Photon pairing instabilities: A microscopic origin for gravitation?*

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We show that general phase-space considerations permit the scattering amplitude for two interacting massless particles to develop a weak-coupling singularity in the ladder approximation. In the case of photon-photon scattering induced by electron vacuum polarization, spin factors prevent such singularities from actually occurring in unaccelerated flat spacetime. However, in the case of conformally flat Riemannian spacetimes, which can be studied using Feynman rules similar to those in the Minkowski-space case, a reevaluation of the photon-photon scattering ladder sum shows that weak-coupling singularities do occur. We conjecture that such instabilities are a general feature of the non-Minkowski case and may provide a microscopic basis for gravitation, with the gravitational fields identified with photon pairing amplitudes of a superconductive type. According to this conjecture, the "graviton" would not be described by a conventional local quantum field.

I. INTRODUCTION

Despite the continued progress in testing and understanding the general theory of relativity at the classical level, attempts to construct a consistent quantized theory of relativity have met with frustration.[1] In the usual approach, where the gravitational fields are treated as local quantum fields, nonrenormalizable infinities are already found at the one-loop level when matter couplings are included. In this paper we explore the possibility of developing an alternative approach to a microscopic theory of gravitation, in which the gravitational fields are composite "pairing" amplitudes which arise as vacuum expectations in a local quantum field theory, but which are not themselves local quantum fields. Thus we have in mind a situation closely analogous to the Ginzburg-Landau-Gor'kov theory of superconductivity, in which the superconducting state is described by an order parameter $\Delta(\vec{r})$ obeying a nonlinear wave equation. Although the order parameter is an off-diagonal expectation of local quantum fields, it is not a quantum field variable and the microscopic Bardeen-Cooper-Schrieffer theory of superconductivity is not obtained by second-quantizing the Ginzburg-Landau equation. In the gravitational analog about which we speculate below, the pairing amplitudes would be off-diagonal expectations of a pair of photon fields, with the pairing interaction arising from the four-photon vacuum polarization interaction in curved or flat accelerated spacetime. In the remainder of this section, we discuss a simple scalar-meson ladder model for the pairing interaction of two massless particles and indicate, on the basis of this model, why vacuum polarization effects cannot produce a photon pairing instability in flat, unaccelerated spacetime.[2] In Sec II we reexamine the photon ladder problem in the special case of conformally flat spacetimes (chosen because, with minor modifications, we can continue to use Minkowski-space Feynman rules) and find evidence suggesting that a pairing instability does occur. In Sec. III we briefly present some speculations (and counter-speculations) on how this instability may become the basis for a microscopic theory of gravitation. Technical details, and some calculations dealing with related issues which are somewhat off the main line of development, are relegated to the appendixes.

As a first orientation, let us consider the simple but unphysical example of a pair of massless scalar particles scattering through the exchange of a scalar particle of mass M, as illustrated in Fig. 1. (For simplicity we neglect crossed diagrams, which only change numerical factors in the calculation to be described. Symmetrization will be properly included in the photon ladder calculations of Sec. II.) For the Nth term in the ladder sum

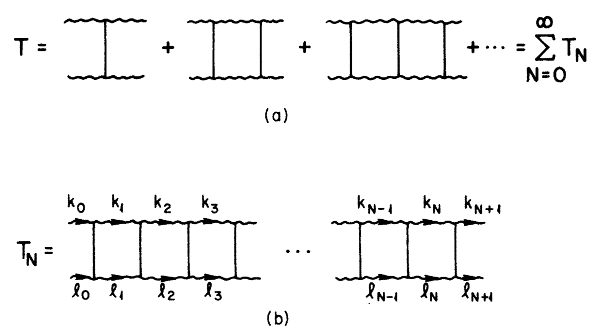

FIG. 1. (a) Scalar-meson ladder series. The wavy lines are massless scalar mesons; the solid lines are scalar mesons of mass M. (b) Momentum labeling for the order-N term of the ladder series.

we get[3]

$$T_N=\left[\prod_{j=1}^{N}\int\frac{d^4k_j}{(2\pi)^4}\int\frac{d^4l_j}{(2\pi)^4}\right]V(k_0l_0k_1l_1)\prod_{j=1}^{N}\left[\frac{i}{k_j^2}\frac{i}{l_j^2}V(k_jl_jk_{j+1}l_{j+1})\right], \tag{1}$$

$$V(k_jl_jk_{j+1}l_{j+1})=(2\pi)^4\delta^4(k_{j+1}+l_{j+1}-k_j-l_j)(ig)^2\frac{i}{(k_{j+1}-k_j)^2-M^2}.$$

In evaluating Eq. (1) it is useful to make the change of variables

$$\begin{aligned}k_j&=\tfrac{1}{2}s_j+r_j,\quad j=0,\ldots,N+1\\ l_j&=\tfrac{1}{2}s_j-r_j\end{aligned} \tag{2}$$

giving (with $s\equiv s_{N+1}$)

$$T_N=(2\pi)^4\delta^4(s-s_0)\left[\prod_{j=1}^{N}\int\frac{d^4r_j}{(2\pi)^4}\right]\frac{-ig^2}{(r_1-r_0)^2-M^2}\left[\prod_{j=1}^{N}\frac{ig^2}{(r_{j+1}-r_j)^2-M^2}\frac{1}{(\frac{1}{2}s+r_j)^2}\frac{1}{(\frac{1}{2}s-r_j)^2}\right]. \tag{3}$$

In order to study the infrared behavior of this integral it suffices to make the approximation

$$(r_{j+1}-r_j)^2-M^2\approx-M^2 \tag{4}$$

in the massive boson propagators (i.e., we treat the interaction of the massless scalars in the local or effective Lagrangian approximation); this introduces logarithmic ultraviolet divergences which we control with a cutoff Λ which we expect to be of order M. Equation (3) then immediately takes the simple form

$$T_N=(2\pi)^4\delta^4(s-s_0)\frac{ig^2}{M^2}L(s)^N, \tag{5}$$

$$\begin{aligned}L(s)&=\int\frac{d^4r}{(2\pi)^4}\frac{-ig^2}{M^2}\frac{1}{(\frac{1}{2}s+r)^2}\frac{1}{(\frac{1}{2}s-r)^2}\\ &=\int_0^1dx\frac{g^2}{16\pi^2M^2}\int_0^{\Lambda^2}\frac{(-r^2)d(-r^2)}{[-r^2-s^2x(1-x)]^2}\\ &=\frac{g^2}{16\pi^2M^2}\ln\left(\frac{\Lambda^2}{-s^2}\right)+(\text{finite terms as }s^2\to0),\end{aligned}$$

and the ladder sum becomes

$$\begin{aligned}T&=\sum_{N=0}^{\infty}T_N\\ &=(2\pi)^4\delta^4(s-s_0)\frac{ig^2}{M^2}\frac{1}{1-\dfrac{g^2}{16\pi^2M^2}\ln\left(\dfrac{\Lambda^2}{-s^2}\right)}.\end{aligned} \tag{6}$$

Before going on to discuss the properties of T, we note for use below that when the massless particles have spin, rather than being scalars, each loop integral contains a polynomial $f(k_j,l_j)$ in the numerator, and the integral $L(s)$ in Eq. (5) is replaced by

$$\begin{aligned}L_f(s)&=\int\frac{d^4r}{(2\pi)^4}(-i)\frac{f(\frac{1}{2}s+r,\frac{1}{2}s-r)}{(\frac{1}{2}s+r)^2(\frac{1}{2}s-r)^2}\\ &=\frac{1}{16\pi^2}\int_0^1dx\,f(sx,s(1-x))\ln\left(\frac{\Lambda^2}{-s^2}\right)\\ &\quad+(\text{finite terms as }s^2\to0).\end{aligned} \tag{7}$$

Turning our attention now to Eq. (6), we see that the denominator in the ladder sum vanishes for

$$s^2=-\Lambda^2\exp(-16\pi^2M^2/g^2), \tag{8}$$

and so for arbitrarily small g^2 there is a (tachyon) pole in the neighborhood of $s^2=0$. Note that such a pole is not present if the massless particles are given a mass μ, since then the $s^2\to0$ limit of T exists and is given by

$$T=(2\pi)^4\delta^4(s-s_0)\frac{ig^2}{M^2}\frac{1}{1-\dfrac{g^2}{16\pi^2M^2}\ln\left(\dfrac{\Lambda^2}{\mu^2}\right)}, \tag{9}$$

which is regular in the limit $g\to0$. The important qualitative difference between the massless and massive cases is a reflection of the fact that phase space for a pair of particles of mass μ is proportional to

$$\rho(s)=\left(\frac{s-4\mu^2}{s}\right)^{1/2} \tag{10}$$

and vanishes at the physical threshold at $s=4\mu^2$. However, when $\mu=0$ Eq. (10) becomes

$$\rho(s)=1 \tag{11}$$

and phase space is nonvanishing[4] down to the threshold at $s=0$. This permits the integral

$$\int_{\text{threshold}}ds'\frac{\rho(s')}{s'-s}(\text{matrix element})^2 \tag{12}$$

to develop a logarithmic divergence at threshold in the massless case, provided that the squared matrix element in the numerator [analogous to the factor f in Eq. (7)] does not vanish at threshold. The phase-space behavior in the massless-particle case is similar to the form of the density of states in the neighborhood of the Fermi surface in a metal, and is one of the principal motivations for our speculations about a superconductive analog involving massless-particle pairing.

Let us now apply the lessons learned from this simple example to the massless particles which actually occur in the real world, photons and neutrinos. The ladder series for photon-photon scattering is illustrated in Fig. 2; the basic interaction mechanism is photon-photon scattering induced by electron vacuum polarization, which for photon wavelengths large compared with the electron Compton wavelength is described by the effective Lagrangian

$$\mathcal{L}_I = \frac{2\alpha^2}{45m^4}(4\mathcal{F}^2 + 7\mathcal{G}^2). \tag{13}$$

Here $\alpha \approx \frac{1}{137}$ is the fine-structure constant, m is the electron mass, and the electromagnetic field invariants $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\begin{aligned} \mathcal{F} &= \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2) = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \\ \mathcal{G} &= \vec{B}\cdot\vec{E} = \tfrac{1}{8}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}, \\ F_{\mu\nu} &= \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}. \end{aligned} \tag{14}$$

Because gauge invariance requires the electromagnetic field amplitudes to couple through their derivatives, when we set up the photon-photon scattering analog of Eq. (1) the vertex factors V are quadrilinear in their momentum arguments (the precise form will be given in Sec. II below),

$$V(k_j l_j k_{j+1} l_{j+1}) \propto k_j l_j k_{j+1} l_{j+1}. \tag{15}$$

But then, according to Eq. (7), for each factor $\ln(\Lambda^2/-s^2)$ arising from a loop integration there are four factors s, which upon contractions over tensor indices give (except near the ends of the ladder) two powers of s^2. Hence the N-loop term in the ladder series contains, in addition to a factor $[\ln(\Lambda^2/-s^2)]^N$, at least $2N$-4 powers[5] of s^2, and so the ladder sum contains no interesting weak-coupling singularities in the vicinity of $s^2 = 0$.

Turning next to the case of neutrino-neutrino (or neutrino-antineutrino) scattering, we again consider the ladder series in Fig. 2, this time with fermion propagators along the side legs and with the interaction supplied by a local four-fermion effective Lagrangian. The vertex factors V are

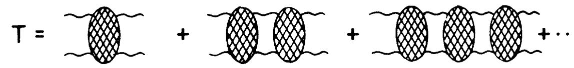

FIG. 2. Ladder series for photon-photon scattering (with the blobs the electron vacuum polarization photon-photon scattering interaction) or for neutrino-neutrino or or neutrino-antineutrino scattering (with the blobs a four-fermion effective interaction).

now constant matrices, with tensor indices contracted from the upper to the lower side of the ladder. The fermion propagators, when rationalized, give numerator factors $\not{k}_j$ ($\not{l}_j$) on the upper (lower) sides of the ladder; according to Eq. (7), in the coefficient of the logarithm coming from each loop integration these factors become an $\not{s}$ on the upper side and an $\not{s}$ on the lower side of the ladder. Commuting or anticommuting all $\not{s}$ factors to the extreme right end of the ladder, we see that the N-loop term in the ladder contains, in addition to a factor $[\ln(\Lambda^2/-s^2)]^N$, at least $N-1$ powers[6] of s^2, and so again there is no weak-coupling singularity in the ladder sum in the vicinity of $s^2 = 0$.

To summarize, while phase-space considerations permit two interacting massless particles to develop a weak-coupling singularity in the ladder approximation, in unaccelerated flat spacetime the spin factors in the photon and neutrino ladders prevent such singularities from actually occurring.

II. PHOTON LADDER-GRAPH SUM IN A CONFORMALLY FLAT METRIC

We turn now to a reexamination of the photon ladder-graph sum in Riemannian spacetimes. We do not consider the general case, but rather, for technical reasons, restrict ourselves to spacetimes described by metrics which are conformally flat,

$$d\tau^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = e^{\psi(x)}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{16}$$

$$\eta_{\mu\nu} = 0, \quad \mu \neq \nu$$

$$\eta_{00} = 1, \quad \eta_{jj} = -1, \quad j = 1, 2, 3.$$

As we shall see, metrics of this type permit the ladder-sum problem to be formulated in terms of flat-space photon propagators, with the effects of the metric transformed to a modification of the interaction vertex. Techniques similar to those of Sec. I can then be used to evaluate the ladder sum in leading-logarithm approximation.

