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Koba-Nielsen-Olesen scaling and phase transitions of the Feynman-Wilson gas

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The implications of Koba-Nielsen-Olesen scaling on the thermodynamic properties of the Feynman-Wilson gas are discussed. Consequences of the existence of a thermodynamic limit as a fundamental demand on hadronic physics at asymptotic energies are deduced. It is generally found that the Feynman-Wilson gas undergoes a phase transition at infinite energies. First-order transitions lead to average multiplicity growth like the rapidity Y, while higher-order transitions correspond to the behavior suggested by the absorptive-model cutting rules in Pomeron calculus. Possible links with the critical phenomena obtained in Reggeon field theories are conjectured.

I. INTRODUCTION

In the present work we investigate the relevance of the Koba-Nielsen-Olesen (KNO) scaling hypothesis[1] to the thermodynamics of the Feynman-Wilson (FW) gas.[2] As has been widely advocated,[3] the FW gas analogy could be employed as a useful mathematical tool in the study of hadronic properties at high energies. In our analysis we shall introduce the requirement of the existence of a thermodynamic limit as a fundamental dynamical assumption, which if supplemented with the KNO scaling hypothesis has several interesting consequences for hadron physics. In this way, the formal mathematical analogy between high-energy physics and statistical mechanics will be given serious physical consideration.

Outside the KNO scaling framework, the FW gas analogy has been extensively explored[4,5,6] and the thermodynamic limit has been considered.[7] In particular, it has been argued[5] that there is experimental evidence that the FW gas is at the critical point at high energies, and this is best revealed from the viewpoint of the grand canonical ensemble.[6]

Since our approach is relevant to the s-channel description of a process, we expect it to be complementary to those emphasizing the t-channel properties, namely the Reggeon field theories (RFT).[8] In particular, we might expect the critical phenomena which are related to the RFT to manifest themselves in the FW-gas thermodynamics. In fact, it has been argued[9] that approaching the critical temperature from above, which in the Pomeron-calculus language means that the Pomeron intercept approaches unity from below, results in the formation of high-mass clusters, droplets, which resembles condensation phenomena in the FW gas. Since the strong-coupling solution of RFT resembles a higher-order phase transition in statistical systems, there have been speculations[10] that we could possibly *derive* the strong-coupling solution of RFT assuming that there is a higher-order phase transition in the FW gas. Indeed, assuming a KNO scaling function without a cutoff, we shall show that the FW gas does have a higher-order phase transition at asymptotic energies, leading to multiplicity and total-cross-section behavior suggested by the absorptive-model (AM) cutting rules in RFT.[11,12] On the contrary, a KNO scaling function with a cutoff at a finite value of $n/\langle n\rangle$ leads to a first-order phase transition and a logarithmic multiplicity growth, as suggested by the simplest two-component model,[3] or the simplified-absorptive-model (SAM) cutting rules in RFT.[11,14]

The plan of this paper is as follows. In Sec. II we introduce our formalism and give a series of statements on the behavior of the FW gas, resulting on rather general grounds from the KNO scaling hypothesis. In Sec. III we specify the asymptotic behavior of the KNO scaling function and make several predictions on the behavior of the average multiplicity and the total cross section. Section IV deals with the large-order multiplicity moments, their connection with the asymptotic behavior of the KNO scaling function in our model, and the relevance of their structure to the RFT results. Finally, our conclusions are given in Sec. V.

II. KNO SCALING AND FEYNMAN-WILSON-GAS THERMODYNAMICS

With the assumption of KNO scaling ($x = n/\langle n\rangle$)

$$\langle n\rangle \sigma_n(Y)/\sigma_t(Y) = \psi(x) , \qquad (1)$$

the generating function

$$Q(z, Y) = \sum_n z^n \sigma_n(Y)/\sigma_t(Y) \ , \tag{2}$$

which plays the role of the grand partition function with z the fugacity and Y the rapidity, has the following integral representation at high energies ($w = \langle n \rangle \ln z$):

$$Q(w) = \int_0^\infty \psi(x) e^{wx}\, dx \ . \tag{3}$$

The moments of the multiplicity $c_p = \langle n^p \rangle / \langle n \rangle^p$ are given by

$$c_p = \int_0^\infty \psi(x) x^p dx \ . \tag{4}$$

The obvious normalization conditions on the KNO scaling function $\psi(x)$ are

$$\int_0^\infty \psi(x)\, dx = \int_0^\infty x\psi(x)\, dx = 1 \ . \tag{5}$$

The existence of the generating function $Q(z, Y)$ for $z > 1$ imposes on $\psi(x)$ the bound

$$\psi(x) < e^{-cx} \tag{6}$$

at large x, for any positive c.

