. That is, (e, g) must be equal to many fundamental charge units.

The above represents our main result, and in the remaining section we will discuss some related questions of interest.

V. BALANCE OF FORCES

It is evident that the results of the preceding section are valid for general motion of the test particle. Let us, however, consider the question of under what conditions the test particle can remain at rest in the dual-charged RN field. We clearly expect such a possibility as the result of a balance between gravitational attraction and electromagnetic repulsion, and it is indeed known that an analogous balance occurs in exact many-body solutions.[5,6]

From the metric (3.1) and fields (3.5) we may write the remaining $x^0=ct$ and $x^1=r$ equations of motion as

$$\begin{aligned} &\frac{d}{ds}[\mu c^2 g_{00}\dot{t}+(ee_0+gg_0)/(rc)]=0\,, \\ &c^2\dot{t}\,g_{00}=D-(ee_0+gg_0)/(\mu r c)\,, \\ &\mu(\ddot{r}+\tfrac{1}{2}g_{00}'g_{00}c^2\dot{t}^2-g_{00}'\dot{r}^2/2g_{00}-g_{00}r\dot{\theta}^2 \\ &\qquad -r\sin^2\theta g_{00}\dot{\phi}^2)=(ee_0+gg_0)g_{00}\dot{t}/(r^2c)\,, \end{aligned} \tag{5.1}$$

where D is a constant to be determined, and a prime denotes differentiation with respect to r. We seek a solution in which $\dot{\phi}=\dot{r}=\dot{\theta}=0$. It is clear that the angular equations (4.2) and (4.3) are trivially satisfied, whereas the radial and time equations imply that for consistency we must have

$$\mu\ddot{r}=\left\{\frac{ee_0+gg_0}{r^2c^3}-\frac{\kappa\mu M}{r^2c^4}\,\frac{(1-\Lambda/mr)[D-(ee_0+gg_0)/\mu rc]}{(1-2m/r+\Lambda/r^2)}\right\}\left(D-\frac{ee_0+gg_0}{\mu rc}\right). \tag{5.2}$$

We solve this by setting the quantity in curly brackets equal to zero and equating coefficients of the powers of r to zero, since we wish the force balance to hold at any value of r. The result is

$$\begin{aligned} &ee_0+gg_0=\mu m c D=\mu\kappa M D/c\,, \\ &m(ee_0+gg_0)=\mu\Lambda D c=\mu\kappa D(e^2+g^2)/c^3\,. \end{aligned} \tag{5.3}$$

In order to have the classical balance of force hold in the nonrelativistic limit we must choose the constant to be $D=c$. This choice of D can be obtained in an alternative way. From (5.1) we see that for r asymptotically large we can identify $\mu c D$ as the total energy of the particle at infinity. Since we are considering a situation where no forces act and the particle is at rest, we can move it adiabatically to any position with no change in its energy, so that $\mu c D$ must be the rest energy μc^2, and $D=c$. We thus see that a force balance can occur as expected, but only if the charges and mass obey

$$e^2+g^2=\kappa M^2\,, \tag{5.4}$$

which we refer to as optimally charged.

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