There are two situations of particular physical interest which lead to conformally flat metrics. First of all, the general Robertson-Walker cosmological metric,

$$d\tau^2 = dt_1^2 - \frac{R^2(t_1)}{R_0^2}\left(\frac{dr_1^2}{1-kr_1^2/R_0^2} + r_1^2 d\theta^2 + r_1^2\sin^2\theta\, d\phi^2\right), \qquad R_0 \equiv R(0), \tag{17}$$

is conformally flat.[7] Although the global transformations which show Eq. (17) to be conformally flat are somewhat complicated, it is easy to give an infinitesimal coordinate transformation which puts Eq. (17) in conformally flat form through second-order terms in the expansion of the metric about the spacetime coordinate origin. Thus defining the standard cosmological parameters H_0, q_0 by[8]

$$\frac{R(t_1)}{R_0} = 1 + H_0 t_1 - \tfrac{1}{2}H_0^2 q_0 t_1^2 + \cdots, \tag{18}$$

we set

$$\begin{aligned} t_1 &= t - \frac{1}{12}\left[(1+2q_0)H_0^2 + \frac{k}{R_0^2}\right]t^3 - \tfrac{1}{2}H_0\vec{r}^{\,2} + \frac{1}{4}\left(H_0^2 - \frac{k}{R_0^2}\right)t\vec{r}^{\,2} + \cdots, \\ \vec{r}_1 &= \vec{r}\left[1 + \frac{1}{4}\left(H_0^2 - \frac{k}{R_0^2}\right)\vec{r}^{\,2} - H_0 t + \left(\tfrac{3}{4}H_0^2 - \frac{1}{4}\frac{k}{R_0^2}\right)t^2\right] + \cdots. \end{aligned} \tag{19}$$

In terms of these new coordinates, the line element of Eq. (17) takes the conformally flat form

$$\begin{aligned} d\tau^2 &= e^{\psi}dx^2, \\ e^{\psi} &= 1 + \frac{1}{2}\left(H_0^2 + \frac{k}{R_0^2}\right)x^2 - \left[(1+q_0)H_0^2 + \frac{k}{R_0^2}\right]t^2 + \cdots, \\ x^2 &= t^2 - \vec{r}^{\,2}. \end{aligned} \tag{20}$$

A second case which yields a conformally flat metric is ordinary unaccelerated flat space

$$d\tau^2 = dx_1^2, \tag{21}$$

which when viewed from the coordinate frame obtained by making the special conformal coordinate transformation[9]

$$x_{1\mu} = \frac{x_\mu + c_\mu x^2}{1 + 2c\cdot x + c^2x^2} \tag{22}$$

has the metric

$$d\tau^2 = e^{\psi}dx^2, \qquad e^{\psi} = (1 + 2c\cdot x + c^2x^2)^{-2}. \tag{23}$$

From the inverse transformation

$$x_\mu = \frac{x_{1\mu} - c_\mu x_1^2}{1 - 2c\cdot x_1 + c^2 x_1^2} \tag{24}$$

we see that the origin $\vec{x}=0$ in the new frame appears, to an observer in the original inertial frame, to be uniformly accelerating when terms of order c^3 and higher are neglected,[9]

$$\vec{x} = 0 \Rightarrow \vec{x}_1 = \vec{c}(t_1^2 - \vec{x}_1^2) = \vec{c}t_1^2 - \vec{c}\,\vec{c}^{\,2}t_1^4 + \cdots. \tag{25}$$

Evidently, a conformally flat spacetime can be written in explicitly conformally flat form in many ways since, in addition to the usual freedom of making Poincaré transformations, one can make arbitrary coordinate transformations of the form of Eq. (22) and still preserve the metric form given in Eq. (16). Of course, under such transformations the conformal factor e^{ψ} will change; for example, apart from homogeneous Lorentz transformations the conformal factor

$$e^{\psi} = (1 + 2c\cdot x + c^2x^2)^{-2}\left\{1 + \frac{1}{2}\left(H_0^2 + \frac{k}{R_0^2}\right)x^2 - \left[(1+q_0)H_0^2 + \frac{k}{R_0^2}\right]t^2 + \cdots\right\} \tag{26}$$

gives the general representation of the Robertson-Walker metric in the form of Eq. (16), through terms of second order in an expansion around a fixed coordinate origin.

In order to evaluate the ladder sum, we must find the appropriate photon Feynman rules when the metric is given by Eq. (16). The simplest way to do this is to write the photon kinetic and effective Lagrangian terms in generally covariant form,

$$\begin{aligned} \mathcal{L} &= \sqrt{g}\left[-\mathfrak{F} + \frac{2\alpha^2}{45m^4}(4\mathfrak{F}^2 + 7\mathcal{G}^2)\right], \\ g &= -\det[g_{\mu\nu}], \\ \mathfrak{F} &= \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \mathcal{G} = \frac{1}{8}\frac{1}{\sqrt{g}}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}, \\ F^{\mu\nu} &= g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma}, \quad F_{\lambda\sigma} = \frac{\partial A_\lambda}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\lambda} = A_{\lambda;\sigma} - A_{\sigma;\lambda}. \end{aligned} \tag{27}$$

When the metric is conformally flat, we have

$$g_{\mu\nu}=e^{\psi}\eta_{\mu\nu}, \quad \sqrt{g}=e^{2\psi}, \tag{28}$$

and it is convenient to introduce rescaled electromagnetic field strength tensors defined by

$$\tilde{F}_{\mu\nu}=\frac{\partial\tilde{A}_{\mu}}{\partial x^{\nu}}-\frac{\partial\tilde{A}_{\nu}}{\partial x^{\mu}}=F_{\mu\nu}, \qquad \tilde{A}_{\mu}=A_{\mu},$$
$$\tilde{F}^{\mu\nu}=\eta^{\mu\lambda}\eta^{\nu\sigma}\tilde{F}_{\lambda\sigma}=e^{2\psi}F^{\mu\nu}. \tag{29}$$

The Lagrangian density then takes the simple form

$$\mathfrak{L}=-\tilde{\mathfrak{F}}+\frac{2\alpha^{2}}{45\tilde{m}(x)^{4}}(4\tilde{\mathfrak{F}}^{2}+7\tilde{\mathfrak{G}}^{2}),$$
$$\tilde{\mathfrak{F}}=\tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \tilde{\mathfrak{G}}=\tfrac{1}{8}\epsilon^{\mu\nu\lambda\sigma}\tilde{F}_{\mu\nu}\tilde{F}_{\lambda\sigma},$$
$$\tilde{m}(x)=m\,e^{\psi(x)/2}, \tag{30}$$

which is identical in form to the Lagrangian density in unaccelerated flat spacetime, except that the "mass" $\tilde{m}$ has become coordinate dependent through the conformal factor $e^{\psi/2}$. [An alternative way of deriving this result, given in Appendix A, is to use general covariance to derive the change in the Minkowski-space coupled Maxwell-Dirac equations when one goes to a conformally flat metric, and then to use the Callan-Symanzik equations to calculate the induced modification in the effective Lagrangian. In Eqs. (27) and (30) we have neglected vacuum polarization correction terms involving covariant derivatives of the field strengths, and also corrections proportional to tensors formed from the curvature tensor and its covariant derivatives (which vanish in the Minkowski-metric case). We argue in Appendix A that such terms should not spoil the ladder-graph arguments given below.] From the Lagrangian density of Eq. (30), we can read off the Feynman rules needed for evaluating the photon ladder-graph sum. Since the kinetic term in Eq. (30) has the usual Minkowski-metric form, the photon propagator continues to be given by the usual expression

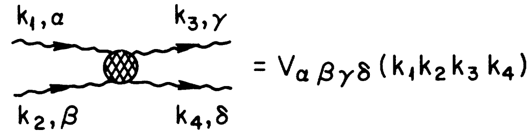

FIG. 3. Vertex part for the photon-photon scattering calculation. The blob indicates the four-photon interaction generated by Eq. (30); the indices α, β, γ, δ label the photon polarization states.

$$iD_{F}(k)_{\mu\nu}=\frac{-i\eta_{\mu\nu}}{k^{2}+i\epsilon}. \tag{31}$$

The vertex Feynman rule corresponding to the diagram of Fig. 3 is just the matrix element of $i\mathfrak{L}_{I}$, with $\mathfrak{L}_{I}$ the interaction term in Eq. (30). Here there is a difference from the usual Minkowski-metric form, since the explicit x dependence of the coefficient $[2\alpha^{2}/(45m^{4})]\exp[-2\psi(x)]$ results in a breakdown of four-momentum conservation. We readily find

$$V_{\alpha\beta\gamma\delta}(k_{1}k_{2}k_{3}k_{4})=i\,\exp\left[-2\psi\left(i\frac{\partial}{\partial K}\right)\right](2\pi)^{4}\delta^{4}(K)k_{1}^{\xi}k_{2}^{\eta}k_{3}^{\lambda}k_{4}^{\sigma}U_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)}, \tag{32}$$
$$K=k_{1}+k_{2}-(k_{3}+k_{4}),$$
$$U_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)}=\frac{4\alpha^{2}}{45m^{4}}\{4[(\eta_{\alpha\beta}\eta_{\xi\eta}-\eta_{\alpha\eta}\eta_{\beta\xi})(\eta_{\gamma\delta}\eta_{\lambda\sigma}-\eta_{\gamma\sigma}\eta_{\delta\lambda})+(\eta_{\alpha\gamma}\eta_{\xi\lambda}-\eta_{\alpha\lambda}\eta_{\gamma\xi})(\eta_{\beta\delta}\eta_{\eta\sigma}-\eta_{\beta\sigma}\eta_{\delta\eta})$$
$$+(\eta_{\alpha\delta}\eta_{\xi\sigma}-\eta_{\alpha\sigma}\eta_{\delta\xi})(\eta_{\beta\gamma}\eta_{\eta\lambda}-\eta_{\beta\lambda}\eta_{\gamma\eta})]$$
$$+7[\epsilon_{\alpha\xi\beta\eta}\epsilon_{\gamma\lambda\delta\sigma}+\epsilon_{\alpha\xi\gamma\lambda}\epsilon_{\beta\eta\delta\sigma}+\epsilon_{\alpha\xi\delta\sigma}\epsilon_{\beta\eta\gamma\lambda}]\}.$$

Since we only plan to work through second-order terms in an expansion of $\exp[-2\psi(x)]$ around $x=0$ [this is the leading order in which interesting deviations from the Minkowski-space case appear; note that there is a small subclass of conformally flat metrics for which the truncated expansion is exact], we write

$$\exp[-2\psi(x)]=1+D_{\mu}x^{\mu}-\tfrac{1}{2}E_{\mu\nu}x^{\mu}x^{\nu}+\cdots. \tag{33a}$$

When the conformal factor is chosen as in Eq. (26), the coefficients D_{μ} and $E_{\mu\nu}$ are given by

$$D_{\mu}=8c_{\mu},$$
$$-\tfrac{1}{2}E_{\mu\nu}=\eta_{\mu\nu}\left[4c^{2}-\left(H_{0}^{2}+\frac{k}{R_{0}^{2}}\right)\right]+2\left[(1+q_{0})H_{0}^{2}+\frac{k}{R_{0}^{2}}\right]\eta_{\mu0}\eta_{\nu0}+24c_{\mu}c_{\nu}. \tag{33b}$$

Finally, we note that because the vertex of Eq. (32) is completely symmetrized, we must include a symmetrization factor of $\frac{1}{2}$ for each photon loop in the ladder to avoid overcounting.

Using the Feynman rules which we have just derived, we may immediately write down an expression for the Nth term in the ladder sum (with $\alpha_f=\alpha_{N+1}$, $\beta_f=\beta_{N+1}$),

$$(T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f}=\frac{1}{2^N}\left[\prod_{j=1}^{N}\int\frac{d^4k_j}{(2\pi)^4}\int\frac{d^4l_j}{(2\pi)^4}\right]V_{\alpha_0\beta_0}{}^{\alpha_1\beta_1}(k_0l_0k_1l_1)\prod_{j=1}^{N}\left[\frac{(-i)}{k_j^2}\frac{(-i)}{l_j^2}V_{\alpha_j\beta_j}{}^{\alpha_{j+1}\beta_{j+1}}(k_jl_jk_{j+1}l_{j+1})\right]. \quad (34)$$

Making the change of variables of Eq. (2), substituting Eq. (32), defining

$$\mathfrak{D}_j=1+iD_\mu\frac{\partial}{\partial s_{j\mu}}+\tfrac{1}{2}E_{\mu\nu}\frac{\partial}{\partial s_{j\mu}}\frac{\partial}{\partial s_{j\nu}}, \quad (35)$$

and isolating the loop integrals by defining[10]

$$L^{\lambda\sigma\xi\eta}(s)=\int\frac{d^4r}{(2\pi)^4}\frac{-i}{(\frac{1}{2}s+r)^2(\frac{1}{2}s-r)^2}(\tfrac{1}{2}s+r)^\lambda(\tfrac{1}{2}s-r)^\sigma(\tfrac{1}{2}s+r)^\xi(\tfrac{1}{2}s-r)^\eta,$$

Eq. (34) may be rewritten in the form (with $k_f=k_{N+1}$, $l_f=l_{N+1}$)

$$(T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f}=\frac{1}{2^N}\left(\prod_{j=1}^{N}\int d^4s_j\right)i(2\pi)^4[\mathfrak{D}_0\delta^4(s_0-s_1)]k_0^{\lambda_0}l_0^{\sigma_0}(k_f)_{\lambda_{N+1}}(l_f)_{\sigma_{N+1}}U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_1\lambda_1)(\beta_1\sigma_1)}$$
$$\times\prod_{j=1}^{N}\{L_{\lambda_j\sigma_j}{}^{\xi_j\eta_j}(s_j)[\mathfrak{D}_j\delta^4(s_j-s_{j+1})]U_{(\alpha_j\xi_j)(\beta_j\eta_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}\}. \quad (36)$$