We now form the quantity

$$\tilde{P}(z, Y) = \frac{1}{Y} \ln Q(z, Y) \ . \tag{7}$$

If the thermodynamic limit

$$\lim_{Y \to \infty} \tilde{P}(z, Y) \equiv P(z) \tag{8}$$

exists, its finite value $P(z)$ is called the "analog pressure" of the FW gas, which is said to undergo a phase transition of order n at $z = z_0$ if the nth derivative of $P(z)$ with respect to z is discontinuous at $z = z_0$, but all lower derivatives of $P(z)$ are continuous there.[13] If some derivative of $P(z)$ diverges at $z = z_0$, we are dealing with a λ transition, as in liquid He^4.[13] As suggested by classical statistical mechanics,[13] we assume that the convergence of $\tilde{P}(z, Y)$ in (8) as well as the convergence of $(\partial/\partial z)\tilde{P}(z, Y)$ is uniform for $z > 1$.

We now study some general implications of the KNO scaling hypothesis on the FW-gas behavior. By inspection of representation (3) of the generating function, several statements based on rather general grounds can be made.

We start by noting that from (2) and (7) the quantity $\tilde{P}(z, Y)$ is a nondecreasing function of z $[\sigma_n(Y) \geq 0]$. Moreover, we have $\tilde{P}(z < 1, Y) < 0$, $\tilde{P}(z = 1, Y) = 0$, and $\tilde{P}(z > 1, Y) > 0$. Assuming that $\langle n \rangle$ increases with Y and integrating (3) by parts, we obtain at high energy for $z < 1$

$$Q(w) = \frac{\psi(0)}{|w|} + O(w^{-2}) \ ; \tag{9}$$

hence, if $\langle n \rangle$ does not increase like a power of the energy, we have for the analog pressure (8)

$$0 \geq P(z) = \lim_{Y \to \infty} \frac{1}{Y} \ln \frac{\psi(0)}{|w|} = 0, \quad 0 < z < 1 \ . \tag{10}$$

On the other hand, we have $P(z = 1_+) = 0$ and $P(z > 1) \geq 0$ (and nondecreasing with z). Hence, provided that the thermodynamic limit exists for $z > 1$, $P(z)$ is a continuous function of z at $z = 1$, but unless $P(z) \equiv 0$ it cannot be analytic there. Thus, we conclude that *at infinite energies, the FW gas always undergoes a phase transition at $z = 1$.* The above discussion leads to the physical picture that as we approach the critical point $z = 1$ from values $z > 1$, droplets are always formed in the FW gas at asymptotic energies, resulting in zero pressure for $z \leq 1$.

Let us now assume that the thermodynamic limit (8) exists for $z > 1$. From (3) we obtain

$$+\infty > P(z) > \lim_{Y \to \infty} \frac{1}{Y} \ln \psi(x_1) \delta x_1 + \lim_{Y \to \infty} \frac{|w| x_1}{Y} \tag{11}$$

for arbitrary x_1 if δx_1 is sufficiently small. We then have the following:

(i) If $\psi(x)$ does not have a cutoff, we can move x_1 to as large values as we like, and inequality (11) necessarily leads to the bound $\langle n \rangle < Y$.