The differential operators $\mathfrak{D}_j$ in this equation act only on the δ^4 functions with which they are bracketed. In evaluating L we keep *all* terms containing a $\ln s^2$ (factors of s^2 in front can be eliminated by the derivatives contained in the factors $\mathfrak{D}$) but drop terms which are polynomials in s^2 [they cannot contribute to the $(\ln s^2)^N$ term in T_N], giving

$$L^{\lambda\sigma\xi\eta}(s)=\frac{1}{(4\pi)^2}\frac{1}{10}\ln\left(\frac{\Lambda^2}{-s^2}\right)Q^{\lambda\sigma\xi\eta}(s), \quad (37)$$

$$Q^{\lambda\sigma\xi\eta}(s)=\tfrac{1}{3}s^\lambda s^\sigma s^\xi s^\eta+\tfrac{1}{6}(\eta^{\lambda\sigma}s^\xi s^\eta+\eta^{\lambda\eta}s^\sigma s^\xi+\eta^{\sigma\xi}s^\lambda s^\eta+\eta^{\xi\eta}s^\lambda s^\sigma)s^2$$
$$-\tfrac{1}{4}(\eta^{\lambda\xi}s^\sigma s^\eta+\eta^{\sigma\eta}s^\lambda s^\xi)s^2+\tfrac{1}{24}(\eta^{\lambda\sigma}\eta^{\xi\eta}+\eta^{\lambda\xi}\eta^{\sigma\eta}+\eta^{\lambda\eta}\eta^{\sigma\xi})(s^2)^2.$$

Finally, it is useful to define

$$M_{(\alpha_j\lambda_j)(\beta_j\sigma_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}(s_j)=L_{(\lambda_j\sigma_j)}{}^{(\xi_j\eta_j)}(s_j)U_{(\alpha_j\xi_j)(\beta_j\eta_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}, \quad (38)$$

allowing us to write T_N in the compact form

$$(T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f}=\frac{1}{2^N}\left(\prod_{j=1}^{N}\int d^4s_j\right)i(2\pi)^4[\mathfrak{D}_0\delta^4(s_0-s_1)]k_0^{\lambda_0}l_0^{\sigma_0}(k_f)_{\lambda_{N+1}}(l_f)_{\sigma_{N+1}}U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_1\lambda_1)(\beta_1\sigma_1)}$$
$$\times\prod_{j=1}^{N}\{M_{(\alpha_j\lambda_j)(\beta_j\sigma_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}(s_j)[\mathfrak{D}_j\delta^4(s_j-s_{j+1})]\}. \quad (39)$$

Because of the presence of the differential operators $\mathfrak{D}_j$, the expression for T_N in Eq. (39) has a very complicated structure. If, for example, we adopt the convention of successively integrating by parts to bring all derivatives to the left so that they act on the initial δ function $\delta^4(s_0-s_1)$, we will find a result containing tensor contractions of derivatives of widely separated M's. However, there is one special situation in which the structure can be made to simplify greatly, so that an evaluation of T_N in leading-logarithm approximation becomes possible. This is obtained by taking the final two photons to be parallel and nearly on-shell,

$$k_f=\theta s,\quad l_f=(1-\theta)s, \quad (40a)$$

while at the same time (to keep the residue of the sum of leading logarithms from vanishing) keeping the initial two photons off-shell,

$$k_0{}^2\neq 0,\qquad l_0{}^2\neq 0. \quad (40b)$$

We then find, as shown in Appendix B, that the only contributions to the leading logarithm come from terms where the derivatives in each $\mathfrak{D}_j$, on integration by parts, act on the factor

$$M_{(\alpha_j\lambda_j)(\beta_j\sigma_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}(s_j)$$

which stands immediately to the left, making possible an inductive evaluation of Eq. (39).

Since the results depend explicitly on the helicity state of the final two photons, we must introduce a notation for this. Let us take the momentum $\vec{s}_f$ of the final two photons as the three-axis, so that

$$s^\mu \approx s^0(1,0,0,1) = (s^0, s^1, s^2, s^3), \tag{41a}$$

and let $\epsilon^{(1)}$, $\epsilon^{(2)}$, and $\epsilon^{(\pm)}$be the polarization vectors

$$\begin{aligned} \epsilon^{(1)} &= (0,1,0,0), \\ \epsilon^{(2)} &= (0,0,1,0), \\ \epsilon^{(\pm)} &= \frac{1}{\sqrt{2}}[\epsilon^{(1)} \pm i\epsilon^{(2)}] \\ &= \frac{1}{\sqrt{2}}(0,1,\pm i,0). \end{aligned} \tag{41b}$$

Then the three possible polarization states of the two parallel photons are described by

$$\begin{aligned} P^{(\pm 2)}_{\alpha\beta} &= \epsilon^{(\pm)*}_\alpha \epsilon^{(\pm)*}_\beta , \\ P^{(0)}_{\alpha\beta} &= \epsilon^{(+)*}_\alpha \epsilon^{(-)*}_\beta + \epsilon^{(-)*}_\alpha \epsilon^{(+)*}_\beta \\ &= \epsilon^{(1)}_\alpha \epsilon^{(1)}_\beta + \epsilon^{(2)}_\alpha \epsilon^{(2)}_\beta . \end{aligned} \tag{42}$$

Denoting the polarization of the initial photons with four-momenta k_0, l_0, by $\epsilon(k_0), \epsilon(l_0)$, the result of Appendix B for the ladder sum in leading-logarithm approximation is

$$\sum_{N=0}^{\infty} \epsilon(k_0)^{\alpha_0} \epsilon(l_0)^{\beta_0} (T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f} P^{(F)}_{\alpha_f\beta_f} = i(2\pi)^4\delta^4(s_0 - s)\{H_0^{(\alpha_0\lambda_0)(\beta_0\sigma_0)} U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_f\lambda_f)(\beta_f\sigma_f)} H^{(F)}_{(\alpha_f\lambda_f)(\beta_f\sigma_f)}\}$$
$$\times\left[1 - \xi(F)\frac{22}{15\times 45}\frac{\alpha^2}{m^4}\frac{1}{(4\pi)^2} s^\mu s^\nu E_{\mu\nu}\ln\left(\frac{\Lambda^2}{-s^2}\right)\right]^{-1}, \tag{43}$$

$$\xi(\pm 2) = 0, \quad \xi(0) = 1,$$

$$H_0^{(\alpha_0\lambda_0)(\beta_0\sigma_0)} = \epsilon(k_0)^{\alpha_0}\epsilon(l_0)^{\beta_0} k_0^{\lambda_0} l_0^{\sigma_0},$$

$$H^{(F)}_{(\alpha_f\lambda_f)(\beta_f\sigma_f)} = P^{(F)}_{\alpha_f\beta_f}(k_f)_{\lambda_f}(l_f)_{\sigma_f}.$$

We see that Eq. (43) shows no instability when the final photon pair helicity is ± 2, while when the photon pair helicity is 0, the ladder sum develops a singularity in the neighborhood of $s^2 = 0$ as long as $E_{\mu\nu}$ is not proportional to the Minkowski metric $\eta_{\mu\nu}$. When $s^\mu s^\nu E_{\mu\nu} > 0$ for real s, the singularity occurs for real values of s; when $s^\mu s^\nu E_{\mu\nu} < 0$ for real s, the singularity occurs for imaginary values of $s = (s^0, \vec{s})$. In both cases the singularity remains at finite s as $E_{\mu\nu} \to 0$, and hence the ladder sum shows an instability for arbitrarily weak coupling. For the particular $E_{\mu\nu}$ given in Eq. (33b), we have

$$\begin{aligned} -\tfrac{1}{2}E_{\mu\nu}s^\mu s^\nu|_{s^2=0} &= 2\left[(1+q_0)H_0^2 + \frac{k}{R_0^2}\right](s^0)^2 + 24(c\cdot s)^2 \\ &= 24(c\cdot s)^2 \quad \text{[accelerated frame conformal factor of Eq. (23)]} \\ &= 2\left[(1+q_0)H_0^2 + \frac{k}{R_0^2}\right](s^0)^2 \text{[purely quadratic Robertson Walker conformal factor of Eq.(20)].} \end{aligned} \tag{44}$$

Thus for the case of flat space viewed from an accelerating frame the singularity always occurs for imaginary s, while in the case of the Robertson-Walker metric the singularity necessarily occurs for imaginary s for $(1+q_0)H_0^2 + k/R_0^2 > 0$, but can occur for real s for $(1+q_0)H_0^2 + k/R_0^2 < 0$.[11]

Continuing our examination of Eq. (43), it is instructive to evaluate the curly bracketed matrix element in the numerator and to compare it with the standard expression for graviton exchange between photons in the kinematic configuration specified by Eq. (40). As shown in Appendix B, the numerator matrix element, in the configuration $F = 0$ for which the ladder sum becomes singular, is given by

$$H_0^{(\alpha_0\lambda_0)(\beta_0\sigma_0)} U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_f\lambda_f)(\beta_f\sigma_f)} H^{(0)}_{(\alpha_f\lambda_f)(\beta_f\sigma_f)} = \frac{11}{45}\frac{\alpha^2}{m^4} T_0^{\mu\nu}(T_f^{(0)})_{\mu\nu}. \tag{45}$$

Here $T_0^{\mu\nu}$ and $(T_f^{(0)})_{\mu\nu}$ are the initial photon and final photon matrix elements of the electromagnetic energy-momentum tensor

$$T_0^{\mu\nu} = -F(k_0)^{\mu\tau}F(l_0)^{\nu}{}_{\tau} + \tfrac{1}{4}\eta^{\mu\nu}F(k_0)^{\tau\sigma}F(l_0)_{\tau\sigma} + (k_0 \leftrightarrow l_0),$$

$$F(k_0)^{\mu\nu} = i[\epsilon(k_0)^{\mu}k_0^{\nu} - \epsilon(k_0)^{\nu}k_0^{\mu}], \tag{46}$$

$$F(l_0)^{\mu\nu} = i[\epsilon(l_0)^{\mu}l_0^{\nu} - \epsilon(l_0)^{\nu}l_0^{\mu}],$$

$$\begin{aligned} T_f^{(0)\mu\nu} &= -F(k_f)^{\mu\tau}F(l_f)^{\nu}{}_{\tau} + \tfrac{1}{4}\eta^{\mu\nu}F(k_f)^{\tau\sigma}F(l_f)_{\tau\sigma} \\ &\quad + (k_f \leftrightarrow l_f) \\ &= 2P^{(0)\tau}{}_{\tau}\theta(1-\theta)s^{\mu}s^{\nu} \\ &= -4\theta(1-\theta)s^{\mu}s^{\nu}. \end{aligned}$$

In the same notation, the graviton exchange matrix element (for general final photon pair helicity F) is proportional to

$$\begin{aligned} \frac{1}{s^2}T_0^{\mu\nu}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})T_f^{(F)\rho\sigma} &= 0, \quad F = \pm 2 \\ &= \frac{2}{s^2}T_0^{\mu\nu}(T_f^{(0)})_{\mu\nu}, \quad F = 0. \end{aligned} \tag{47}$$

Thus, for the $F = \pm 2$ case for which no ladder-graph instability is found, the analogous graviton exchange amplitude vanishes. [The vanishing of the graviton exchange amplitude in this case is just a reflection of the fact that the matrix element for a real (helicity ± 2) graviton to decay into two photons is zero.] For the $F = 0$ case, where there is a ladder-graph instability, the numerator matrix element in the ladder-graph sum is identical in form to the residue at the graviton pole in the analogous graviton exchange amplitude.

In Appendix C we give a related calculation, in which we examine photon scattering in a strong, constant background electromagnetic field, in Minkowski spacetime. Again a weak-coupling singularity is found in the ladder approximation. Finally, in Appendix D we formulate a simple phonon scattering problem, which again develops a weak-coupling singularity in the ladder approximation.

III. SOME SPECULATIONS (AND COUNTERSPECULATIONS)

From the calculations of the preceding section, we see that while the photon-photon scattering ladder sum does not develop interesting singularities in Minkowski spacetime, in a general conformally flat metric weak-coupling singularities are found in the ladder sum, with the structure of the singularities bearing some intriguing resemblances to graviton exchange amplitudes. While we have concentrated on the conformally flat case for technical reasons, we suspect that the ladder-graph singularities found there are a general feature of the non-Minkowski case. The results of Sec. II naturally suggest the speculation that, in some sense yet to be made precise, gravitation is associated with a photon pairing phenomenon. The speculation is supported by the fact that even in the case of flat space viewed from an accelerating frame (where, according to the equivalence principle, a gravitational field is present) an instability is found in the ladder approximation. We note, however, that the ladder sums do not yield the covariant propagator for a zero-mass spin-2 particle; thus our speculation is that just as in the case of superconductivity,[12] singularities in the ladder-sum signal the instability of the conventionally assumed vacuum state, but that the ladder approximation is not sufficiently good to give the properties of the true ground state.

Presumably what is now needed is a reanalysis of the Lagrangian of Eq. (27) using the extended Hartree-Fock approximation; the idea would be to assume a general background metric, to introduce photon pairing amplitudes $\langle E^i(x)E^j(x)\rangle$, $\langle E^i(x)B^j(x)\rangle$, $\langle B^i(x)B^j(x)\rangle$, and then to solve for the photon Green's functions in an appropriately linearized modification of Eq. (27), looking for a prediction of nonvanishing pairing amplitudes which could, in some fashion, be identified with the gravitational fields associated with the background metric. This would evidently be a variant on the superconductive-type pairing calculation, in which the pairing amplitudes would themselves, in a self-consistent manner, be the source of the pairing interaction. It is hoped that terms involving solely the pairing amplitudes would then generate the gravitational kenetic Lagrangian. The gravitational coupling to matter generated by the metric would, as usual, involve the matter energy-momentum tensor, as required by tree graph S-matrix arguments[13] which apply even when the massless spin-2 "graviton" is of composite origin. However, since the gravitational fields would be pairing amplitudes of superconductive type, the "gravitons" would not scatter electromagnetically (as might be expected for an ordinary electromagnetic bound state), and the field commutation arguments used to discredit the neutrino theory of light[14] would not apply. Work along the general lines outlined above is now in progress.[15,16]

We conclude with some cautionary remarks which are a possible counter to the above speculations. We note that the logarithmic term in the denominator of Eq. (43) is much less than unity unless $|s^2|$ is very small, of the order

$$|s^2| \sim m^2 \exp\left[-\frac{15\times 45}{22}\frac{m^4}{\alpha^2}\frac{(4\pi)^2}{|s^{\mu}s^{\nu}E_{\mu\nu}|}\right]. \tag{48}$$

If we associate a characteristic length l with this value of s^2 by writing $l^{-2} \sim |s^2|$, then we readily see that l is much larger than the radius of the universe. Hence the singularities in the ladder sum are of interest *only* if they signal the presence of a vacuum state which differs from the normal one by observable amounts; if this is not the case the singularities in themselves apparently pose no problem since, in scattering amplitudes smeared with physically realizable wave packets, their effects would be undetectable. An even more serious objection, perhaps, is that our focus on ladder graphs is simplistic; N-loop nonladder graphs generated by the Lagrangians of Eqs. (27) or (30), such as those illustrated in Fig. 4, also behave as $(\ln s^2)^N$ as $s^2 \to 0$. These additional contributions could significantly alter the behavior of the photon-photon scattering amplitude from what is found in the ladder approximation. Again, as we have already emphasized, the ultimate significance of the ladder approximation singularities which we have found can only be determined by a reanalysis of our model Lagrangians from a self-consistent extended Hartree-Fock point of view.