(ii) If $\psi(x)$ has a cutoff, x_1 may only be finite and we obtain the bound $\langle n \rangle \leq Y$. Moreover, assuming a cutoff at $x = x_c$ in $\psi(x)$ we can explicitly show that the equality sign survives in this bound, while the FW gas undergoes a first-order phase transition at $z = 1$. Indeed, integrating (3) by parts, we obtain for $z > 1$

$$\ln Q(z, Y) = w x_c + O(\ln w) \ . \tag{12}$$

Hence, if we impose the condition of the existence of thermodynamic limit we obtain $\langle n \rangle \sim Y$ and $P(z) \sim \ln z$, which gives a first-order phase transition at $z = 1$. Note that this is precisely the answer obtained from a simple two-component model,[3] with a correlation-free multiperipheral component, leading to a Poisson-type multiplicity distribution, and a diffractive component responsible for the first-order phase transition. Because of KNO scaling we obtain $\langle n^p \rangle \sim Y^p$, which is the result of the SAM cutting rules of Caneschi and Jengo.[11,14] The conclusion of this discussion is that *a necessary condition for the existence of a thermodynamic limit for $z > 1$ is the bound $\langle n \rangle \leq Y$.* The equality sign appears when the KNO scaling function $\psi(x)$ has a cutoff at a finite value of x, and then the FW gas has a first-order phase transition.

Lastly, we have for $z \geq 1$

$$\frac{\partial}{\partial z} P(z) = \lim_{Y \to \infty} \frac{\int_0^\infty \psi(x) x \langle n \rangle z^{-1} e^{wx} dx}{Y \int_0^\infty \psi(x) e^{wx} dx} \ , \tag{13}$$

which, because of the normalization (5) of $\psi(x)$, gives

$$\frac{\partial}{\partial z} P(z=1_+) = \lim_{Y\to\infty} \frac{\langle n\rangle}{Y} . \qquad (14)$$

Thus, provided that the thermodynamic limit exists, because of our previous discussion, the limit of the first derivative of the analog pressure with respect to z, as z approaches unity from the right, is always zero, unless $\psi(x)$ has a cutoff when it has a nonzero finite value. Hence, we conclude that *first-order phase transitions in the FW gas at* $z=1$ *at infinite energies are only allowed if the KNO scaling function* $\psi(x)$ *has a cutoff.* If $\psi(x)$ does not have a cutoff the FW gas always undergoes a higher-order phase transition at $z=1$.

To complete this section we mention that a particular Van der Waals model of the FW gas (first-order phase transition) with $\langle n\rangle \sim Y$ has been considered by Arnold and Thomas.[5] But this model is outside our framework since it does not have KNO scaling.

III. HIGHER-ORDER PHASE TRANSITIONS

Having established that the FW gas exhibits phase transitions at asymptotic energies and that first-order transitions correspond to a KNO scaling function with a cutoff, we now study in detail the nature of the transition obtained in the case that there is no cutoff.

It is convenient to write

$$\psi(x) = g(x)e^{-f(x)} , \qquad (15)$$

where $g(x)$ is bounded by a power of x from above and below, and because of the bound (6), $f(x)$ must be increasing faster than x at large x. Note that such an asymptotic behavior of $\psi(x)$ guarantees unique solution of the moment problem.[1] By standard steepest descent[15] we find for $z>1$

$$\ln Q(w) = wx_0 - f(x_0) + \ln\left[\frac{2\pi}{f''(x_0)}\right]^{1/2} \sum_{n=0}^{\infty} \frac{g^{(2n)}(x_0)}{(2n)!![f''(x_0)]^n} , \qquad (16)$$

where the saddle point x_0 is defined by $w=f'(x_0)$. Since $f(x)$ is increasing faster than x at large x, $f'(x)$ is an increasing function of x and the saddle point x_0 moves to infinity when the energy becomes very large. Therefore *only the asymptotic behavior of* $\psi(x)$ *is relevant to the calculation of thermodynamic quantities at asymptotic energies.* In particular, *the critical exponents are independent of the detailed behavior of* $\psi(x)$ *at finite* x.

We next specify the asymptotic behavior of the function $f(x)$:

$$f(x) = \alpha x^k, \quad k>1 . \qquad (17)$$

We shall show that the power k is connected to the inverse of the critical exponent η defined in RFT. Introducing (17) into the asymptotic expansion (16), we obtain for $w>0$

$$\ln Q(w) = \alpha(k-1)\left(\frac{w}{\alpha k}\right)^{k/(k-1)} + O(\ln w) . \qquad (18)$$

Requiring in order for this analysis to make sense the existence of the thermodynamic limit (8) as $Y\to\infty$, we are led, together with the assumption $P(z)\neq 0$, to

$$\langle n\rangle \sim Y^{1-\eta}, \quad 0<\eta=\frac{1}{k}<1 . \qquad (19)$$

From this relation and KNO scaling we obtain

$$\langle n^p\rangle \sim Y^{p(1-\eta)}, \quad 0<\eta<1 . \qquad (20)$$

It is worth noting that this behavior coincides with the result of the AM cutting rules in RFT, at least to the lowest order in the ϵ expansion,[11] but disagrees with the Abramovskii-Gribov-Kancheli (AGK) cutting rules[16] which lead to the nonperturbative result[11] (see also Ref. 12).

$$\langle n^p\rangle \sim Y^{p(1+\eta)}, \quad \eta>0 \qquad (21)$$

which, anyway, because of inequality (11), does not allow for a thermodynamic limit. Note that the critical exponent η is identified as the inverse of the power k appearing in the asymptotic behavior (17) of $f(x)$. If we also assume Feynman scaling, (19) means

$$\sigma_t \sim Y^{\eta}, \quad 0<\eta<1 . \qquad (22)$$

From (8) and (18) we obtain the analog pressure for $z>1$:

$$P(z) \sim (\ln z)^{\nu}, \quad \nu=\frac{1}{1-\eta}=\frac{k}{k-1}>1 . \qquad (23)$$

Recall that for $z\leq 1$, we generally have $P(z)=0$. Hence, if ν is not an integer we have a λ transition at $z=1$ [see Fig. 1(a)], while if ν is an integer we have an ordinary phase transition of order $\nu \geq 2$ [see Fig. 1(b)]. Note that first-order phase transitions are not allowed, in agreement with our general argument of the preceding section. But if $k\to\infty$ [$\psi(x)$ falls very rapidly to zero], we have $\eta\to 0$, $\nu\to 1$ and we obtain a first-order phase transition together with the results we had from general arguments in the preceding section, where we considered a KNO scaling function with an explicit cutoff. If the second derivative of the pressure with respect to z has a discontinuity at the critical point $z=1$, we have $1<\nu\leq 2$, $0<\eta\leq\frac{1}{2}$, $k\geq 2$. Note that the values of the critical exponent η obtained by ϵ expansion of the loop expansion in RFT fall in this range.[9,10] If the discontinuity is exhibited in a higher-order derivative we have $\nu>2$, $\frac{1}{2}<\eta<1$,

$1 < k < 2$, and this range of the η values is consistent with the high-temperature expansion results in RFT.[9,10] The connection between the allowed values of the exponents k, η, ν and the nature of the corresponding phase transition is illustrated in Fig. 2.

If instead of (17) we assume that $f(x)$ has the more general asymptotic behavior

$$f(x) = \alpha x^k (\ln x)^\lambda, \quad k > 1, \quad -\infty < \lambda < +\infty \tag{17'}$$

the requirement of the existence of thermodynamic limit leads to

$$\langle n^p \rangle \sim Y^{p(1-\eta)} (\ln Y)^{\lambda \eta p} , \tag{20'}$$

but the pressure (23) remains unchanged. This means that the leading contribution to $\langle n^p \rangle$ is basically unchanged, and we have the same critical behavior of the FW gas at $z = 1$. Note that if $k = 1$ and $\lambda > 0$ the generating function $Q(w)$ exists, but the thermodynamic limit (8) does not exist for any Y dependence of $\langle n \rangle$. This means that models with a KNO scaling function of the form

$$\psi(x) = g(x) e^{-\alpha x (\ln x)^\lambda}, \quad \lambda > 0 \tag{24}$$

and in particular the model of Ref. 17, which has $\lambda = 1$, cannot be accommodated in our scheme.