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APPENDIX A: ALTERNATIVE DERIVATION OF THE LAGRANGIAN OF EQ. (30)

We give here an alternative derivation of the Lagrangian of Eq. (30), obtained by applying general covariance to the Maxwell-Dirac equations and then using the Callan-Symanzik equations to determine the effective Lagrangian implied by the modified equations of motion. We also discuss two related issues. First, in connection with the kinetic term $\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, we show that one gets different answers depending on whether one defines asymptotic photon states relative to a general conformal factor $e^{(1/2)\psi}$ or to a Minkowski conformal factor of 1, but that the two answers become the same if the Callan-Symanzik function $\beta(\alpha)$ for electrodynamics vanishes at the physical coupling α.[17] Second, we discuss the effect of vacuum polarization corrections, neglected in writing Eqs. (27) and (30), which are proportional to tensors constructed from covariant derivatives

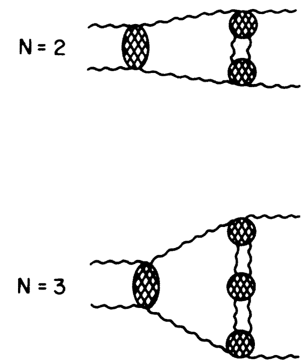

FIG. 4. Some typical nonladder diagrams which can make contributions proportional to $(\ln s^2)^N$ in N-loop order.

of the field strengths or the curvature tensor and its covariant derivatives.

Our starting point is the Dirac equation written in generally covariant form,[18]

$$i\gamma^\nu\left(\frac{\partial}{\partial x^\nu} - \Gamma_\nu\right)\phi = m_0\phi, \tag{A1}$$

with the γ^ν coordinate-dependent Dirac matrices satisfying

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{A2}$$

The additional coordinate-dependent matrices Γ_ν are determined by

$$\frac{\partial\gamma_\mu}{\partial x^\nu} - \Gamma^\eta_{\mu\nu}\gamma_\eta = [\Gamma_\nu, \gamma_\mu], \tag{A3}$$

with $\Gamma^\eta_{\mu\nu}$ the affine connection. Although we are interested in the special case of a conformally flat metric, it is convenient to start from the somewhat more general situation of a general orthogonal metric[19] (repeated lower indices will be understood not to be summed)

$$g_{\mu\nu} = G_\mu(x)\eta_{\mu\nu}, \qquad G_\mu > 0. \tag{A4}$$

Letting $\tilde{\gamma}^\mu$ be the ordinary Dirac matrices which satisfy

$$\tilde{\gamma}_\mu = \eta_{\mu\nu}\tilde{\gamma}^\nu, \quad \{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\eta_{\mu\nu}, \tag{A5}$$

we can satisfy Eq. (A2) with

$$\gamma_\mu = G_\mu^{1/2}\tilde{\gamma}_\mu. \tag{A6}$$

Using Eq. (A6) and the affine connection

$$\Gamma^\sigma_{\lambda\mu} = \tfrac{1}{2}G_\sigma^{-1}\left(\frac{\partial G_\mu}{\partial x^\lambda}\,\delta^\sigma_\mu + \frac{\partial G_\lambda}{\partial x^\mu}\,\delta^\sigma_\lambda - \eta_{\lambda\mu}\eta^{\sigma\tau}\frac{\partial G_\mu}{\partial x^\tau}\right), \tag{A7}$$

we readily find that the unique solution of Eq. (A3) satisfying

$$\mathrm{tr}(\Gamma_\nu)=0 \tag{A8a}$$

is

$$\Gamma_\nu=-\tfrac{1}{4}\gamma_\nu\sum_{\mu\neq\nu}\gamma_\mu G_\nu{}^{-1}\frac{\partial G_\nu}{\partial x_\mu}. \tag{A8b}$$

The condition of Eq. (A8a) just guarantees that there is no part of Γ_ν which could be reinterpreted as an electromagnetic potential. Substituting into Eq. (A1), we get as the Dirac equation for a general orthogonal metric

$$i\left[\gamma^\nu\frac{\partial}{\partial x^\nu}+\tfrac{1}{4}\gamma^\nu\frac{\partial}{\partial x^\nu}\left(\ln\prod_{\mu\neq\nu}G_\mu\right)\right]\phi=m_0\phi. \tag{A9}$$

Specializing now to the case of a conformally flat metric with

$$G_\mu=e^\psi,\quad g_{\mu\nu}=e^\psi\eta_{\mu\nu} \tag{A10}$$

we get

$$ie^{-\psi/2}\left(\tilde\gamma^\nu\frac{\partial}{\partial x^\nu}+\tfrac{3}{4}\tilde\gamma^\nu\frac{\partial\psi}{\partial x^\nu}\right)\phi=m_0\phi. \tag{A11}$$

Introducing the electromagnetic field by the standard prescription, we now find for the coupled Maxwell-Dirac equations in a conformally flat metric

$$\begin{aligned}&ie^{-\psi/2}\left(\tilde\gamma^\nu\frac{\partial}{\partial x^\nu}+ie_0\tilde\gamma^\nu A_\nu+\tfrac{3}{4}\tilde\gamma^\nu\frac{\partial\psi}{\partial x^\nu}\right)\phi=m_0\phi,\\&\frac{\partial}{\partial x^\mu}(e^{2\psi}F^{\mu\nu})=-e^{2\psi}J^\nu,\\&\frac{\partial}{\partial x^\lambda}F_{\mu\nu}+\frac{\partial}{\partial x^\nu}F_{\lambda\mu}+\frac{\partial}{\partial x^\mu}F_{\nu\lambda}=0,\\&J^\nu=-e_0\bar\phi\gamma^\nu\phi,\quad F_{\nu\lambda}=\frac{\partial A_\nu}{\partial x^\lambda}-\frac{\partial A_\lambda}{\partial x^\nu},\\&F^{\nu\lambda}=g^{\nu\alpha}g^{\lambda\beta}F_{\alpha\beta}.\end{aligned} \tag{A12}$$

Making now the rescalings of Eq. (29),

$$A_\mu=\tilde A_\mu,\quad F_{\mu\nu}=\tilde F_{\mu\nu},\quad F^{\mu\nu}=e^{-2\psi}\tilde F^{\mu\nu}, \tag{A13a}$$

as well as introducing a rescaled electron field $\tilde\phi\equiv\Phi$,

$$\phi=e^{-(3/4)\psi}\Phi,\quad \bar\phi=e^{-(3/4)\psi}\bar\Phi, \tag{A13b}$$

we find that the coupled Maxwell-Dirac equations take the form

$$\begin{aligned}&i\left(\tilde\gamma^\nu\frac{\partial}{\partial x^\nu}+ie_0\tilde\gamma^\nu\tilde A_\nu\right)\Phi=m_0e^{\psi/2}\Phi,\\&\frac{\partial}{\partial x^\mu}\tilde F^{\mu\nu}=-\tilde J^\nu,\\&\frac{\partial}{\partial x^\lambda}\tilde F_{\mu\nu}+\frac{\partial}{\partial x^\nu}\tilde F_{\lambda\mu}+\frac{\partial}{\partial x^\mu}\tilde F_{\nu\lambda}=0,\\&\tilde J^\nu=-e_0\bar\Phi\tilde\gamma^\nu\Phi.\end{aligned} \tag{A14}$$

Apart from the alteration in the electron bare mass term, these are just the equations of Minkowski-space quantum electrodynamics. Since the rescalings of Eq. (A13) also convert the kinetic Lagrangian terms to their Minkowski-space form,

$$\begin{aligned}&\sqrt{g}\,F_{\mu\nu}F^{\mu\nu}\to\tilde F_{\mu\nu}\tilde F^{\mu\nu},\\&\sqrt{g}\,\bar\phi\gamma^\nu\frac{\partial}{\partial x^\nu}\phi\to\bar\Phi\tilde\gamma^\nu\frac{\partial}{\partial x^\nu}\Phi,\end{aligned} \tag{A15}$$

the rescaled fields satisfy the standard canonical commutation relations.

Continuing from Eq. (A14), we wish next to find the modification in the photon effective Lagrangian resulting from the presence of the coordinate dependent factor $e^{\psi/2}$ in the electron bare mass term. If we denote by $\mathcal{L}_{(2)}$ and $m^{-4}\mathcal{L}_{(4)}$ the terms in $\mathcal{L}$ which are respectively of second and fourth degree in the electromagnetic fields, then we expect that these will become functionals of ψ and its derivatives as well as of the fields and their derivatives. That is, we have

$$\begin{aligned}&\mathcal{L}_{(2)}=\mathcal{L}_{(2)}\left[\tilde F^r,\frac{1}{m}\partial\tilde F^r,\frac{1}{m^2}\partial\partial\tilde F^r,\ldots;\psi,\frac{1}{m}\partial\psi,\frac{1}{m^2}\partial\partial\psi,\ldots\right],\\&\frac{1}{m^4}\mathcal{L}_{(4)}=\frac{1}{m^4}\mathcal{L}_{(4)}\left[\tilde F^r,\frac{1}{m}\partial\tilde F^r,\frac{1}{m^2}\partial\partial\tilde F^r,\ldots;\psi,\frac{1}{m}\partial\psi,\frac{1}{m^2}\partial\partial\psi,\ldots\right],\end{aligned} \tag{A16}$$

where the necessary electron mass factors have been inserted to make the $\mathcal{L}_{(j)}$'s dimensionless functionals of their arguments. In writing Eq. (A16) we have included a superscript r on the field strengths to stress the fact that the effective Lagrangian, which is a mnemonic for calculating renormalized Green's functions, involves renormalized field strengths $\tilde F^r$, rather than the unrenormalized field strengths $\tilde F$ which have appeared above in the fundamental Lagrangian and equations of motion. If we now consider both the electromagnetic fields and the conformal factor ψ to be slowly varying on the scale of an electron Compton wavelength, then it is reasonable to develop Eq. (A16) in an expansion in powers of $1/m$ and to keep only the leading term. (Effects of nonleading terms on photon propagation will be discussed briefly below.) Thus we approximate

$$\mathcal{L}_{(2)} \approx \mathcal{L}_{(2)}[\tilde{F}^r;\psi], \qquad \frac{1}{m^4}\mathcal{L}_{(4)} \approx \frac{1}{m^4}\mathcal{L}_{(4)}[\tilde{F}^r;\psi], \tag{A17}$$

and so the problem of finding the ψ dependence of the effective Lagrangian reduces to that of finding the effects of the bare mass rescaling $m_0 \to m_0 e^{\psi/2}$ with ψ a *constant*.

Thus simplified, the problem is evidently one for which the Callan-Symanzik equations are suited. In applying them, we follow the procedure of Ref. 17 and take e_0, m_0, and the cutoff implicit in the renormalization constants as independent variables, so that consequently e and m (which are functions of e_0, m_0, and the cutoff) are dependent variables. The renormalized field strengths $\tilde{F}^r_{\mu\nu}$ are also independent variables (they are nothing more than kinematic quantities of the form $k_\mu\epsilon_\nu - k_\nu\epsilon_\mu$, with k a photon four-momentum and ϵ a photon polarization), and so are unaffected by differentiation with respect to m_0. Now for any function $f(m_0)$ we evidently have

$$f(m_0 e^{\psi/2}) = \exp\left(\tfrac{1}{2}\psi\, m_0 \frac{\partial}{\partial m_0}\right) f(m_0), \tag{A18}$$

so the effective Lagrangian with m_0 rescaled is obtained from the original effective Lagrangian for bare mass m_0 by acting with the differential operator

$$D_\psi = \exp\left(\tfrac{1}{2}\psi\, m_0 \frac{\partial}{\partial m_0}\right). \tag{A19}$$

But now using the chain rule

$$m_0 \frac{\partial}{\partial m_0} = m_0 \frac{\partial m}{\partial m_0}\frac{\partial}{\partial m} + m_0 \frac{\partial \alpha}{\partial m_0}\frac{\partial}{\partial \alpha}, \tag{A20}$$

and introducing the definitions[17]

$$1+\delta(\alpha) \equiv \frac{m}{m_0}\frac{\partial m_0}{\partial m}, \qquad \beta(\alpha) = \frac{1}{\alpha}\, m \frac{\partial \alpha}{\partial m} = \frac{1}{\alpha}[1+\delta(\alpha)]\, m_0 \frac{\partial \alpha}{\partial m_0} \tag{A21}$$

we can write

$$m_0 \frac{\partial}{\partial m_0} = \frac{1}{1+\delta(\alpha)}\left[m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial \alpha}\right], \tag{A22}$$

which expresses the differential operator in a form where we can readily evaluate its action on renormalized quantities,

$$D_\psi = \exp\left\{\tfrac{1}{2}\psi \frac{1}{1+\delta(\alpha)}\left[m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial \alpha}\right]\right\}. \tag{A23}$$

Acting on the standard renormalized effective Lagrangian (with superscripts r now suppressed)

$$\mathcal{L} = -\tilde{\mathfrak{F}} + \frac{2\alpha^2}{45\,m^4}(4\tilde{\mathfrak{F}}^2 + 7\tilde{\mathfrak{G}}^2), \tag{A24}$$

we get, to leading order in α,

$$D_\psi \mathcal{L} = \exp\left(\tfrac{1}{2}\psi m \frac{\partial}{\partial m}\right)\mathcal{L} = -\tilde{\mathfrak{F}} + \frac{2\alpha^2}{45(m e^{\psi/2})^4}(4\tilde{\mathfrak{F}}^2 + 7\tilde{\mathfrak{G}}^2), \tag{A25}$$

in agreement with Eq. (30) of the text.

So far everything is nicely consistent. However, when we attempt to discuss the kinetic term to nonleading orders in α, the question of how asymptotic photon states are defined comes in. According to the standard renormalization prescription, the kinetic term in $\mathcal{L}$, when $\psi = 0$, is given by

$$\mathcal{L}_{\text{kin}} = -\tilde{\mathfrak{F}} \tag{A26}$$

to all orders in the fine-structure constant α. Hence acting with D_ψ we get

$$\mathcal{L}^{\psi}_{\text{kin}} = D_\psi \mathcal{L}_{\text{kin}} = -\tilde{\mathfrak{F}} \tag{A27}$$

to all orders in α, in agreement with the result which would be obtained by the general covariance argument of Sec. II. However, there is an alternative recipe for calculating $\mathcal{L}_{\text{kin}}$ for the system of Eqs. (A14), which gives a different answer. Let us rewrite the Lagrangian for Eqs. (A14) according to

$$\begin{aligned}\mathcal{L} &= \bar{\Phi}[i\tilde{\gamma}\cdot(\partial + ie_0\tilde{A}) - m_0 e^{\psi/2}]\Phi - \tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \\ &= \bar{\Phi}[i\gamma\cdot(\partial + ie_0\tilde{A}) - m_0]\Phi - \tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - m_0\bar{\Phi}\Phi\Psi, \\ \Psi &= e^{\psi/2} - 1.\end{aligned} \tag{A28}$$

In the first line of Eq. (A28) we are regarding m_0 as a parameter to be rescaled *everywhere* where it appears (including in charge renormalization counterterms) by a factor $e^{\psi/2}$; this is the point of view which leads to Eq. (A27), and corresponds to defining asymptotic photon states relative to a metric with asymptotic conformal factor $e^{\psi/2} \neq 1$. In the second line of Eq. (A28), we have put all dependence on ψ into a fictitious scalar field Ψ, the effects of which are to be calculated as a perturbation series around the original theory, with charge renormalization counterterms evaluated using m_0 as the bare mass; this corresponds to an adiabatic switching off of the conformal factor, so that asymptotic states are defined relative to a Minkowski conformal factor of 1. In this case, defining

$$i\mathcal{L}_S = -i m_0 \bar{\Phi}\Phi\Psi, \tag{A29}$$

those effective Lagrangian terms involving Ψ but not its derivatives are obtained by inserting in all T products a factor

$$\exp\left(i\int d^4x\,\mathcal{L}_S\right) \underset{\substack{\text{derivatives}\\ \text{of }\Psi\\ \text{neglected}}}{=} \exp\left[\Psi\int d^4x(-im_0\bar{\Phi}\Phi)\right] = \sum_{n=0}^{\infty}\frac{\Psi^n}{n!}\left[\int d^4x(-im_0\bar{\Phi}\Phi)\right]^n. \tag{A30}$$

The two-photon matrix element with Eq. (A30) inserted is

$$\mathcal{L}_{\rm kin} = -\tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\sum_{n=0}^{\infty}\frac{\Psi^n}{n!}\alpha\tilde{\Gamma}_{\gamma\gamma S^n}(0,\alpha), \tag{A31}$$

with $\tilde{\Gamma}_{\gamma\gamma S^n}(q^2/m^2,\alpha)$ the renormalized amplitude with two external photons carrying respectively four-momenta q and $-q$ and with n scalar insertions $-im_0\bar{\Phi}\Phi$, each carrying zero four-momentum. (In our definition of $\tilde{\Gamma}_{\gamma\gamma S^n}$ the charge factors associated with the external photon vertices have been stripped off, hence the explicit factor of α in front.) Let us now proceed to get a low-energy theorem for $\tilde{\Gamma}_{\gamma\gamma S^n}(0,\alpha)$. Noting that

$$m_0\frac{\partial}{\partial m_0}\frac{i}{\not{p}-m_0} = \frac{i}{\not{p}-m_0}(-im_0)\frac{i}{\not{p}-m_0} = \text{insertion of } -im_0\bar{\Phi}\Phi, \tag{A32}$$

we expect that $\tilde{\Gamma}_{\gamma\gamma S^n}$ will be generated from the inverse photon propagator by application of $m_0^n(\partial/\partial m_0)^n$. This can be explicitly checked in the $n=1$ case, by using the Callan-Symanzik equation for the photon propagator[20] in the form

$$\frac{1}{1+\delta(\alpha)}\left(m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right)\left[\frac{1}{\alpha}+\pi_c(q^2)\right] = \tilde{\Gamma}_{\gamma\gamma S}(q^2/m^2,\alpha), \tag{A33}$$

which using Eq. (A22) and the relations[20]

$$\pi_c(q^2)=\pi(q^2)-\pi(0),\qquad \frac{1}{\alpha}-\pi(0)=\frac{1}{\alpha_0}, \tag{A34}$$

can be rewritten as

$$m_0\frac{\partial}{\partial m_0}\left[\frac{1}{\alpha_0}+\pi(q^2)\right] = \tilde{\Gamma}_{\gamma\gamma S}(q^2/m^2,\alpha). \tag{A35}$$

Applying higher powers of the operator $\partial/\partial m_0$ we get

$$m_0^n\left(\frac{\partial}{\partial m_0}\right)^n\left[\frac{1}{\alpha_0}+\pi(q^2)\right] = \tilde{\Gamma}_{\gamma\gamma S^n}(q^2/m^2,\alpha), \tag{A36}$$

which on writing

$$m_0^n\left(\frac{\partial}{\partial m_0}\right)^n = \sum_{j=1}^{n}C_{nj}\left(m_0\frac{\partial}{\partial m_0}\right)^j, \tag{A37a}$$

$$C_{nn}=1,\quad C_{n\,n-1}=-\tfrac{1}{2}n(n-1),\ \ldots$$

and reexpressing in terms of renormalized quantities gives

$$\sum_{j=1}^{n}C_{nj}\left\{\frac{1}{1+\delta(\alpha)}\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right]\right\}^j\left[\frac{1}{\alpha}+\pi_c(q^2)\right] = \tilde{\Gamma}_{\gamma\gamma S^n}(q^2/m^2,\alpha). \tag{A37b}$$

Let us now set $q^2=0$. Since $\pi_c(0)=0$ independent of the values of m and α, we get the low-energy theorem[21]

$$\sum_{j=1}^{n}C_{nj}\left[\frac{1}{1+\delta(\alpha)}\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right]^j\frac{1}{\alpha} = \tilde{\Gamma}_{\gamma\gamma S^n}(0,\alpha), \tag{A38}$$

which on substitution into Eq. (A31) gives for the term in the effective Lagrangian involving two photon field strength factors[21]

$$\mathcal{L}_{\rm kin}^{\psi} = -\mathfrak{F}\sum_{n=0}^{\infty}\sum_{j=1}^{n}\frac{\Psi^n}{n!}\alpha C_{nj}\left[\frac{1}{1+\delta(\alpha)}\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right]^j\frac{1}{\alpha}. \tag{A39}$$

This recipe *differs* from that obtained from Eq. (A26) by acting with the operator D_ψ, or by using the general covariance argument of Sec. II, unless[22]

$$\beta(\alpha)=0. \tag{A40}$$

Note that the condition of Eq. (A40) involves the physical, rather than the asymptotic, coupling constant, because it was obtained by a low-energy-theorem argument. We have conjectured[17] on other grounds that $\beta(\alpha)$ vanishes; if this is so, then the same photon kinetic effective Lagrangian is obtained irrespective of whether the photon asymptotic states are defined relative to a conformal factor of $e^{\psi/2}$ or of 1. Henceforth, we will continue to use the result of Eq. (A27) for $\mathcal{L}_{\rm kin}^{\psi}$.

We turn finally to the higher terms in the $1/m$ expansion of Eq. (A16), which we have neglected up to this point. Such corrections to the interaction Lagrangian $m^{-4}\mathcal{L}_{(4)}$ should clearly have a negligible effect, for wavelengths $\gg m^{-1}$, on the ladder sum argument given in the text. However, since the quadratic Lagrangian $\mathcal{L}_{(2)}$ determines the photon propagator, and since we have seen in Sec. I that even a very small mass term in the propagator spoils the argument for a pairing instability, we must address the question of what the effects of higher-order terms in $1/m$ in $\mathcal{L}_{(2)}$ might be. We do not have a way to do this in general, but can make some useful statements in a number of special cases relevant to the discussion of Sec. II. To proceed, let us return to the generally covariant form of the Lagrangian given in Eq. (27), before conformal rescalings were made. In this language, higher terms in the $1/m$ expansion of $\mathcal{L}_{\rm kin}$ will involve covariant derivatives of the field strengths, or the curvature tensor and its covariant derivatives, or both.[23] To discuss effects of covariant derivatives of the field strengths, let us consider the special case of a conformal factor $e^{\psi}=(1+2c\cdot x+c^2x^2)^{-2}$, which as we have noted in Sec. II corresponds to flat spacetime viewed from a noninertial coordinate system. In this case the curvature tensor vanishes identically, so terms involving the curvature tensor and its covariant derivatives are absent. Furthermore, the photon propagator in an inertial frame has the spectral form

$$D_F(x_1-y_1)=\frac{1}{(x_1-y_1)^2}+\int_{4m^2}^{\infty}d(\rho^2)c(\rho^2)\Delta_F(\rho^2,(x_1-y_1)^2), \tag{A41}$$

from which by making the transformation of Eq. (22), using the relation[24]

$$(x_1-y_1)^2=e^{(1/2)\psi(x)}e^{(1/2)\psi(y)}(x-y)^2, \tag{A42}$$

and rescaling out the conformal factors, we get the photon propagator corresponding to the Lagrangian of Eq. (A28) in the form

$$\tilde{D}_F(x-y)=\frac{1}{(x-y)^2}+e^{(1/2)\psi(x)}e^{(1/2)\psi(y)}\int_{4m^2}^{\infty}d(\rho^2)c(\rho^2)\Delta_F(\rho^2,(x-y)^2e^{(1/2)\psi(x)}e^{(1/2)\psi(y)}). \tag{A43}$$

In the text, we have kept only the $1/(x-y)^2$ term in each photon leg of the ladder sum; when Fourier-transformed these become the $1/k^2$ propagators which led to the pairing instability. Since the Fourier transform of the spectral term in Eq. (A43) with respect to the difference variable $x-y$ is finite at $k^2=0$, we can see no way for this neglected term to remove the singularity at $k^2=0$ arising from the first term.

In cases where the curvature tensor is nonvanishing it is hard to make general arguments. There will now be additional terms in the kinetic Lagrangian of the form

$$\tilde{F}_{\mu\nu}\tilde{F}^{\lambda\sigma}T^{\mu\nu}{}_{\lambda\sigma},\quad T^{\mu\nu}{}_{\lambda\sigma}=\frac{c}{m^2}R^{\mu\nu}{}_{\lambda\sigma}+\cdots \tag{A44}$$

as well as more complicated terms involving R and derivatives of the field strengths. We confine ourselves here to remarking that in the approximation used in Sec. II of expanding the metric in powers of x, $g_{\mu\nu}=\eta_{\mu\nu}+O(\xi x)+O(\xi^2x^2)$, with ξ a smallness parameter, and dropping cubic and higher terms in ξ, the curvature tensor $R^{\mu\nu}{}_{\lambda\sigma}$ will be a constant. The effects of the term in Eq. (A44) will then be to give the photon a refractive index n different from 1 (not a mass); in spherically symmetric metrics the refractive index will be independent of photon polarization and will simply change the photon propagator to

$$\frac{-i\eta_{\mu\nu}}{k_0^2n^2-\vec{k}^2+i\epsilon}\,. \tag{A45}$$

This can be changed to the usual photon propagator by rescaling $k_0\to k_0/n$, which results in no substantial change in the ladder sum argument. To summarize, while we have not dealt with the higher $1/m$ terms in the kinetic Lagrangian in a general way, we believe that the arguments just given support the view that the possibility of a pairing instability is a general geometric property associated with massless particle propagation, and is unaffected by radiative corrections to the massless particle propagator, or equivalently, to the kinetic term in the effective Lagrangian.

APPENDIX B: PHOTON LADDER-GRAPH CALCULATION IN A CONFORMALLY FLAT METRIC

We give here details of the inductive argument used to evaluate the ladder sum discussed in Sec. II, and the algebraic reduction of its numerator. We wish to consider the action of the matrix $M_{(\alpha\lambda)(\beta\sigma)}{}^{(\alpha'\lambda')(\beta'\sigma')}(s)$ defined in Eq. (38) on a parallel photon pair wave function $H_{(\alpha'\lambda')(\beta'\sigma')}(s')$. We note to begin with that M has the following symmetry properties in its final four indices:

M is antisymmetric in $\alpha'\lambda'$,

antisymmetric in $\beta'\sigma'$,

symmetric under

pair interchange $(\alpha'\lambda')\leftrightarrow(\beta'\sigma')$. (B1)

Hence when M is contracted with a second M [or with U of Eq. (32), which has the same final index symmetries] on the left, we can drop terms in the right-hand M which are symmetric in $\alpha\lambda$, symmetric in $\beta\sigma$, or antisymmetric under the pair interchange $(\alpha\lambda)\leftrightarrow(\beta\sigma)$. We will consistently use these facts to simplify the expressions which follow.