IV. STRUCTURE OF THE MULTIPLICITY MOMENTS

In this section we examine in some detail the structure of the energy-independent quantities c_p, which, having assumed KNO scaling, are given by Eq. (4). Their form for large p reflects the asymptotic behavior of $\psi(x)$, which, as we have shown, is connected to the critical phenomena in

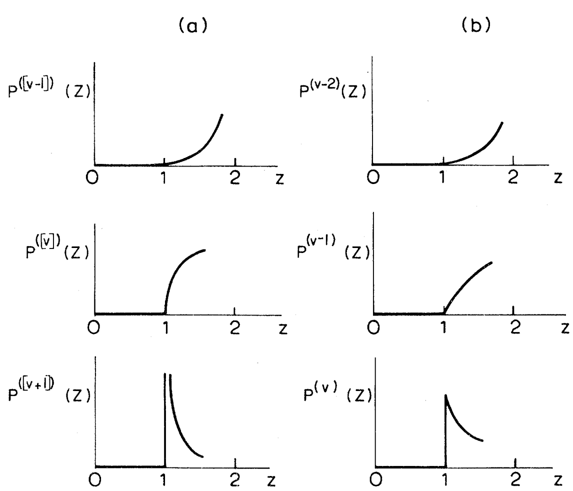

FIG. 1. Derivatives of the Feynman-Wilson-gas analog pressure near the critical point $z = 1$. (a) noninteger ν corresponding to λ transition, (b) integer ν corresponding to νth-order ordinary phase transition.

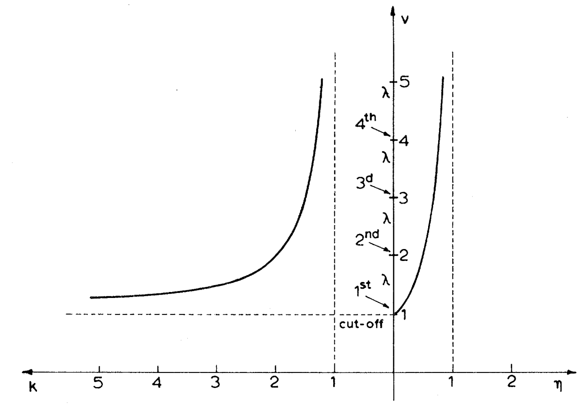

FIG. 2. Illustration of the connection between the exponents k, η, ν and the nature of the various phase transitions of the Feynman-Wilson gas.

the FW gas. On the other hand, the Reggeon field theories predict energy-independent c_p, which if compared with our results can possibly provide links with our s-channel approach.

Writing

$$c_p = \phi(p) e^{p h(p)} , \tag{25}$$

where $\phi(p)$ is polynomially bounded, the inverse Mellin transform of (4) reads

$$t \psi(t) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \phi(p) e^{p[h(p) - \ln t]} dp . \tag{26}$$

Clearly, if $h(p)$ is finite for $p \to \infty$, $\psi(t)$ has a cutoff at a finite value of t since c may be moved as far to the right as we like, but if $h(p) \to \infty$ for $p \to \infty$ no such cutoff exists. Conversely, assuming that $\psi(x)$ has a cutoff at $x = x_c$ and integrating (4) by parts, we obtain

$$c_p \sim x_c^p \phi(p) , \tag{27}$$

namely, (25) in the special case $h(p) = \ln x_c$.

After these general comments on the connection of the asymptotic behavior of $\psi(x)$ with the form of c_p, we assume the general form (15) for $\psi(x)$, and by standard steepest-descent methods[15] we obtain from (4)

$$c_p = y_0^p e^{-f(y_0)} \left[\frac{2\pi}{f''(y_0) + p/y_0^2} \right]^{1/2} \times \sum_{n=0}^{\infty} \frac{g^{(2n)}(y_0)}{(2n)!! [f''(y_0) + p/y_0^2]^n} , \tag{28}$$

where the saddle point y_0 is defined by $y_0 f'(y_0) = p$. Hence, with (17) and (17'), provided that $g(x)$ is bounded by a power of x from above and below, the leading contribution to c_p for large p is

$$c_p \sim (\text{const})^p p^{\eta p} \tag{29}$$

and

$$c_p \sim (\text{const})^p p^{\eta p} / (\ln p)^{\lambda \eta p} , \tag{29'}$$

respectively. Since we have $\eta < 1$, it is easy to verify that the sufficient condition

$$\sum_{p=0}^{\infty} (c_p)^{-1/2p} = \infty \quad (30)$$

for the uniqueness of solution of the moment problem[1] is fulfilled in both cases. Note that the behavior (29) or (29′) corresponds to the case $h(p) \to \infty$ for $p \to \infty$, previously discussed.