Let us now postulate $H_{(\alpha'\lambda')(\beta'\sigma')}(s')$ to have the following properties:

$$\text{I.}\quad s'^{(1)}H_{(\alpha'\lambda')(\beta'\sigma')}\propto\begin{cases}0 \text{ or}\\ s'_{(1')}s'_{\beta'}s'_{\sigma'},\end{cases}$$

$$s'^{(2)}H_{(\alpha'\lambda')(\beta'\sigma')}\propto\begin{cases}0 \text{ or}\\ s'_{(2')}s'_{\alpha'}s'_{\lambda'},\end{cases} \tag{B2}$$

$$\text{II.}\quad \eta^{(1)(2)}H_{(\alpha'\lambda')(\beta'\sigma')}\propto\begin{cases}0 \text{ or}\\ s'_{(1')}s'_{(2')},\end{cases}$$

where

(1) = any index from first pair $(\alpha'\lambda')$; $(1')$ = remaining index in the first pair;

(2) = any index from second pair $(\beta'\sigma')$; $(2')$ = remaining index in the second pair. Obviously, II implies that

$$\eta_{(1)(2)}\eta_{(1')(2')}H^{(\gamma\lambda)(\delta\sigma)}(s')=0 \Rightarrow (\eta_{\gamma\delta}\eta_{\lambda\sigma}-\eta_{\gamma\sigma}\eta_{\delta\lambda})H^{(\gamma\lambda)(\delta\sigma)}(s')=0, \tag{B3}$$

and it is also easy to see (by taking s' as an axis) that I implies that

$$\epsilon_{\gamma\lambda\delta\sigma}H^{(\gamma\lambda)(\delta\sigma)}(s')=0, \tag{B4}$$

allowing us to drop the first term in the square bracket multiplied by 4 and the first term in the square bracket multiplied by 7 in Eq. (32) for U, before evaluating M from Eq. (38). Substituting Eqs. (32) and (37) into Eq. (38), reexpressing terms quadratic in the Levi-Civita tensor ϵ in terms of the Minkowski metric η, and contracting with H, we find

$$\begin{aligned}M_{(\alpha\lambda)(\beta\sigma)}{}^{(\alpha'\lambda')(\beta'\sigma')}(s)H_{(\alpha'\lambda')(\beta'\sigma')}(s')=\frac{4\alpha^2}{45m^4}\frac{1}{(4\pi)^2}\tfrac{1}{10}\ln\left(\frac{\Lambda^2}{-s^2}\right)&\\ \times[\tfrac{1}{3}s_\lambda s_\sigma s^\xi s^\eta & \quad (1)\\ +\tfrac{1}{6}(\eta_{\lambda\sigma}s^\xi s^\eta+\eta_\lambda^\eta s_\sigma s^\xi+\eta_\sigma^\xi s_\lambda s^\eta+\eta^{\xi\eta}s_\lambda s_\sigma)s^2 & \quad (2)\\ -\tfrac{1}{4}(\eta_\lambda^\xi s_\sigma s^\eta+\eta_\sigma^\eta s_\lambda s^\xi)s^2 & \quad (3)\\ +\tfrac{1}{24}(\eta_{\lambda\sigma}\eta^{\xi\eta}+\eta_\lambda^\xi\eta_\sigma^\eta+\eta_\lambda^\eta\eta_\sigma^\xi)(s^2)^2] & \quad (4)\\ \times\{-6[H^S_{(\alpha\xi)(\beta\eta)}+H^S_{(\xi\alpha)(\eta\beta)}-H^S_{(\alpha\xi)(\eta\beta)}-H^S_{(\xi\alpha)(\beta\eta)}]&\\ -14(-\eta_{\alpha\beta}H^S_{\eta\xi}+\eta_{\alpha\eta}H^S_{\beta\xi}+\eta_{\xi\beta}H^S_{\eta\alpha}-\eta_{\xi\eta}H^S_{\alpha\beta})\},&\end{aligned} \tag{B5}$$

with

$$\begin{aligned}&H^S_{(\alpha\gamma)(\beta\sigma)}=\tfrac{1}{2}[H_{(\alpha\lambda)(\beta\sigma)}(s')+H_{(\beta\sigma)(\alpha\lambda)}(s')],\\ &H^S_{\eta\xi}=H^S_{(\theta\eta)}{}^{(\theta}{}_{\xi)}+H^S_{(\eta\theta)(\xi}{}^{\theta)}-H^S_{(\theta\eta)(\xi}{}^{\theta)}-H^S_{(\eta\theta)}{}^{(\theta}{}_{\xi)}=H^S_{\xi\eta}\propto s'_\eta s'_\xi \text{ by II.}\end{aligned} \tag{B6}$$

On contracting terms (1) through (4) with the quantity in curly brackets in Eq. (B5), we find that every term contains a double zero when s' is set equal to s. Thus, when we integrate the operator $\mathfrak{D}_N$ off to the left, a nonvanishing contribution to the ladder sum results only when both derivatives $\partial/\partial s_N$ in $\mathfrak{D}_N$ act on the factor $M(s_N)$ standing immediately to the left. Acting on Eq. (B5) with $\frac{1}{2}E_{\mu\nu}(\partial/\partial s_\mu)(\partial/\partial s_\nu)$ (the single derivative term $D_\mu\partial/\partial s_\mu$ makes no contribution) and then setting $s'=s$ we get

$$\frac{4\alpha^2}{45m^4}\frac{1}{(4\pi)^2}\tfrac{1}{10}\ln\left(\frac{\Lambda^2}{-s^2}\right)[H^{(1)}_{(\alpha\lambda)(\beta\sigma)}+H^{(2)}_{(\alpha\lambda)(\beta\sigma)}+H^{(3)}_{(\alpha\lambda)(\beta\sigma)}+H^{(4)}_{(\alpha\lambda)(\beta\sigma)}], \tag{B7a}$$

with the respective contributions of terms (1) through (4) given by

$$H^{(1)}_{(\alpha\lambda)(\beta\sigma)} = -\tfrac{6}{3} s_\lambda s_\sigma [H^S_{(\alpha\xi)(\beta\eta)} + H^S_{(\xi\alpha)(\eta\beta)} - H^S_{(\alpha\xi)(\eta\beta)} - H^S_{(\xi\alpha)(\beta\eta)}]E^{\xi\eta} + \tfrac{14}{3} s_\lambda s_\sigma \eta_{\alpha\beta} H^S_{\xi\eta} E^{\xi\eta},$$

$$\begin{aligned} H^{(2)}_{(\alpha\lambda)(\beta\sigma)} = -\tfrac{6}{6}\{ & \eta_{\lambda\sigma}[H^S_{(\alpha e)(\beta s)} + H^S_{(\alpha s)(\beta e)} + H^S_{(e\alpha)(s\beta)} + H^S_{(s\alpha)(e\beta)} - H^S_{(\alpha e)(s\beta)} - H^S_{(\alpha s)(e\beta)} - H^S_{(e\alpha)(\beta s)} - H^S_{(s\alpha)(\beta e)}] \\ & + s_\sigma[H^S_{(\alpha e)(\beta\lambda)} + H^S_{(e\alpha)(\lambda\beta)} - H^S_{(\alpha e)(\lambda\beta)} - H^S_{(e\alpha)(\beta\lambda)}] + s_\lambda[H^S_{(\alpha\sigma)(\beta e)} + H^S_{(\sigma\alpha)(e\beta)} - H^S_{(\alpha\sigma)(e\beta)} - H^S_{(\sigma\alpha)(\beta e)}]\} \\ & - \tfrac{14}{6}[\eta_{\lambda\sigma}(s_\alpha H^S_{\beta e} + s_\beta H^S_{e\alpha} - 4eH^S_{\beta\alpha}) - \eta_{\alpha\beta}(s_\sigma H^S_{\lambda e} + s_\lambda H^S_{e\sigma})], \end{aligned} \tag{B7b}$$

$$\begin{aligned} H^{(3)}_{(\alpha\lambda)(\beta\sigma)} = \tfrac{6}{4}\{ & s_\sigma[H^S_{(\alpha\lambda)(\beta e)} + H^S_{(\lambda\alpha)(e\beta)} - H^S_{(\alpha\lambda)(e\beta)} - H^S_{(\lambda\alpha)(\beta e)}] + s_\lambda[H^S_{(\alpha e)(\beta\sigma)} + H^S_{(e\alpha)(\sigma\beta)} - H^S_{(\alpha e)(\sigma\beta)} - H^S_{(e\alpha)(\beta\sigma)}]\} \\ & + \tfrac{14}{4}[s_\sigma(-\eta_{\alpha\beta} H^S_{e\lambda} + \eta_{\lambda\beta} H^S_{e\alpha}) + s_\lambda(-\eta_{\alpha\beta} H^S_{\sigma e} + \eta_{\alpha\sigma} H^S_{\beta e})], \end{aligned}$$

$$\begin{aligned} H^{(4)}_{(\alpha\lambda)(\beta\sigma)} = -\tfrac{6}{6} e[& \eta_{\lambda\sigma} H^S_{\alpha\beta} + H^S_{(\alpha\lambda)(\beta\sigma)} + H^S_{(\lambda\alpha)(\sigma\beta)} - H^S_{(\alpha\lambda)(\sigma\beta)} - H^S_{(\lambda\alpha)(\beta\sigma)} + H^S_{(\alpha\sigma)(\beta\lambda)} + H^S_{(\sigma\alpha)(\lambda\beta)} - H^S_{(\alpha\sigma)(\lambda\beta)} - H^S_{(\sigma\alpha)(\beta\lambda)}] \\ & - \tfrac{14}{6} e(-4\eta_{\lambda\sigma} H^S_{\beta\alpha} - 2\eta_{\alpha\beta} H^S_{\lambda\sigma} + \eta_{\alpha\sigma} H^S_{\beta\lambda} + \eta_{\lambda\beta} H^S_{\sigma\alpha}), \end{aligned}$$

with

$$\begin{aligned} & e = s^\mu s^\nu E_{\mu\nu}, \\ & H^S_{(\alpha\lambda)(\beta e)} = H^S_{(\alpha\lambda)(\beta\sigma)} e^\sigma, \quad H^S_{(\alpha s)(\beta e)} = H^S_{(\alpha\lambda)(\beta\sigma)} s^\lambda e^\sigma, \quad \text{etc.} \\ & e^\sigma = 2E^{\sigma\mu} s_\mu . \end{aligned} \tag{B8}$$

It is now straightforward to verify that $H^{(1)}, \ldots, H^{(4)}$ each have properties I and II of Eq. (B2). So we have a procedure which can be used recursively, starting with the differential operator $\mathfrak{D}_N$ furthest to the right, then proceeding to $\mathfrak{D}_{N-1}$ and so forth, which makes each $\mathfrak{D}_j$ act only on the $M(s_j)$ standing immediately to the left.

Now let us take the factor H on the extreme right to be $H^{(F)}_{(\alpha\lambda)(\beta\sigma)}$ of Eq. (43). Since this satisfies properties I and II, the inductive argument which we have just developed applies. Substituting into Eq. (B7), and simplifying by using the right-index symmetries of the M standing to the left, we find

$$\frac{4\alpha^2}{45m^4}\frac{1}{(4\pi)^2}\tfrac{1}{10}\ln\left(\frac{\Lambda^2}{-s^2}\right)\theta(1-\theta)[eP^{(F)}_{\alpha\beta}s_\lambda s_\sigma(-2-4+12-6) + e\eta_{\alpha\beta}s_\lambda s_\sigma P^{(F)}{}_\tau{}^\tau(\tfrac{14}{3}+\tfrac{28}{3}-28+\tfrac{53}{3})]. \tag{B9}$$

Now for $F=\pm 2$ we have $P^{(F)}{}_\tau{}^\tau = 0$, and since the first term in square brackets in Eq. (B9) vanishes by detailed cancellations among the contributions of terms (1) through (4), we simply get zero. On the other hand, for $F=0$ we have $P^{(F)}{}_\tau{}^\tau = -2$. Although $\eta_{\alpha\beta}$ is not identical to $P^{(0)}_{\alpha\beta}$, it is easy to see that

$$\begin{aligned} & \eta_{\alpha\beta} = -P^{(0)}_{\alpha\beta} - \hat{s}_\alpha\hat{s}_\beta + \hat{s}_\alpha v_\beta + v_\alpha\hat{s}_\beta, \\ & \hat{s} = (1,0,0,1) = s/s^0, \quad v = (1,0,0,0), \end{aligned} \tag{B10}$$

and since the three terms involving $\hat{s}$ each make a contribution symmetric in α, λ, in β, σ or in both pairs of indices, we can drop them and make the replacement $\eta_{\alpha\beta} \to -P^{(0)}_{\alpha\beta}$. Thus for $F=0$, Eq. (B9) becomes

$$\frac{4\alpha^2}{45m^4}\frac{1}{(4\pi)^2}\tfrac{1}{5}\ln\left(\frac{\Lambda^2}{-s^2}\right)H^{(0)}_{(\alpha\lambda)(\beta\sigma)}s^\mu s^\nu E_{\mu\nu}\tfrac{11}{3}. \tag{B11}$$

Hence, via our inductive argument we have shown that[25]

$$\begin{aligned} \epsilon(k_0)^{\alpha_0}\epsilon(l_0)^{\beta_0}(T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f}P^{(F)}_{\alpha_f\beta_f} = {} & i(2\pi)^4\delta^4(s_0-s)[H_0^{(\alpha_0\lambda_0)(\beta_0\sigma_0)}U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_f\lambda_f)(\beta_f\sigma_f)}H^{(F)}_{(\alpha_f\lambda_f)(\beta_f\sigma_f)}] \\ & \times\left[\xi(F)\frac{22}{15\times 45}\frac{\alpha^2}{m^4}\frac{1}{(4\pi)^2}s^\mu s^\nu E_{\mu\nu}\ln\left(\frac{\Lambda^2}{-s^2}\right)\right]^N, \end{aligned} \tag{B12}$$

$$\xi(\pm 2) = 0, \quad \xi(0) = 1,$$

from which Eq. (43) immediately follows.

To derive the result of Eq. (45) for the residue of the ladder sum, we note that it is just the initial photon to final photon matrix element of

$$\begin{aligned} & \mathcal{L}_I = \frac{2\alpha^2}{45m^4}(4\mathfrak{F}^2 + 7\mathfrak{G}^2), \\ & \mathfrak{F} = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \mathfrak{G} = \tfrac{1}{8}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}. \end{aligned} \tag{B13}$$

Defining the additional electromagnetic Lorentz-invariant

$$\mathcal{R} = F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\sigma F_\sigma{}^\mu \tag{B14a}$$

which is related to $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{R} = 4\mathcal{G}^2 + 8\mathcal{F}^2, \tag{B14b}$$

it is convenient to rewrite Eq. (B13) in the form

$$\mathcal{L}_I = \frac{2\alpha^2}{45m^4}\left[-\tfrac{10}{16}(FF)^2 + \tfrac{7}{4}(FFFF)\right], \quad (FF) = F_\mu{}^\nu F_\nu{}^\mu, \quad (FFFF) = F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\sigma F_\sigma{}^\mu . \tag{B15}$$

Let us now substitute $F = F_1 + F_2 + F_3 + F_4$, and keep the terms quadrilinear in 1234. (We will identify 1 and 2 with the initial photons, 3 and 4 with the final photons.) Then we get for the matrix element of $\mathcal{L}_I$

$$\mathfrak{M} = \frac{4\alpha^2}{45m^4}\, 4\{-\tfrac{10}{16}[(F_1F_2)(F_3F_4) + (F_1F_3)(F_2F_4) + (F_1F_4)(F_2F_3)] + \tfrac{7}{4}[(F_1F_2F_3F_4) + (F_1F_2F_4F_3) + (F_1F_3F_2F_4)]\}. \tag{B16}$$

Restricting ourselves now to the case where photons 3 and 4 are parallel and have helicity zero, as specified in Eqs. (40a) and (42), we make the replacement

$$F_{3\alpha\beta}F_{4\gamma\delta} \to i^2\theta(1-\theta)[s_\alpha\epsilon_\beta^{(1)}s_\gamma\epsilon_\delta^{(1)} + s_\alpha\epsilon_\beta^{(2)}s_\gamma\epsilon_\delta^{(2)} + \text{antisymmetrizations}], \tag{B17a}$$

which on use of Eq. (B10) simplifies to

$$F_{3\alpha\beta}F_{4\gamma\delta} \to \theta(1-\theta)(s_\alpha s_\gamma\eta_{\beta\delta} - s_\beta s_\gamma\eta_{\alpha\delta} - s_\alpha s_\delta\eta_{\beta\gamma} + s_\beta s_\delta\eta_{\alpha\gamma}). \tag{B17b}$$

Substituting this expression into Eq. (B16) we readily find

$$\mathfrak{M} = \frac{44\alpha^2}{45m^4}(F_{2\delta}{}^\alpha F_1^{\delta\gamma} + F_{1\delta}{}^\alpha F_2^{\delta\gamma})\theta(1-\theta)s_\alpha s_\gamma, \tag{B18}$$

and since $s^2 = 0$ we are free to add a multiple of $\eta^{\alpha\gamma}$ to the tensor multiplying $s_\alpha s_\gamma$, giving

$$\mathfrak{M} = \frac{11\alpha^2}{45m^4}T_{12}^{\alpha\gamma}[-4\theta(1-\theta)s_\alpha s_\gamma], \tag{B19}$$

with $T_{12}^{\alpha\gamma}$ the initial photon matrix element of the electromagnetic energy-momentum tensor. Changing to the notation used in Sec. II, this immediately gives the result quoted in Eqs. (45) and (46).

APPENDIX C: PHOTON-PHOTON SCATTERING IN A CONSTANT BACKGROUND ELECTROMAGNETIC FIELD IN MINKOWSKI SPACETIME

In this Appendix we discuss photon-photon scattering as described by the Heisenberg-Euler effective Lagrangian in Minkowski spacetime. We show that, in the presence of a constant background electromagnetic field, the eighth-order effective Lagrangian term provides an interaction which leads to a singularity in the ladder sum for photon-photon scattering.

We start from the Heisenberg-Euler Lagrangian in the form[26]

$$\mathcal{L} = -\mathcal{F} - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-m^2s}[(es)^2L - 1 - \tfrac{2}{3}(es)^2\mathcal{F}], \tag{C1}$$

$$L = i\mathcal{G}\frac{\cosh\{es[2(\mathcal{F}+i\mathcal{G})]^{1/2}\} + \cosh\{es[2(\mathcal{F}-i\mathcal{G})]^{1/2}\}}{\cosh\{es[2(\mathcal{F}+i\mathcal{G})]^{1/2}\} - \cosh\{es[2(\mathcal{F}-i\mathcal{G})]^{1/2}\}},$$

and expand out to eighth order, giving

$$\mathcal{L} = -\mathcal{F} + \frac{2\alpha^2}{45m^4}(4\mathcal{F}^2 + 7\mathcal{G}^2) - \frac{2\pi}{315}\frac{\alpha^3}{m^8}(128\mathcal{F}^3 + 208\mathcal{F}\mathcal{G}^2) + \frac{32\pi^2}{945}\frac{\alpha^4}{m^{12}}(384\mathcal{F}^4 + 704\mathcal{F}^2\mathcal{G}^2 + 152\mathcal{G}^4) + \cdots. \tag{C2}$$

Let us now substitute $F_{\mu\nu} \to F_{\mu\nu} + f_{\mu\nu}$, with F a constant background electromagnetic field and with f a wave field. A "pairing" interaction leading to a $(\ln s^2)^N$ contribution to the N-loop ladder term T_N can come only from terms in $\mathcal{L}$ where each f is contracted, through an $\mathcal{F}$ or $\mathcal{G}$ structure, with a background field factor F, since $f_{1\mu\nu}f_2^{\mu\nu} = \epsilon^{\mu\nu\lambda\sigma}f_{1\mu\nu}f_{2\lambda\sigma} = 0$ when $f_{1,2}$ are the field strengths for parallel photons. Hence the α^4

term in Eq. (C2) gives a pairing interaction, and we readily find, in the notation of Sec. II,

$$V_{\alpha\beta\gamma\delta}(k_1k_2k_3k_4) = i(2\pi)^4\delta^4(K)k_1^\xi k_2^\eta k_3^\lambda k_4^\sigma U^T_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)}\,, \tag{C3}$$

$$U^T_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)} = U_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)} + U^{\text{pairing}}_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)}\,,$$

with the lowest-order contribution U given by Eq. (32) of Sec. II and with U^{pairing} given by

$$\begin{aligned} U^{\text{pairing}}_{(\alpha\xi)(\beta\eta)(\gamma\lambda)(\delta\sigma)} = \frac{16\pi^2\alpha^4}{945m^{12}}\,[&768\times 24F_{\alpha\xi}F_{\beta\eta}F_{\gamma\lambda}F_{\delta\sigma} \\ &+704\times 2(F_{\alpha\xi}F_{\beta\eta}\hat F_{\gamma\lambda}\hat F_{\delta\sigma}+F_{\alpha\xi}F_{\gamma\lambda}\hat F_{\beta\eta}\hat F_{\delta\sigma}+F_{\alpha\xi}F_{\delta\sigma}\hat F_{\beta\eta}\hat F_{\gamma\lambda}+F_{\beta\eta}F_{\gamma\lambda}\hat F_{\alpha\xi}\hat F_{\delta\sigma} \\ &+F_{\beta\eta}F_{\delta\sigma}\hat F_{\alpha\xi}\hat F_{\gamma\lambda}+F_{\gamma\lambda}F_{\delta\sigma}\hat F_{\alpha\xi}\hat F_{\beta\eta})+19\times 24\hat F_{\alpha\xi}\hat F_{\beta\eta}\hat F_{\gamma\lambda}\hat F_{\delta\sigma}]\,, \end{aligned} \tag{C4}$$

$$\hat F_{\mu\nu} = \epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}\,.$$

For the N-loop term in the ladder sum we find, when refractive index complications are ignored (see below),

$$\begin{aligned} (T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f} &= \frac{1}{2^N}\,i(2\pi)^4\delta^4(s_0-s)k_0^{\lambda_0}l_0^{\sigma_0}(k_f)_{\lambda_{N+1}}(l_f)_{\sigma_{N+1}} \\ &\times U^T_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_1\lambda_1)(\beta_1\sigma_1)}\prod_{j=1}^{N}[M^T_{(\alpha_j\lambda_j)(\beta_j\sigma_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}(s)]\,, \end{aligned} \tag{C5}$$

$$M^T_{(\alpha_j\lambda_j)(\beta_j\sigma_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}(s) = L_{(\lambda_j\sigma_j)}{}^{(\xi_j\eta_j)}(s)U^T_{(\alpha_j\xi_j)(\beta_j\eta_j)}{}^{(\alpha_{j+1}\lambda_{j+1})(\beta_{j+1}\sigma_{j+1})}\,,$$

$$L^{\lambda\sigma\xi\eta}(s) = \frac{1}{(4\pi)^2}\,\tfrac{1}{30}\ln\left(\frac{\Lambda^2}{-s^2}\right)s^\lambda s^\sigma s^\xi s^\eta + (\text{terms which vanish at } s^2=0)\,.$$

Again taking the two photons at the right end of the ladder to be parallel, so that $k_f \propto l_f \propto s$, and contracting with polarization vectors, we find

$$\begin{aligned} \sum_{N=0}^{\infty}\epsilon(k_0)^{\alpha_0}\epsilon(l_0)^{\beta_0}(T_N)_{\alpha_0\beta_0}{}^{\alpha_f\beta_f}P^{(F)}_{\alpha_f\beta_f} &= i(2\pi)^4\delta^4(s_0-s)\,[H_0^{(\alpha_0\lambda_0)(\beta_0\sigma_0)}U_{(\alpha_0\lambda_0)(\beta_0\sigma_0)}{}^{(\alpha_f\lambda_f)(\beta_f\sigma_f)}H^{(F)}_{(\alpha_f\lambda_f)(\beta_f\sigma_f)}] \\ &\times\left[1-\frac{\lambda^{(F)}}{2m^{12}}\,\frac{\alpha^4}{945}\,\tfrac{1}{30}\ln\left(\frac{\Lambda^2}{-s^2}\right)\right]^{-1}, \end{aligned} \tag{C6}$$

where we have assumed $P^{(F)}_{\alpha\beta}$ to be an eigenvector of the following matrix $t_{\alpha\beta,\gamma\delta}$ with eigenvalue $\lambda^{(F)}$:

$$\begin{aligned} t_{\alpha\beta,\gamma\delta} &= 768\times 24F_{\alpha s}F_{\beta s}F_{\gamma s}F_{\delta s} \\ &+704\times 2(F_{\alpha s}F_{\beta s}\hat F_{\gamma s}\hat F_{\delta s}+F_{\alpha s}F_{\gamma s}\hat F_{\beta s}\hat F_{\delta s}+F_{\alpha s}F_{\delta s}\hat F_{\beta s}\hat F_{\gamma s}+F_{\beta s}F_{\gamma s}\hat F_{\alpha s}\hat F_{\delta s}+F_{\beta s}F_{\delta s}\hat F_{\alpha s}\hat F_{\gamma s}+F_{\gamma s}F_{\delta s}\hat F_{\alpha s}\hat F_{\beta s}) \\ &+19\times 24\hat F_{\alpha s}\hat F_{\beta s}\hat F_{\gamma s}\hat F_{\delta s}\,, \end{aligned} \tag{C7}$$

$$F_{\alpha s} = F_{\alpha\beta}s^\beta, \quad \hat F_{\alpha s} = \hat F_{\alpha\beta}s^\beta\,;$$

$$t_{\alpha\beta,}{}^{\gamma\delta}P^{(F)}_{\gamma\delta} = \lambda^{(F)}P^{(F)}_{\alpha\beta}\,.$$

To show that there are some nonvanishing eigenvalues, we consider the special case where the external field is a pure magnetic field transverse to $\vec s$,

$$\begin{aligned} &s = s^0(1,1,0,0)\,, \\ &\bar B = F_{12} = -F_{21} \neq 0, \quad F_{\mu\nu}=0, \quad (\mu\nu)\neq(12),(21)\,, \end{aligned} \tag{C8a}$$

for which $t_{\alpha\beta,\gamma\delta}$ becomes

$$\begin{aligned} t_{\alpha\beta,\gamma\delta} = (s^0\bar B)^4\,[&768\times 24\delta_{\alpha 2}\delta_{\beta 2}\delta_{\gamma 2}\delta_{\delta 2} \\ &+704\times 8(\delta_{\alpha 2}\delta_{\beta 2}\delta_{\gamma 3}\delta_{\delta 3}+\delta_{\alpha 2}\delta_{\beta 3}\delta_{\gamma 2}\delta_{\delta 3}+\delta_{\alpha 2}\delta_{\beta 3}\delta_{\gamma 3}\delta_{\delta 2}+\delta_{\alpha 3}\delta_{\beta 2}\delta_{\gamma 2}\delta_{\delta 3}+\delta_{\alpha 3}\delta_{\beta 2}\delta_{\gamma 3}\delta_{\delta 2}+\delta_{\alpha 3}\delta_{\beta 3}\delta_{\gamma 2}\delta_{\delta 2}) \\ &+19\times 24\times 16\delta_{\alpha 3}\delta_{\beta 3}\delta_{\gamma 3}\delta_{\delta 3}]\,. \end{aligned} \tag{C8b}$$

It is easily verified, for example, that $P_{\gamma\delta} = \delta_{\gamma 2}\delta_{\delta 3} + \delta_{\gamma 3}\delta_{\delta 2}$ is an eigenvector, with $704\times 16(s^0\bar B)^4$ as eigenvalue; two other eigenvectors of the form $P_{\gamma\delta} = A\delta_{\gamma 2}\delta_{\delta 2} + B\delta_{\gamma 3}\delta_{\delta 3}$ can be found, and also have nonvanishing real eigenvalues.