Having calculated c_p we can now continue the comparison of our results with the recent findings of Reggeon field theories. As already mentioned, the results obtained with the AGK cutting rules cannot be accommodated into our scheme in the sense that they do not allow for the thermodynamic limit.

With the AM cutting rules, to the lowest order in the ϵ expansion c_p are calculated by Caneschi and Jengo[11] to be

$$c_p = 1 + \frac{\epsilon}{12}\left\{p \sum_{l=3}^{p}\left[\frac{1}{l} - \frac{2}{l(l-1)}\right] + \frac{p}{2} - 1 + \sum_{l=2}^{p}\frac{1}{l}\right\}. \quad (31)$$

Their conjecture $c_p \sim e^{c\epsilon p}$ for large p is not consistent with the assumption of the existence of a thermodynamic limit since (i) from our analysis such a behavior implies a KNO scaling function with a cutoff, which in turn leads to $\langle n\rangle \sim Y$, but (ii) to lowest order in ϵ the AM cutting rules lead to the result (20) with $\eta = \frac{1}{12}\epsilon > 0$,[8] which allows for the thermodynamic limit. Instead, replacing the sums in (31) with integrals for large p, one obtains

$$c_p \sim 1 + \eta p \ln p \quad (31')$$

and is tempted to conjecture that higher orders in ϵ will provide the remaining terms to resurrect the dominant contribution to (29) by exponentiation, while leaving the behavior (20) unaffected.

Finally, let us mention that the SAM-cutting-rules result to first order in ϵ,[11] namely $c_p = 1$,[14] is equivalent to a Poisson distribution,[1] which again implies that $\langle n\rangle \sim Y$, consistent with the SAM result as mentioned in Sec. II.

V. CONCLUSIONS

We have taken seriously the analogy between statistical mechanics and high-energy multihadron physics assuming that the latter possesses a thermodynamic limit at infinite energies. We armed our formalism with the KNO scaling hypothesis which leads to an easily handled integral representation for the generating function. We insisted on comparisons between our results and the findings of Reggeon field theories. Our main results may be classified according to the behavior of the KNO scaling function $\psi(x)$ for large x:

(i) If $\psi(x)$ has a cutoff at a finite value of x, then the Feynman-Wilson gas has a first-order phase transition at $z=1$ at infinite energies as suggested by the simplest two-component model. The multiplicity grows like Y, as suggested by the SAM cutting rules in Pomeron calculus. The same result is also suggested by ϕ^4 considerations in Reggeon field theories relevant to the s-channel discontinuity of the scattering amplitude. We find that the moments of the distribution behave like $c_p \sim (\text{const})^p$ for large p, and assuming Feynman scaling we are led to a constant total cross section.

(ii) If $\psi(x)$ does not have a cutoff, but obeys the bound $\psi(x) < e^{-cx}$ for any positive c, for large x, we find that the Feynman-Wilson gas undergoes a higher-order phase transition at $z=1$, at infinite energies, which in general is a λ transition as in liquid He^4. The requirement of the existence of a thermodynamic limit puts on the average multiplicity the bound $\langle n\rangle < Y$. We calculated $\langle n^p\rangle \sim c_p Y^{p(1-\eta)}$, $0<\eta<1$, and assuming Feynman scaling we found $\sigma_t \sim Y^{\eta}$. These results are similar to those obtained by the AM cutting rules in Pomeron calculus, at least to the lowest order in the ϵ expansion. They also follow from ϕ^3 considerations in Reggeon field theories relevant to the calculation of the s-channel discontinuity of the scattering amplitude. The dominant contribution to the moments of the distribution has, for large p, the form $c_p \sim p^{\eta p}$, and we were tempted to associate this behavior with the lowest-order-in-ϵ AM-cutting-rules result $c_p \sim 1 + \eta p \ln p$.

We believe our approach provides interesting links of the critical phenomena obtained in Reggeon field theories from t-channel-unitarity considerations, with the critical behavior of the Feynman-Wilson gas at asymptotic energies.

ACKNOWLEDGMENT

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