The calculation described above is oversimplified in one important respect: As is well known,[27] a photon in a constant electromagnetic background field propagates with polarization-dependent refractive indices which are different from unity. For example, in the case of a photon propagating normal to a constant magnetic field, the modes with photon magnetic field vector respectively parallel to and perpendicular to the constant field have refractive indices

$$n_{\parallel} = 1 + \frac{4\alpha^2 \bar{B}^2}{90\pi m^4}, \qquad n_{\perp} = 1 + \frac{7\alpha^2 \bar{B}^2}{90\pi m^4}. \tag{C9}$$

Since these differ, we must really regard $\parallel$ and $\perp$ polarized photons as distinct particles, and restrict all the photons in the ladder sum to have one definite polarization to make the argument for a singularity. In the case described by Eqs. (C8) and (C9), this is easily done by picking out of $t_{\alpha\beta,\gamma\delta}$ the parts that scatter two parallel (perpendicular) polarized photons into two parallel (perpendicular) polarized photons,

$$t^{\parallel\to\parallel}_{\alpha\beta,\gamma\delta} = (s^0 \bar{B})^4\, 768\times 24\,\delta_{\alpha 2}\delta_{\beta 2}\delta_{\gamma 2}\delta_{\delta 2}, \qquad t^{\perp\to\perp}_{\alpha\beta,\gamma\delta} = (s^0 \bar{B})^4\, 19\times 24\times 16\,\delta_{\alpha 3}\delta_{\beta 3}\delta_{\gamma 3}\delta_{\delta 3}. \tag{C10}$$

The denominator in Eq. (C6) is then replaced by

$$1 - \frac{\lambda_{\parallel,\perp}}{2m^{12}} \frac{\alpha^4}{945} \frac{1}{30} \ln\left[\frac{\Lambda^2}{-(s^0)^2 n_{\parallel,\perp}{}^2 + \vec{s}^{\,2}}\right], \tag{C11}$$

$$\lambda_{\parallel} = 768\times 24 (s^0 \bar{B})^4, \qquad \lambda_{\perp} = 19\times 24\times 16 (s^0 \bar{B})^4,$$

and evidently a weak-coupling singularity is still present in both polarization cases.

APPENDIX D: A SIMPLE PHONON SCATTERING MODEL

In this appendix we describe a simple phonon scattering model which shows ladder-graph singularities analogous to those found in Appendixes B and C. The model is motivated by considering the situation in which two longitudinally polarized phonons multiply forward scatter through the four-phonon anharmonic interaction. Because the interaction vanishes[28] as any of the four-phonon momenta approach zero, one effectively has a quadrilinear gradient coupling, suggesting the model Hamiltonian (we take the phonon velocity ω/k as unity)

$$H_0 = \frac{1}{2}\int d^3x\, [\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) + \pi(x)^2], \qquad H_I = \frac{1}{24}\int d^3x\, g[\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x)]^2, \qquad H = H_0 + H_I, \tag{D1}$$

together with the canonical commutation relations

$$[\phi(\vec{x}t), \phi(\vec{y}t)] = [\pi(\vec{x}t), \pi(\vec{y}t)] = 0, \qquad [\phi(\vec{x}t), \pi(\vec{y}t)] = i\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\theta(k_D - k). \tag{D2}$$

The model is automatically ultraviolet-finite because of the presence of the Debye frequency k_D as a short wavelength cutoff.

The Feynman rules corresponding to Eqs. (D1) and (D2) are

$$\text{phonon propagator} \quad \frac{i}{k_0{}^2 - \vec{k}^2 + i\epsilon}\,\theta(k_D - k),$$

$$\text{four-phonon vertex} \quad -i\frac{g}{3}(\vec{k}_1\cdot\vec{k}_2\vec{k}_3\cdot\vec{k}_4 + \vec{k}_1\cdot\vec{k}_3\vec{k}_2\cdot\vec{k}_4 + \vec{k}_1\cdot\vec{k}_4\vec{k}_2\cdot\vec{k}_3)(2\pi)^4\delta^4(k_3 + k_4 - k_1 - k_2). \tag{D3}$$

Remembering the symmetrization factor of $\frac{1}{2}^N$ in the N-loop ladder term, the phonon-phonon scattering ladder sum denominator analogous to that of Eqs. (43) and (C6) is

$$1 + \frac{g}{32\pi^2} k^4 \frac{1}{30} \ln\left[\frac{\Lambda^2}{-(k_0{}^2 - k^2)}\right], \tag{D4}$$

with Λ a cutoff of the order of k_D. For arbitrarily small g, Eq. (D4) vanishes for $k_0 \sim k$, $\arg k = 0$, $\pi/2$, π, $3\pi/2$ for $g<0$, and for $k_0 \sim k$, $\arg k = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ for $g>0$, so again there is a weak-coupling singularity in the ladder approximation. Evidently, the model of Eq. (D1) is only one of a large class of models which can be generated by applying derivative couplings to a basic ϕ^4 interaction. When there are no derivatives, the model shows weak-coupling singularities only when $g<0$ and then is essentially the negative-coupling ϕ^4 model first discussed by Symanzik[29] as an example of an asymptotically free field theory.

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[1]R. P. Feynman, Acta. Phys. Polon. 24, 697 (1963); B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967); in *Relativity, Groups and Topology* (Gordon and Breach, London, 1964); G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré 20, 69 (1974); S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. 32, 245 (1974); Phys. Rev. D 10, 401 (1974); 10, 411 (1974); S. Deser, H.-S. Tsao, and P. van Nieuwenhuizen, *ibid.* 10, 3337 (1974).

[2]Ladder-graph instabilities, related to the work described in Sec. I, in the context of the $1/N$ expansion in ϕ^4 field theory and in asymptotically free field theory models, have been noted by D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974); S. Coleman, R. Jackiw, and H. D. Politzer, *ibid.* 10, 2491 (1974); and H. J. Schnitzer, *ibid.* 10, 1800 (1974). Recent attempts to build strong-interaction theories by analogy with superconductivity theory are described by T. Eguchi and H. Sugawara, *ibid.* 10, 4257 (1974), and A. Chakrabarti and B. Hu, Phys. Rev. D 13, 2347 (1976); the basic early reference here is Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961). For an attempt to get the graviton as a collective effect in the Nambu–Jona-Lasinio model, see P. R. Phillips, Phys. Rev. 146, 966 (1966); see also a discussion of the photon as a collective excitation by J. D. Bjorken, Ann. Phys. (N.Y.) 24, 174 (1963).

[3]We follow the metric and γ-matrix conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Appendix A.

[4]Equation (11) is of course just a reflection of the well-known fact that the phase space for a massless particle to decay into two massless particles is nonvanishing, as expressed by the easily verified identity [which shows an obvious structural resemblance to Eq. (7)]

$$(|\vec{k}_1||\vec{k}_2|)^{-1}\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2)\delta(|\vec{k}|-|\vec{k}_1|-|\vec{k}_2|)$$
$$=2\pi\int_0^1 dx\,\delta^3(\vec{k}_2-\vec{k}x)\delta^3(\vec{k}_1-\vec{k}(1-x)).$$

A case in which the matrix element for such a process is nonvanishing is photon splitting in a strong magnetic field; see S. L. Adler, Ann. Phys. (N.Y.) 67, 599 (1971), for a detailed discussion.

[5]The momenta k_0, l_0, k_{N+1}, l_{N+1} and the four polarization indices at the ends of the ladder could each, in principle, absorb one factor of s.

[6]A residual factor of $\not{s}$ could remain at the right-hand ends of the upper and lower sides of the ladder.

[7]L. Infeld and A. Schild, Phys. Rev. 68, 250 (1945); G. E. Tauber, J. Math. Phys. 8, 118 (1967).

[8]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 441. H_0 and q_0 are, respectively, the Hubble constant and the deceleration parameter.

[9]T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. 34, 442 (1962); D. G. Boulware, L. S. Brown, and R. D. Peccei, Phys. Rev. D 2, 293 (1970). The higher-order terms in c_μ in Eq. (25) can be interpreted as relativistic corrections to uniformly accelerated motion [T. Fulton, F. Rohrlich, and L. Witten, Nuovo Cimento 26, 652 (1962)]; possible difficulties with the notion that special conformal transformations are "acceleration transformations" are discussed by Kastrup [H. A. Kastrup, Phys. Rev. 150, 1183 (1966).]

[10]In evaluating L an ultraviolet cutoff of order m is understood. Note that the photon-photon scattering box diagram vanishes rapidly for photon four-momenta $\gg m$; it is only in the effective Lagrangian approximation that the ladder sum is ultraviolet-divergent.

[11]In the absence of a cosmological term in Einstein's field equations, in the present matter-dominated era the cosmological parameters appearing in Eq. (44) are related by $k/R_0^2=(2q_0-1)H_0^2$ [see S. Weinberg, Ref. 8, p. 476]. Hence $(1+q_0)H_0^2+k/R_0^2=3q_0H_0^2$ and has the sign of the deceleration parameter q_0.

[12]For a discussion of the failure of the ladder approximation in the case of superconductivity, see J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964), p. 164–169.

[13]S. Weinberg, Phys. Rev. 135, B1049 (1964); 138, 988 (1965); R. P. Feynman, California Institute of Technology Lecture Notes, 1962 (unpublished); D. G. Boulware and S. Deser, Ann. Phys. (N.Y.) 89, 193 (1975).

[14]M. H. L. Pryce, Proc. R. Soc. London A165, 247 (1938); K. M. Case, Phys. Rev. 106, 1316 (1957).

[15]One possibility under investigation is that, in Riemannian normal coordinates, the pairing amplitude proportional to $\frac{1}{2}\langle F_{\mu\nu}F_{\lambda\sigma}+F_{\lambda\sigma}F_{\mu\nu}\rangle$ is to be identified with the curvature tensor $R_{\mu\nu\lambda\sigma}$. If this can be made to work self-consistently, then the vacuum expectation of the electromagnetic kinetic Lagrangian $-\frac{1}{4}\sqrt{g}F_{\mu\nu}F^{\mu\nu}$ could give a gravitational kinetic Lagrangian proportional to $\sqrt{g}R$. See S. L. Adler, following paper, Phys. Rev. D 14, 379 (1976).

[16]Some related questions are the following: (i) Do neutrino pairing effects occur in external gravitational fields? (ii) Does the quartic anharmonic phonon-phonon scattering interaction induce phonon pairing effects? (iii) Do pairing effects occur in non-Abelian gauge theories with massless quanta? Furthermore, if gravitation is related to electron vacuum polarization forces operating at a characteristic distance m^{-1}, then gravitational effects should show qualitative departures from the Einstein theory predictions at wavelengths shorter than m^{-1}, at temperatures greater than $5\times10^{9}\,{}^0$K, and possibly (as suggested by the calculation of Appendix C) in the presence of very strong electric and magnetic fields.

[17]S. L. Adler, Phys. Rev. D 5, 3021 (1972). We are grateful to Dr. S. Joglekar for clarifying the relation between the definition of asymptotic photon states and the calculation of $\mathfrak{L}_{\rm kin}$.

[18]V. Bargmann, Sitzber. Deut. Akad. Wiss. Berlin, Math.—Naturw. Kl. 1932, 346 (1932); L. Parker, Phys. Rev. D 3, 346 (1971).

[19]If some of the G_μ's are negative, we can make them positive by altering the signature of the Minkowski metric $\eta_{\mu\nu}$.

[20]Our notation here is that of Ref. 17, except that we denote the bare charge appearing in the Lagrangian and the corresponding fine-structure constant by e_0, α_0 rather than e_b, α_b. Equation (A33) follows immediately from Eqs. (12) and (43) of Ref. 17, while Eq. (A34) is a consequence of Eqs. (7b) and (13) of Ref. 17.

[21]Since $\beta(\alpha)=2\alpha/(3\pi)+\cdots$, to leading order in α the $n=1$ term of Eq. (A39) is $\Psi\tilde{\mathfrak{F}}2\alpha/(3\pi)$, in agreement with the expression given by Schwinger for the lowest-

order effective Lagrangian for scalar-meson decay into two photons [J. Schwinger, Phys. Rev. 82, 664 (1951), Eqs. (5.1) and (5.6) with $g/M=1$]. The $n=1$ case of the low-energy theorem of Eq. (A38) is just an analog, for the vertex (scalar) $\rightarrow 2\gamma$, of the Adler-Bardeen low-energy theorem for the vertex (pseudoscalar) $\rightarrow 2\gamma$ derived from the theory of the axial-vector anomaly. The derivation of Eq. (A38) from the Callan-Symanzik equations is analogous to the derivations of the Adler-Bardeen theorem from the Callan-Symanzik equations given by A. Zee, Phys. Rev. Lett. 29, 1198 (1972), and by J. Lowenstein and B. Schroer, Phys. Rev. D 7, 1929 (1973).

[22]Similar statements should hold in other renormalizable field theory models.

[23]The only tensors which can be formed from the metric tensor and its derivatives are obtained as contractions of the metric tensor, the curvature tensor, and covariant derivatives of the curvature tensor. See A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1961), p. 36.

[24]See, for example, D. G. Boulware, L. S. Brown, and R. D. Peccei, Phys. Rev. D 2, 293 (1970), Eq. (15). In the transformations of Eqs. (A41)–(A43) we cavalierly neglect the issue of Green's function boundary conditions, so these are very schematic arguments at best.

[25]We replace $\mathfrak{D}_0\delta^4(s_0-s_1)$ by $\delta^4(s_0-s_1)$, since the nonderivative term in $\mathfrak{D}$ survives here and we are treating D_μ and $E_{\mu\nu}$ as small quantities.

[26]See, for example, J. Schwinger, Phys. Rev. 82, 664 (1951).

[27]For a detailed discussion, see S. L. Adler, Ref. 4, Sec. 3A.

[28]R. E. Peierls, *Quantum Theory of Solids* (Clarendon Press, Oxford, 1956), p. 37. We wish to thank R. F. Dashen for a discussion about this.

[29]K. Symanzik, Lett. Nuovo Cimento 6, 77 (1973).